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Cores and Compactness of Infinite Directed Graphs

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In this paper we define the property of homomorphic compactness for digraphs. We prove that if a digraph H is homomorphically compact then H has a core, although the converse does not hold. We also examine a weakened compactness condition and show that when this condition is assumed, compactness is equivalent to containing a core. We use this result to prove that if a digraph H of size κ is not

1. INTRODUCTION

A directed graph or digraph G is a pair $(V(G), E(G))$ where $V(G)$ is a set called the *vertex-set* of G and $E(G)$ is a subset of $V(G) \times V(G)$ called the *edge-set* of G. Hence, the digraphs we consider in this paper may have loops but do not have multiple edges. As usual we denote the edge (u, v) by uv. The *in-neighborhood* of a vertex $v \in V(G)$ is the set $\{u: uv \in E(G)\}$ and is denoted $N^-(v)$. The *out-neighborhood* of v is the set $N^+(v) = \{u: vu \in E(G)\}$. We define an equivalence relation $\equiv$ on the vertex-set of a digraph by $u \equiv v$ if and only if $N^+(u) = N^+(v)$ and $N^-(u) = N^-(v)$. If both $N^-(v)$ and $N^+(v)$ are finite for each $v \in V(G)$ then we say that G is *locally finite*.

Given $u, v \in V(G)$, we define the *distance from u to v* in G, denoted $d(u, v)$, to be the number of edges in a shortest oriented path from u to v. If $u \in V(G)$ and $S \subseteq V(G)$, we define $d(u, S) = \min_{v \in S} d(u, v)$.

By convention we will consider the *size* (or *cardinality*) of a digraph to be the size of its vertex-set, so $|G|$ is defined to be equal to $|V(G)|$. Also, if κ is any cardinal we denote by κ^+ the smallest cardinal that is larger than κ.

If G is a subdigraph of H we write $G \subseteq H$. If $S \subseteq V(G)$, then we denote by $G[S]$ the subdigraph of G induced by S.

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Let G and H be digraphs. A *homomorphism* from G to H is a mapping $f: V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$ for all $u, v \in V(G)$. We will often use the notation $f: G \to H$ when f is a homomorphism from G to H. We write $G \to H$ to indicate that a homomorphism from G to H exists. We denote by $f(V(G))$ the set $\{f(v): v \in V(G)\}$. The set $f(V(G))$ is also called the *range of* f or *range*(f). We denote by $f(G)$ the digraph with vertex-set $f(V(G))$ and edge-set $\{f(u)f(v): uv \in E(G)\}$. We call $f(G)$ the *image of* G *under* f. A homomorphism is said to *preserve non-edges* if $uv \notin E(G)$ implies $f(u)f(v) \notin E(H)$ for all $u, v \in V(G)$. Observe that if a homomorphism f is a bijection and preserves non-edges then f is an isomorphism. If $G \to H$ and $H \to G$ then we write $G \leftrightarrow H$ and say that G and H are *homomorphically equivalent* or simply *equivalent*.

A homomorphism from a digraph G to itself is called an *endomorphism of* G. An endomorphism which is not a surjection is called a *proper* endomorphism. An isomorphism from G to itself is called an *automorphism*. We use the standard notation $f|_S$ to indicate the restriction of a function f to a subset S of its domain. If f is an endomorphism of G and $f|_{f(V(G))}$ is the identity mapping then f is called a *retraction* of G.

2. CORES

The notation of a *core* of a digraph has been used often in the study of homomorphic properties of finite digraphs [1, 8, 12, 14, 15, 20]. In [2], the present author investigated various questions relating to cores in infinite digraphs. The two standard definitions of a core are the following.

1. G is a core if G admits no proper endomorphism [12].
2. G is a core if G admits no proper retraction [14].

A digraph H is said to be a core of a digraph G if $H \subseteq G$, $G \to H$, and H is a core. We say that G *has a core* if such an H exists.

In [2], we showed that the standard definitions of a core, while equivalent for finite digraphs, are not equivalent when applied to infinite digraphs. Furthermore, cores defined in either of the standard ways lose many of their attractive properties in the infinite case. We were led to propose the following definition of a core:

DEFINITION 1. A digraph G is a core if every endomorphism of G is an automorphism of G.

Clearly this condition is stronger than (1) and (2). However, it is a simple matter to verify that they are all equivalent for finite digraphs.

It is also easy to see that every finite digraph has a core. Unfortunately, this is not true for infinite digraphs. The natural problem to consider is that of characterizing infinite digraphs which have cores. We will provide a partial solution using the property of homomorphic compactness, which we will define in the next section.

We will require two of the results from [2].

LEMMA 2. *If a digraph G is not a core then there is either an endomorphism of G which is not a surjection, or there is an endomorphism of G which does not preserve non-edges.*

LEMMA 3. *If f is an endomorphism of a digraph G such that f preserves non-edges, and $f(u)=f(v)$ for some $u, v \in V(G)$, then $u \equiv v$.*

3. COMPACTNESS

Let G and H be digraphs. We say that H is *compact with respect to* G if either $G \to H$ or there exists a finite subdigraph G' of G such that $G' \nrightarrow H$. We say that H is α-*compact* if H is compact with respect to G for all G with $|G| \leqslant \alpha$. If H is compact with respect to every G, then we say that H is *compact*. Equivalently, we might say that H is compact if for every digraph G, we have $G \to H$ if and only if $G' \to H$ for every finite subdigraph G' of G. If every finite subdigraph of G admits a homomorphism to H, but $G \nrightarrow H$, then we say G is a *certificate of non-compactness* for H. Observe that if G is a certificate of non-compactness for H then some connected component of G is a certificate of non-compactness for H.

If two digraphs H and H' are homomorphically equivalent and G is any digraph, then clearly H is compact with respect to G if and only if H' is compact with respect to G. Also note that if H is α-compact and $\beta < \alpha$ then H is β-compact.

The property of homomorphic compactness itself is of considerable interest. A well-known result of de Bruijn and Erdös [5] states that an infinite digraph G is n-colourable if and only if every finite subdigraph of G is n-colourable. This is equivalent to the statement that the complete digraph K_n is compact. A more general result, proved in [10], can be used to show that all finite digraphs are compact. Later on, we will give another proof that all finite digraphs are compact. However, our general method will allow us to give examples of several large classes of infinite digraphs which are compact.

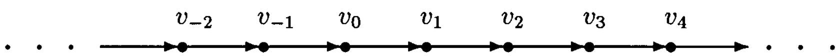

FIG. 1. The Line.

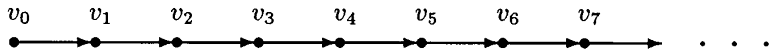

FIG. 2. The Ray.

EXAMPLE 4. The Line (Fig. 1) is compact.

It is well known that a digraph G admits a homomorphism to the Line if and only if every oriented cycle C in G admits a homomorphism to the line. Thus, if G does not admit a homomorphism to the line then there is some finite subdigraph (an oriented cycle) of G which admits no homomorphism to the line. That the Line is compact also follows from results proved later in this paper.

EXAMPLE 5. The Ray (Fig. 2) is not compact.

The Line is a certificate of non-compactness for the Ray, since every finite subdigraph of the Line admits a homomorphism to the Ray, but clearly the Line admits no homomorphism to the Ray. In fact, the Line demonstrates that the Ray is not $\aleph_0$-compact.

In the next section we will show that every compact digraph contains a core. In pursuit of this result we will also give some results relating to other special digraphs necessarily contained in compact digraphs.

In Section 5 we examine digraphs H which are $|H|$-compact. We prove that if H is a $|H|$-compact core then H is compact. We use this result to prove that a digraph H is compact if and only if it is $|H|^+$-compact.

In Section 6 we will describe some large classes of compact digraphs, and thereby also describe classes of infinite digraphs with cores.

4. COMPACT DIGRAPHS HAVE CORES

The problem of characterizing cores is a difficult one even in the case of finite digraphs, and has been solved only for undirected graphs with independence number at most two [12]. For infinite digraphs we also have the related problem of characterizing those digraphs which have cores. In this section we provide a partial characterization by showing that every compact digraph has a core. We begin with the following lemma:

LEMMA 6. *If H is an $|H|$-compact digraph then there exists a digraph $H' \supseteq H$ such that $V(H') = V(H)$, $H' \to H$, and every endomorphism of H' preserves non-edges.*

Proof. Suppose that H is $|H|$-compact but no such H' exists. We define a transfinite sequence $\{H_\alpha\}$ of superdigraphs of H on the same vertex-set

$V(H)$. Each of these will have the property that $H_\alpha \to H$. Let $H_0 = H$, and if H_α is defined for some ordinal α, define $H_{\alpha+1}$ as follows. Since $H_\alpha \to H$, by assumption there is an endomorphism f of H_α which does not preserve non-edges.

Define $H_{\alpha+1}$ by

$$V(H_{\alpha+1}) = V(H_\alpha)$$

$$E(H_{\alpha+1}) = \{uv: f(u) f(v) \in E(H_\alpha)\}$$

Clearly $f: H_{\alpha+1} \to H_\alpha$ is a homomorphism, and so by transitivity we have $H_{\alpha+1} \to H$. Also, since f does not preserve non-edges in H_α, it must be the case that $E(H_\alpha)$ is properly contained in $E(H_{\alpha+1})$.

If λ is a limit ordinal and H_α is defined, with $V(H_\alpha) = V(H)$, for all $\alpha < \lambda$, then we define H_λ by

$$V(H_\lambda) = V(H)$$

$$E(H_\lambda) = \bigcup_{\alpha < \lambda} E(H_\alpha).$$

To show that $H_\lambda \to H$, let K be any finite subdigraph of H_λ. Then K has only finitely many edges, so $K \subseteq H_\alpha$ for some $\alpha < \lambda$, and so $K \to H$. Since $|H_\lambda| = |H|$ and H is $|H|$-compact, we have $H_\lambda \to H$.

Now for each α, the cardinality of $E(H_\alpha)$ can be no more than $|H_\alpha| = |H|$. At each step in the above induction we add at least one new edge to $H_{\alpha+1}$, and we never remove edges once added, and so $|E(H_\alpha)| \geqslant |\alpha|$. This yields a contradiction when $|\alpha| > |H|$. ∎

We now apply this lemma to obtain some useful results relating to subdigraphs of compact digraphs.

LEMMA 7. *Let H be a digraph which is $|H|$-compact. If G is a subdigraph of H such that $H \to G$ and every endomorphism of G is a surjection, then G is a core of H.*

Proof. Suppose G is not a core. Since every endomorphism of G is a surjection, by Lemma 2 it must be the case that some endomorphism of G does not preserve non-edges.

We will show that for every superdigraph H' of H such that $H' \to H$ there is an endomorphism of H' which does not preserve non-edges, in order to obtain a contradiction to Lemma 6. Thus, let H' be a superdigraph of H which admits a homomorphism to H. Then H' admits a homomorphism to G.

Now let $h: H' \to G$ be a homomorphism, and let f be an endomorphism of G which does not preserve non-edges. There exist $u, v \in V(G)$ such that

$uv \notin E(G)$ but $f(u)f(v) \in E(G)$. Since $G \subseteq H'$ and every endomorphism of G is a surjection, the mapping h must be a surjection, and so both u and v have pre-images u' and v' under h in H'. Now $u'v'$ cannot be an edge of H' since then h would not be a homomorphism, but $(f \circ h)(u')(f \circ h)(v')$ is an edge of H'. Thus the composition $f \circ h$ is an endomorphism of H' which does not preserve non-edges. ∎

According to the above lemma, in order to show that a compact digraph H has a core we need only to show that H contains a subdigraph G such that $H \to G$ and every endomorphism of G is a surjection. Our next two lemmas will show that a compact digraph does indeed contain such a subdigraph.

We will require the following definition.

Definition 8. Let K be a digraph. Let S be a set containing one vertex from each equivalence class of $V(K)$ under $\equiv$. We define a new digraph K^r to be the subdigraph of K induced by S. The digraph K^r is called the *reduced* digraph of K.

It is a simple matter to verify that $K \to K^r$, and that no two vertices of K^r are equivalent. We will refer to the endomorphism which maps every vertex of K to the representative of its equivalence class as the *canonical* homomorphism from K to K^r.

Lemma 9. *If H is an $|H|$-compact digraph, then there is a subdigraph G of H such that $H \to G$, and every endomorphism of G is a injection which preserves non-edges.*

Proof. Suppose H is $|H|$-compact but no such G exists. As in the previous lemma we will show that for every superdigraph H' of H such that $H' \to H$, there is an endomorphism of H' which does not preserve non-edges, contradicting Lemma 6.

Let H' be a superdigraph of H such that $H' \to H$, and let $f: H' \to H$ be a homomorphism. Let $G = (f(H'))^r$. By definition f is a surjection from H' to $f(H')$, and the canonical homomorphism from $f(H')$ to G is a surjection as well. Thus the composition of these homomorphisms is a surjection from H' to G. Call this composition f'.

Now $G \subseteq H$ and $H \to G$ because $H \subseteq H'$ and $H' \to G$, so by assumption there is an endomorphism g of G which is either not an injection or does not preserve non-edges. However, observe that if g is not an injection then g does not preserve non-edges either, since in that case there exist $u, v \in V(G)$ such that $g(u) = g(v)$. But in G no two vertices are equivalent, and so Lemma 3 implies that g does not preserve non-edges.

Thus, g is an endomorphism of G which does not preserve non-edges. Look at the composition $g \circ f'\colon H' \to G$. This homomorphism does not preserve non-edges since g does not preserve non-edges and f' is a surjection. Also, $G \subseteq H \subseteq H'$, so $g \circ f'$ is an endomorphism of H' which does not preserve non-edges. ▮

This lemma is used to prove the following important result.

THEOREM 10. *If H is an $|H|^+$-compact digraph then H has a core.*

Proof. Let H be a $|H|^+$-compact digraph. Since H is also $|H|$-compact, by the preceding lemma we know that H contains a subdigraph H_0 such that $H \to H_0$, and every endomorphism of H_0 is an injection and preserves non-edges. We need only show that every endomorphism of H_0 is a surjection.

Suppose that H_0 admits a non-surjective endomorphism. Note that H_0 is $|H_0|^+$-compact, since $H_0 \leftrightarrow H$ and $|H_0|^+ \leqslant |H|^+$. As in Lemma 6 we will obtain a contradiction by constructing a transfinite sequence of digraphs $\{H_\alpha\}$, for $\alpha \leqslant |H_0|^+$. For each $\alpha > 0$ the digraph H_α will be a proper super-digraph of H_0. Each H_α will have the following properties: every endomorphism of H_α is an injection and preserves non-edges, $|H_\alpha| = |H_0|$, and H_α is isomorphic to a subdigraph of H_0. This last property is somewhat counter-intuitive, but is quite possible when dealing with infinite digraphs. In the construction we are about to present, unlike that in Lemma 6, the vertex-set will not remain constant.

We have already defined H_0, which satisfies the above conditions by assumption.

Now suppose that H_α is defined and that H_α has the required properties. We claim that there must exist a non-surjective endomorphism of H_α. When $\alpha = 0$ this is true by assumption. If $\alpha > 0$ then we know that there is a homomorphism $h\colon H_\alpha \to H_0$ because H_α is isomorphic to a subdigraph of H_0. But H_0 is a proper subdigraph of H_α, so h is a non-surjective endomorphism of H_α.

We now define $H_{\alpha+1}$.

Let $I = \{v \in V(H_\alpha)\colon h^{-1}(v) = \varnothing\}$, i.e., I is the set of all vertices which have no pre-image under h. Obviously I is nonempty. Let $I' = \{v'\colon v \in I\}$ be a set of new vertices. We will define $H_{\alpha+1}$ in two steps. First we define

$$V(H_{\alpha+1}) = V(H_\alpha) \cup I'.$$

Clearly $|H_{\alpha+1}| = |H_\alpha| = |H_0|$. We now define a mapping $h'\colon V(H_{\alpha+1}) \to V(H_\alpha)$ by $h'|_{V(H_\alpha)} = h$, and $h'(v') = v$ for all $v' \in I'$. We may now define

$$E(H_{\alpha+1}) = \{uv\colon h'(u)\, h'(v) \in E(H_\alpha)\}.$$

It follows immediately from the definition of $H_{\alpha+1}$ that h' is a homomorphism from $H_{\alpha+1}$ to H_α.

The homomorphism h is an injection and preserves non-edges because it is an endomorphism of H_α. The homomorphism h' is also an injection, since no two vertices in I' are mapped to the same vertex, and no vertex in I' is mapped to a vertex that is the image of a vertex under h. Also, h' preserves non-edges, by definition of $E(H_{\alpha+1})$. Furthermore, h' is a surjection, since we explicitly added a pre-image for each vertex of H_α to $H_{\alpha+1}$. Thus, h' is an isomorphism between $H_{\alpha+1}$ and H_α, and so $H_{\alpha+1}$ satisfies all of the required properties.

If $\lambda \leqslant |H_0|^+$ is a limit ordinal and H_α is defined for all $\alpha < \lambda$, then we define H_λ by

$$V(H_\lambda) = \bigcup_{\alpha<\lambda} V(H_\alpha)$$

$$E(H_\lambda) = \bigcup_{\alpha<\lambda} E(H_\alpha).$$

Note that $\{V(H_\alpha)\}_{\alpha<\lambda}$ is an increasing nested sequence of sets, all of cardinality $|H_0|$, and so $|V(H_\lambda)| \leqslant |H_0|^+$. Thus, H_0 is compact with respect to H_λ. Any finite subdigraph G of H_λ must be contained in some H_α with $\alpha < \lambda$, and so $G \to H_0$. Therefore $H_\lambda \to H_0$.

We will now show that H_λ satisfies the required properties.

We first show that every endomorphism of H_λ is an injection and preserves non-edges. Suppose that f is an endomorphism of H_λ such that either f is not an injection or f does not preserve non-edges. Then there exist $u, v \in V(H_\lambda)$ such that $f(u) = f(v)$ or such that $uv \notin E(H_\lambda)$ but $f(u) f(v) \in E(H_\lambda)$. Furthermore, there must exist some $\alpha < \lambda$ such that $u, v \in V(H_\alpha)$.

Look at $f|_{H_\alpha}: H_\alpha \to H_\lambda$. Since $u, v \in V(H_\alpha)$, $f|_{H_\alpha}$ is either not an injection or does not preserve non-edges. We also know that there exists a homomorphism $g: H_\lambda \to H_0$, and $H_0 \subset H_\alpha$. Thus, the composition $g \circ f|_{H_\alpha}: H_\alpha \to H_\alpha$ is an endomorphism of H_α which is either not an injection or does not preserve non-edges. But no such endomorphism exists, and so every endomorphism of H_λ must be an injection and preserve non-edges.

In particular, let $g: H_\lambda \to H_0$, so g is an endomorphism of H_λ. Then g is an injection and preserves non-edges. Hence, $g(H_\lambda)$ is a subdigraph of H_0 which is isomorphic to H_λ. It follows that $|H_\lambda| = |H_0|$, and so H_λ satisfies the required properties.

But at each inductive step in our construction at least one new vertex is added to the digraph. Eventually, for some limit ordinal $\lambda \leqslant |H_0|^+$, it must be the case that $|H_\lambda| \geqslant |H_0|^+$. Thus we obtain a contradiction, and so we may conclude that every endomorphism of H_0 is a surjection. Thus, H_0 is a core of H. ▮

COROLLARY 11. *Every compact digraph has a core.*

5. $|G|$-COMPACT DIGRAPHS

One might hope that the converse of Theorem 10 might also be true. Unfortunately, this is not the case. Consider, for example, the following digraph. We first define a sequence of finite digraphs H_i, $i \geqslant 1$. Let p_n denote the nth odd prime, and let C_n be the directed cycle of length p_n. It is easy to verify that $C_n \nrightarrow C_m$ when $n \neq m$, and that each C_n is a core. To construct H_i, begin with $V(H_i) = \{v_0, ..., v_{i+1}\}$ and $E(H_i) = \{v_0 v_j : 1 \leqslant j \leqslant i+1\}$. Now for $1 \leqslant j \leqslant i$, attach a copy of C_{2j} to v_j by identifying one of the vertices of the cycle with v_j. Finally, attach a copy of C_{2i+1} to v_{i+1} in like manner.

It is a simple exercise to verify that each H_i is a core and that $H_i \nrightarrow H_j$ when $i \neq j$.

We now define H to be the disjoint union of the digraphs H_i. Then H is a core, since each component of H is a core and there is no homomorphism from any component of H to any other. However, H is not compact. Consider the digraph G obtained by taking $V(G) = \{v_0, v_1, v_2, ...\}$ and $E(G) = \{v_0 v_i : 1 \leqslant i\}$, and attaching a copy of C_{2i} to v_i for each $i \geqslant 1$. Any finite subdigraph G' of G will admit a homomorphism to H, since it can contain vertices from at most finitely many of the cycles in G, and therefore will be a subdigraph of some H_i. However, $G \nrightarrow H$, since G is connected and no component of H contains cycles of all lengths p_{2i}.

Note that in this example the digraph H is not $|H|$-compact. In fact in this case both H and G were countable. The property of being $|H|$-compact turns out to be quite strong. In this section we will show that if H is an $|H|$-compact core then H is compact.

We begin with some definitions, leading up to a very useful lemma.

Let G and H be digraphs. Let $l: V(G) \to \mathscr{P}(V(H))$ be a mapping from $V(G)$ to the power set of $V(H)$, called a *list-assignment* for G (with respect to H). An *l-list-homomorphism* $f: G \to H$ is a homomorphism from G to H such that for each $v \in V(G)$ we have $f(v) \in l(v)$ (cf. [7]). We will often wish to apply the same lists to subdigraphs G' of G. By convention we will say that $f: G' \to H$ is an l-list-homomorphism if f is an $l|_{V(G')}$-list-homomorphism.

We will say that a list-assignment $l\colon V(G) \to \mathscr{P}(V(H))$ has property R if for each $v \in V(G)$ either $|l(v)| = 1$ or $l(v) = V(H)$. We will say that a digraph H is *α-R-list-compact* if for every digraph G with $|G| \leqslant \alpha$ and every list-assignment l for G with respect to H for which R holds, either there is an l-list-homomorphism from G to H, or there is a finite subdigraph $G' \subseteq G$ for which no l-list homomorphism exists. If H is α-R-list-compact for every cardinal α then H is R-list-compact.

An l-list-homomorphism where l has the property R is more commonly known as a *precolouring-extension* [4]. However, the more general notion of a list-colouring will be useful to us later on.

When $l(v)$ is a singleton we will occasionally abuse notation and write $u = l(v)$ when $u \in l(v)$.

LEMMA 12. *Let H be a core and $\alpha \geqslant |H|$ be an infinite cardinal. Then H is α-R-list-compact if and only if H is α-compact.*

Proof. If H is α-R-compact then H is α-compact, since for any input digraph G we may define a list-assignment l by $l(v) = V(H)$ for each $v \in V(G)$.

Now suppose H is an α-compact core with $\alpha \geqslant |H|$, and let G be any digraph with $|G| \leqslant \alpha$ (we assume $V(G)$ and $V(H)$ are disjoint). Let l be any list-assignment for G with respect to H such that R holds for l, and let $S = \{v \in V(G)\colon |l(v)| = 1\}$. Suppose also that every finite subdigraph of G admits an l-list-homomorphism to H. We construct a new digraph G^* by taking a copy of G and a copy of H, and identifying $w \in V(H)$ with all $v \in V(G)$ such that $l(v) = \{w\}$. It will be useful to formally define G^* as follows. We define a mapping $s\colon V(G) \cup V(H) \to V(G) \cup V(H)$ by setting $s(v) = l(v)$ for all $v \in S$, and $s(v) = v$ otherwise. Now we define $V(G^*) = s(V(H) \cup V(G))$ and $E(G^*) = \{s(u)\, s(v)\colon uv \in E(G) \cup E(H)\}$. Note that $|G^*| \leqslant \alpha$.

We say that a vertex $v \in V(G^*)$ is a G-vertex if $v = s(u)$ for some $u \in V(G)$. We say that an edge $uv \in E(G^*)$ is a G-edge if $uv = s(x)\, s(y)$ for some $xy \in E(G)$.

We will show that $G^* \to H$. Let K be a finite subdigraph of G^*. Of course K contains only finitely many G-vertices and G-edges. Thus, there is a finite subset A of $V(G)$ such that every G-edge of K and every G-vertex of K has a pre-image in A under s. Let B be the set of all vertices of K which are not G-vertices. Note that each $v \in B$ is its own unique pre-image under s so $V(K) \subseteq s(A \cup B)$.

Let $G' = G[A] \cup H[B]$. By assumption there exists an l-list-homomorphism $f\colon G[A] \to H$, which we can extend to a homomorphism $g\colon G' \to H$ by applying the identity mapping to all $v \in B$.

We now use the homomorphism g to define a homomorphism $h\colon K \to H$. Let v be a vertex of K. Let $v' \in A \cup B$ be some pre-image of v under s, and define $h(v) = g(v')$. Note that h is independent of the choice of v', since if $v \in s(S)$ then $g(v') = l(v')$ for all pre-images v' of v, and if $v \notin s(S)$ then v has a unique pre-image under s.

If uv is an edge of K then there exist $u', v' \in V(G')$ such that $s(u') = u$, $s(v') = v$, and $u'v' \in E(G')$. We know that $h(u) = g(u')$ and $h(v) = g(v')$ by the above remark. But g is a homomorphism so $g(u')\, g(v') \in E(H)$, and so $h(u)\, h(v) \in E(H)$. Thus, $h\colon K \to H$ is a homomorphism, and so by α-compactness of H we know that $G^* \to H$.

Now let $f\colon G^* \to H$ be a homomorphism. Since H is a core, $f|_{V(H)}$ is an automorphism of H. Let $g = (f|_{V(H)})^{-1}$. Then $(g \circ f)\colon G^* \to H$ is a homomorphism and $(g \circ f)|_{V(H)}$ is the identity. It is obvious from the definition of s that $s\colon (G \cup H) \to G^*$ is a homomorphism and $s|_{V(H)}$ is the identity. So $(g \circ f \circ s)\colon (G \cup H) \to H$ is a homomorphism from $G \cup H$ to H. In particular $(g \circ f \circ s)|_{V(G)}\colon G \to H$. But $s(v) = l(v)$ for all $v \in S$, and $l(v) \in V(H)$ for all $v \in S$, so $(g \circ f)(l(v)) = l(v)$ for all $v \in S$. Thus, $(g \circ f \circ s)(v) = l(v)$ for all $v \in S$, and so $(g \circ f \circ s)|_{V(G)}$ is an l-list-homomorphism from G to H. ∎

We will use this lemma to prove a very interesting sufficient condition for compactness. First we require a definition. Let G, H, and K be digraphs with $G \subseteq H$, and let $g\colon G \to K$ and $h\colon H \to K$ be homomorphisms. We say that h is an *extension* of g if $h|_G = g$.

THEOREM 13. *Let H be a core. If H is $|H|$-compact then H is compact.*

Proof. Suppose that H is a $|H|$-compact core and H is not compact. Let κ be the least cardinal such that H is not κ-compact, and let G be a certificate of non-compactness for H with $|G| = \kappa$. We assume $V(G) = \{\alpha\colon \alpha < \kappa\}$, and for each ordinal $\alpha < \kappa$ we define G_α to be the subdigraph of G induced by $\{\beta\colon \beta \leqslant \alpha\}$. Each G_α has fewer than κ vertices, and so G_α admits a homomorphism to H. For each ordinal $\alpha < \kappa$ we will construct a homomorphism $f_\alpha\colon G_\alpha \to H$ which will be an extension of each f_β with $\beta < \alpha$. Also, every f_α will have the property that for each β with $\alpha \leqslant \beta < \kappa$, there exists a homomorphism $g_\beta\colon G_\beta \to H$ which is an extension of f_α.

We will define the f_α inductively. Let γ be an ordinal smaller than κ and suppose that we have defined a homomorphism f_α satisfying the required properties for each $\alpha < \gamma$. Note that this condition is trivially satisfied when $\gamma = 0$. We proceed to define f_γ.

We claim that there exists a vertex $v_0 \in V(H)$ such that for all β with $\gamma \leqslant \beta < \kappa$, there exists a homomorphism $g_\beta\colon G_\beta \to H$ such that g_β is an extension of f_α for all $\alpha < \gamma$, and $g_\beta(\gamma) = v_0$.

If no such vertex exists then for all $v \in V(H)$ there must exist a β with $\gamma \leqslant \beta < \kappa$ such that there is no homomorphism $g_\beta: G_\beta \to H$ which is an extension of every f_α with $\alpha < \gamma$, and for which $g_\beta(\gamma) = v$.

We can rephrase this last statement in terms of list-homomorphisms. Let v be an arbitrary vertex of H. We will define a list-assignment r_v for G with respect to H. Let $r_v(\alpha) = \{f_\alpha(\alpha)\}$ for $0 \leqslant \alpha < \gamma$, $r_v(\gamma) = \{v\}$, and $r_v(\alpha) = V(H)$ for each α with $\gamma < \alpha < \kappa$. The above statement is equivalent to the assertion that for some β with $\gamma \leqslant \beta < \kappa$ there is no r_v-list-homomorphism from G_β to H. But H is $|\beta|$-compact, so by Lemma 12 H is $|\beta|$-R-list-compact. Also, R holds for r_v. Therefore, there must be some finite subdigraph G_v of G_β such that there is no r_v-list-homomorphism from G_v to H.

Note that γ must be a vertex of G_v, for otherwise there would be an $\alpha < \gamma$ such that $V(G_v) \subseteq \{\delta: \delta \leqslant \alpha \text{ or } \gamma < \delta \leqslant \beta\}$. But the previously defined homomorphism $f_\alpha: G_\alpha \to H$ has an extension to a homomorphism $g_\beta: G_\beta \to H$. Since $\gamma \notin V(G_v)$, we see that $g_\beta|_{G_v}$ is an r_v-list-homomorphism from G_v to H.

We now define $G^* = \bigcup_{v \in V(H)} G_v$. We claim that for each $\alpha < \gamma$ there is a homomorphism from G^* to H which is an extension of f_α. The digraph H is $|H|$-compact so by Lemma 12 H is $|H|$-R-list-compact. Clearly $|G^*| \leqslant |H|$, and so H is R-list-compact with respect to G^*. Now if G' is any finite subdigraph of G^*, then G' is a finite subdigraph of G, and so $G' \subseteq G_\beta$ for some $\beta < \kappa$. For each $\alpha < \gamma$ we define a list assignment l_α for G with respect to H by $l_\alpha(\delta) = \{f_\alpha(\delta)\}$ for $\delta \leqslant \alpha$, and $l_\alpha(\delta) = V(H)$ otherwise. By assumption, for each $\alpha < \gamma$ there is an l_α-list-homomorphism from G_β to H, and so for each $\alpha < \gamma$ there is an l_α-list-homomorphism from G' to H. Thus, since H is $|H|$-R-list-compact, for each $\alpha < \gamma$ there is an l_α-list-homomorphism from G^* to H. Clearly such a homomorphism is an extension of f_α.

We claim that in fact there is one homomorphism $g: G^* \to H$ such that g is an l_α-list-homomorphism for all $\alpha < \gamma$. Let us define a list-assignment s for G^* with respect to H by $s(\alpha) = \{f_\alpha(\alpha)\}$ for $\alpha < \gamma$ and $s(\alpha) = V(H)$ otherwise. We will show that there is an s-list-homomorphism from G^* to H, which clearly will be an extension of each f_α for $\alpha < \gamma$. Let G' be a finite subdigraph of G^*. We claim there is an s-list-homomorphism from G' to H. Since G' is finite, there must be some $\alpha < \gamma$ such that for each $\delta \in V(G')$, either $\delta \leqslant \alpha$ or $\delta \geqslant \gamma$. Also, there is some $\beta < \kappa$ such that $G' \subseteq G_\beta$. Thus we may find a homomorphism $f: G_\beta \to H$ which is an extension of f_α, and so $f|_{G'}$ is an s-list-homomorphism from G' to H. Thus, by $|H|$-R-list-compactness of H, there is an s-list-homomorphism from G^* to H. This is the required homomorphism g.

Now let $v = g(\gamma)$. Then $g|_{G_v}$ is an r_v-list homomorphism from G_v to H, a contradiction. Hence, our claim is proven. And so there must be some

$v_0 \in V(H)$ such that for all β with $\gamma \leqslant \beta < \kappa$, there exists a homomorphism $g_\beta\colon G_\beta \to H$ such that g_β is an extension of f_α for each $\alpha < \gamma$, and $g_\beta(\gamma) = v_0$. Thus, we simply define f_γ by $f_\gamma(\alpha) = f_\alpha(\alpha)$ for each $\alpha < \gamma$ and $f_\gamma(\gamma) = v_0$.

By this method we construct the functions f_α for each $\alpha < \kappa$.

To complete the proof, we define a mapping $h\colon G \to H$ by $h(\alpha) = f_\alpha(\alpha)$ for each $\alpha < \kappa$. This is clearly a homomorphism, contradicting the choice of G. We may therefore conclude that H is compact. ▮

COROLLARY 14. *Let H be a digraph. If H is $|H|^+$-compact then H is compact.*

Proof. Let H be a $|H|^+$-compact digraph. By Theorem 10 H has a core K. Then K is $|H|^+$-compact, since $K \leftrightarrow H$. Also, $|K| \leqslant |H| \leqslant |H|^+$ so K is $|K|$-compact, and so by Theorem 13, K is compact. But $H \leftrightarrow K$ so H is also compact. ▮

These results are somewhat surprising. They imply that if a digraph is not compact, then we need only look at digraphs of the next larger cardinality to find a certificate of non-compactness, without having to assume the continuum hypothesis. Furthermore, if a core is not compact, then there is a certificate of non-compactness of the same cardinality. For digraphs which are not cores this is false. For example, a complete digraph of size κ is κ-compact but not κ^+-compact.

6. FAMILIES OF COMPACT DIGRAPHS

So far we have seen that compact digraphs have some interesting properties. The problem now is to determine which digraphs are compact. In particular, we would like to construct some broad families of compact digraphs, or perhaps give some general sufficient conditions for a digraph to be compact.

All of the results in this section are proved using compactness properties of topological spaces, which suggests that there is a close relationship between topological and homomorphic compactness. The precise nature of this relationship, however, remains unclear. In this section we first prove a technical lemma, which will be used to exhibit several large families of compact digraphs. Before we begin we will need some terminology.

We will first generalize the notion of a list-homomorphism. Let G and H be digraphs, and let $l\colon V(G) \to \mathscr{P}(V(H))$ be a list-assignment for G with respect to H. An *l-list-mapping* is a mapping $f\colon V(G) \to V(H)$ such that $f(v) \in l(v)$ for each $v \in V(G)$. An l-list-homomorphism, then, is an l-list-mapping which is also a homomorphism.

Suppose G, H, and l are as above. We define $\mathscr{S} = \prod_{v \in V(G)} l(v)$, i.e., $\mathscr{S}$ is the product of the sets $l(v)$. There is an obvious one-to-one correspondence between l-list-mapping and elements of $\mathscr{S}$. We will therefore consider such a mapping to be identical to the corresponding element of $\mathscr{S}$.

We will make use of a classic result in topology, which we state now.

THEOREM 15 (Tychonoff). *The product of compact topological spaces is compact.*

The original proof of this may be found in [19], but the reader may prefer [13]. Note that the proof of Tychonoff's theorem requires the Axiom of Choice. The property of compact topological spaces which we require is the following: given any collection $\mathscr{C}$ of closed sets in a compact topological space, if the intersection of any finite subcollection of $\mathscr{C}$ is nonempty, then the intersection of all of the sets in $\mathscr{C}$ is nonempty. We may now state our result.

LEMMA 16. *Let G and H be digraphs. Suppose there exists a function $l: V(G) \to \mathscr{P}(V(H))$, and for each $v \in V(G)$ a compact topology $\mathscr{T}_v$ on $l(v)$, such that for every finite subdigraph $G' \subseteq G$:*

- *there exists an l-list-homomorphism $f: G' \to H$, and*
- *the set $\{g: g$ is an l-list-mapping from $V(G)$ to $V(H)$ and $g|_{G'}$ is a homomorphism$\}$ is closed in the product topology $\mathscr{T} = \prod_{v \in V(G)} \mathscr{T}_v$ on $\mathscr{S} = \prod_{v \in V(G)} l(v)$.*

Then $G \to H$.

Proof. Suppose that the conditions of the lemma are satisfied. Let $\mathscr{T}$ be the product topology on $\mathscr{S}$. By Tychonoff's theorem $\mathscr{T}$ is compact. For each finite subdigraph $G' \subseteq G$, let $F_{G'} \subseteq \mathscr{S}$ be the set of all l-list-mappings $h: V(G) \to V(H)$ such that $h|_{G'}$ is a homomorphism. Each $F_{G'}$ is a nonempty closed set in the topological space $(\mathscr{S}, \mathscr{T})$. We claim that the intersection of the collection $\{F_{G'}: G'$ is a finite subdigraph of $G\}$ is nonempty. Since $\mathscr{T}$ is compact it suffices to show that for any finite collection $G_1, ..., G_n$ of finite subdigraphs of G, the intersection $\bigcap_{i=1}^{n} F_{G_i}$ is nonempty. But given any finite collection $G_1, ..., G_n$ of finite subdigraphs of G, the digraph $G' = \bigcup_{i=1}^{n} G_i$ is a finite subdigraph of G, and so there is an l-list-homomorphism $f: G' \to H$. Let g be any l-list-mapping from G to H such that $g|_{G'} = f$. Then it will be the case that $g|_{G_i}$ is an l-list-homomorphism for each $1 \leqslant i \leqslant n$. Therefore $\bigcap_{i=1}^{n} F_{G_i}$ is non-empty. And so our claim is proved.

Now any element of the intersection of the collection $\{F_{G'}: G'$ is a finite subdigraph of $G\}$ is an l-list-homomorphism from G to H, and so we conclude that $G \to H$. ∎

Tychonoff's theorem is an extremely useful tool in proving many different types of compactness theorems. In fact, the above lemma is a generalization of compactness results such as those found in [9, 17, 18]. These papers exploit the fact that if X is a finite set and $\mathscr{T}$ is the discrete topology on X, then $(X, \mathscr{T})$ is compact and every subset of X is closed. This same property of finite sets will be the basis of the proofs of our first two corollaries.

COROLLARY 17. *Any finite digraph is compact.*

Proof. Suppose H is finite, and let G be any digraph such that all finite subdigraphs $G' \subseteq G$ admit homomorphisms to H. For each $v \in V(G)$ let $l(v) = V(H)$ and let $\mathscr{T}_v$ be the discrete topology on $l(v)$. It is a simple matter to verify that the conditions of Lemma 16 are satisfied. Thus $G \to H$ and so H is compact. ∎

This result is not particularly surprising, and is a generalization of the well known compactness theorem for chromatic number [5]. This result can also be proved using a similar result found in [11], which states that a finite subdigraph H of a digraph G is a retract of G if and only if H is a retract of every finite subdigraph of G which contains H.

Our next result characterizes a large class of infinite digraphs which are compact. We denote by $Aut(H)$ the automorphism group of a digraph H.

THEOREM 18. *Let H be a locally finite digraph. If there are only finitely many orbits in $Aut(H)$ then H is compact.*

Proof. Let H be a digraph satisfying the conditions above, and let G be any digraph such that every finite $G' \subseteq G$ admits a homomorphism to H. We may assume without loss of generality that G is connected. Define $A \subseteq V(H)$ to be a set containing one vertex from each orbit of $Aut(H)$, so A is finite. Let v_0 be some fixed vertex in $V(G)$ and define $l(v_0) = A$. Clearly for any subdigraph $G' \subseteq G$ (not necessarily finite), there is a homomorphism $f\colon G' \to H$ if and only if there is such a homomorphism with $f(v_0) \in l(v_0)$.

We now define a finite set $l(u) \subseteq V(H)$ for each $u \in V(G)$. Given $u \in V(G)$, let $l(u) = \{y \in V(H)\colon d(y, A) \leqslant d(u, v_0)\}$. Since H is locally finite, $l(u)$ will be finite.

We will show that every finite subdigraph of G admits an l-list-homomorphism to H. Let G' be a finite subdigraph of G. Let G'' be a finite subdigraph of G containing G', and such that for any $u, v \in V(G')$, the distance between u and v in G'' is the same as their distance in G. Such a digraph is easily constructed by adding to G' a shortest oriented path from u to v in G, for each $u, v \in V(G')$. Since G'' is finite we have $G'' \to H$. And so if we choose a homomorphism $f\colon G'' \to H$ such that $f(v_0) \in A$, then $f(u) \in l(u)$ for all $u \in V(G')$. So $f|_{G'}$ will be an l-list-homomorphism from G' to H.

If we now assign the discrete topology to each set $l(v)$, we may apply Lemma 16 to conclude $G \to H$, and so H is compact. ∎

Before proceeding, we note that applications of Lemma 16 may often be simplified by the following observation. Let G, H, $l(v)$ and $\mathscr{T}_v$ be given as usual. Suppose that G' is a finite subdigraph of G, and let $\mathscr{S}' = \prod_{v \in V(G')} l(v)$ and $\mathscr{T}' = \prod_{v \in V(G')} \mathscr{T}_v$. It follows from our definitions that the set $\{g\colon g$ is a mapping from $V(G)$ to $V(H)$ and $g|_{G'}$ is an l-list-homomorphism$\}$ is closed in $(\mathscr{S}, \mathscr{T})$ if and only if the set $X = \{g\colon g$ is an l-list-homomorphism from G' to $H\}$ is closed in $(\mathscr{S}', \mathscr{T}')$. But X is closed if and only if its complement X^c is open. Because $(\mathscr{S}', \mathscr{T}')$ is a finite product, an open set in $(\mathscr{S}', \mathscr{T}')$ is just a product $\prod_{v \in V(G')} O_v$ where each O_v is open in $(l(v), \mathscr{T}_v)$. Thus, it is sufficient to show that if an l-list-mapping $f\colon V(G') \to V(H)$ is not a homomorphism, then for each $v \in V(G')$, there is an open set N_v in $l(v)$ containing $f(v)$ such that no $g \in \prod_{v \in V(G')} N_v$ is a homomorphism.

This observation immediately yields the following:

Theorem 19. *Let H be a digraph. Suppose we may define a compact topology on $V(H)$ with the property that whenever $uv \notin E(H)$ then there exist open sets O_u and O_v containing u and v such that $xy \notin E(H)$ for each $x \in O_u$ and $y \in O_v$. Then H is compact.*

This observation is also useful in proving our next result. We use $\mathfrak{R}$ to denote the set of real numbers.

Theorem 20. *Let $\mathscr{M} = (M, d)$ be a metric space, and let C be a compact subset of $\mathfrak{R}$. Define a digraph H by $V(H) = M$ and $E(H) = \{uv\colon d(u, v) \in C\}$. If either*

(i) *$\mathscr{M}$ is compact, or*

(ii) *every closed and bounded subspace of $\mathscr{M}$ is compact and $Aut(H)$ has only finitely many orbits,*

then H is compact.

Proof. Let H be a digraph as defined above and let G be any digraph. Assume without loss of generality that G is connected. Suppose that every finite subdigraph of G admits a homomorphism to H. We will define for each $v \in V(G)$ a set $l(v) \subseteq V(H)$ and a compact topology T_v on $l(v)$ so that every finite subdigraph of G admits an l-list-homomorphism to H.

Case 1. (i) holds.

For all $v \in V(G)$ let $l(v) = V(H)$ and let T_v be the metric topology given by $\mathscr{M}$ on $l(v)$. Every finite subdigraph of G clearly admits an l-list-homomorphism to H.

Case 2. (ii) holds.

Let $A \subseteq V(H)$ contain exactly one element from each orbit of $Aut(G)$. Choose some arbitrary $v_0 \in V(G)$ and define $l(v_0) = A$. Now note that the set C must be closed and bounded, so let $r = \max\{x\colon x \in C\}$. For any $v \in V(G) - \{v_0\}$, let k be the length of a shortest oriented path from v to v_0 in G. Let $l(v) = \{w\colon d(w, A) \leqslant kr\}$. Now for all $v \in V(G)$ let $\mathscr{T}_v$ be the metric topology given by $\mathscr{M}$ on $l(v)$. Since $l(v)$ is closed and bounded, the topology $\mathscr{T}_v$ is compact. It is clear that every finite subdigraph of G admits an l-list-homomorphism to H.

Having defined our lists and topologies in one of the above ways, it remains only to show that for any finite $G' \subseteq G$, the set of mappings $f\colon G \to H$ such that $f|_{G'}$ is an l-list-homomorphism is a closed subset of $\mathscr{S} = \prod_{v \in V(G)} l(v)$ under the product topology $\mathscr{T} = \prod_{v \in V(G)} \mathscr{T}_v$. To do this, we will show that given G' and an l-list-mapping $f\colon V(G') \to V(H)$ which is not a homomorphism, there exists a neighborhood $N_v \subseteq l(v)$ of $f(v)$ such that no $g \in \prod_{v \in V(G')} N_v$ is a homomorphism.

Suppose that $f\colon V(G') \to V(H)$ is not a homomorphism. Then there exist $u, v \in V(G')$ such that $uv \in E(G')$ but $f(u) f(v) \notin E(H')$. Then, recalling that $f(u)$ and $f(v)$ are points in the metric space $\mathscr{M}$, we know that $d(f(u), f(v)) \notin C$. But C is closed, so there exists an $\varepsilon > 0$ such that for all $x \in \mathfrak{R}$, $|x - d(f(u), f(v))| < \varepsilon$ implies that $x \notin C$. Therefore, for all $r, s \in M$, if $d(f(u), r) < \varepsilon/2$ and $d(f(v), s) < \varepsilon/2$, it must be the case that $|d(r, s) - d(f(u), f(v))| < \varepsilon$, applying the triangle inequality. So if we let N_u and N_v be the neighbourhoods of radius $\varepsilon/2$ around $f(u)$ and $f(v)$, respectively, in $\mathscr{T}_u$ and $\mathscr{T}_v$, then no vertex in N_u is adjacent to any vertex in N_v in H. Now define $N_w = l(w)$ for all $w \neq u$, v in $V(G')$. No $g \in \prod_{w \in V(G')} N_w$ will be a homomorphism, since $g(u)\, g(v) \notin E(H)$. It follows that the set $\{f \in \mathscr{S}\colon f|_{G'}$ is an l-list-homomorphism$\}$ is closed in $(\mathscr{S}, \mathscr{T})$, and so Lemma 16 applies. We conclude that $G \to H$. ▮

Note that since the distance function d is symmetric, all edge-sets in these digraphs will be symmetric, and so they may be considered to be graphs. We will continue to regard them as digraphs, although for simplicity in our diagrams we will often draw pairs of directed edges as a single undirected edge.

This last result allows us to construct some particularly interesting compact digraphs. Define a digraph D by $V(D) = \mathfrak{R}^2$, i.e. points in the plane, and $E(D) = \{uv\colon d(u, v) = 1\}$, where d is the usual metric on $\mathfrak{R}^2$. This

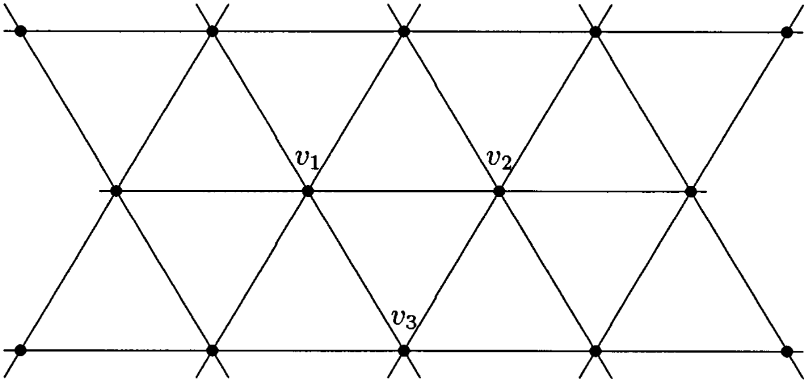

FIGURE 3

digraph has been studied extensively in the literature [6, 10], and has several interesting open problems associated with it. For example, it is quite simple to show that $4 \leqslant \chi(D) \leqslant 7$, but no improvement on these bounds is known. The properties of this digraph which are of interest to us are given by the following theorem.

THEOREM 21. *D is a compact core.*

Proof. That D is compact is a simple corollary of Theorem 20, since $\{1\}$ is a compact subset of $\mathfrak{R}$, D is vertex-transitive, and closed bounded sets in $\mathfrak{R}^2$ are compact. The fact that D is a core is more difficult to prove. We will prove the stronger claim that any endomorphism of D is a rigid transformation of $\mathfrak{R}^2$.

For any three vertices $\{v_1, v_2, v_3\} \subset V(D)$, the vertices $\{v_1, v_2, v_3\}$ induce a K_3 in D if and only if the corresponding points in $\mathfrak{R}^2$ are the vertices of an equilateral triangle with side length one. Since the homomorphic image of K_3 must be another K_3, any endomorphism of D must be a rigid transformation of these three points. Thus, to prove our claim it suffices to show that any endomorphism of D which fixes $\{v_1, v_2, v_3\}$ pointwise must be the identity.

Let f be an endomorphism of D which fixes $\{v_1, v_2, v_3\}$. We will first show that D must fix the vertices of the triangular lattice containing $\{v_1, v_2, v_3\}$ (see Fig. 3).

We will do this by showing that if any triangle $\{u_1, u_2, u_3\}$ in the lattice is fixed, then the lattice point which is the unique common neighborhood of u_2 and u_3 other than u_1 must also be fixed. Then u_2, u_3, and their common neighbour form a new fixed triangle in the lattice. By repeatedly applying this process to the new fixed triangles every vertex in the lattice may be fixed.

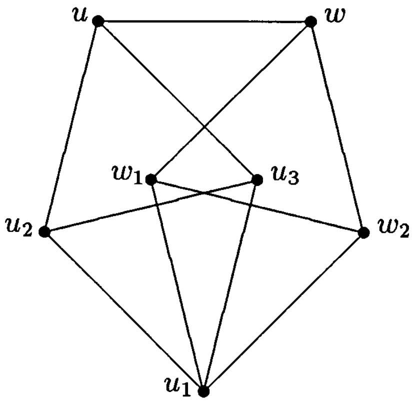

FIGURE 4

Suppose that a triangle $\{u_1, u_2, u_3\}$ is fixed by f. There is a subdigraph of D containing $\{u_1, u_2, u_3\}$ as indicated in Fig. 4. Here u is the common neighbour of u_2 and u_3 not equal to u_1. Since $\{u_1, u_2, u_3\}$ are fixed, $f(u) = u$ or $f(u) = u_1$, so either $d(u_1, f(u)) = \sqrt{3}$ or $d(u_1, f(u)) = 0$.

The triangles $\{u_1, w_1, w_2\}$ and $\{w, w_1, w_2\}$ must map to triangles with the image of the edge $w_1 w_2$ in common, and so $d(u_1, f(w)) = \sqrt{3}$ or $d(u_1, f(w)) = 0$. So if $f(u) = u_1$ then it must be the case that either $d(f(u), f(w)) = \sqrt{3}$ or $d(f(u), f(w)) = 0$. But $d(f(u), f(w))$ must be 1. Thus, $f(u) = u$, and so we may conclude that f fixes the entire lattice.

We must now show that f fixes every point in the plane. Suppose there is a vertex v such that $f(v) \neq v$. We claim that there is a lattice vertex w and

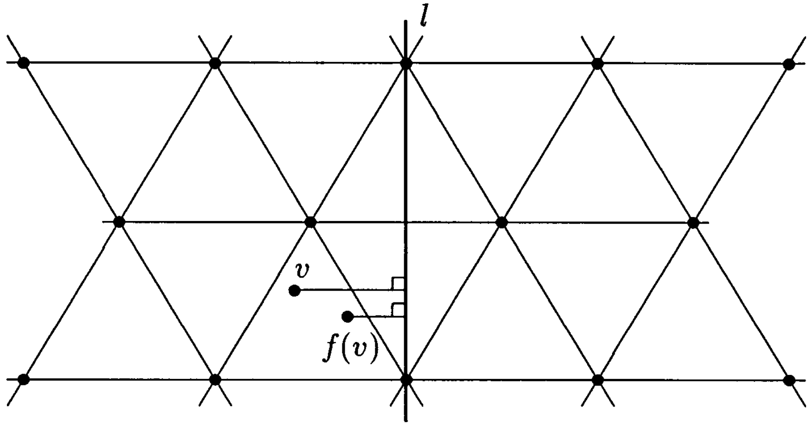

FIGURE 5

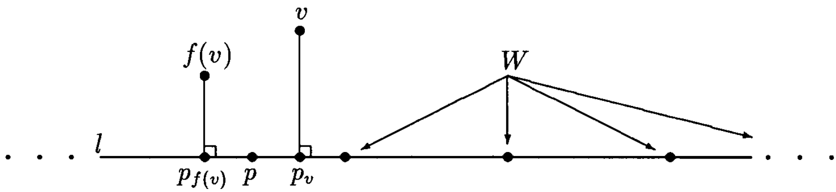

FIGURE 6

an integer k such that $d(w, v) \leqslant k$ and $d(w, f(v)) > k$. To see this, let l be a line which is perpendicular to some edge of the triangular lattice, and which contains a lattice point. Then l contains infinitely many lattice points which are spaced at a distance of $\sqrt{3}$ from each other. Furthermore, choose l so that the perpendicular projections of v and $f(v)$ on l are distinct (Fig. 5).

Let p_v and $p_{f(v)}$ be the perpendicular projections of v and $f(v)$ onto l. Let p be a point on l midway between p_v and $p_{f(v)}$ and let $r = d(p, p_v)$. Let W be the set of lattice points on l which are outside the interval $[p_v, p_{f(v)}]$ and are on the same side of the interval as p_v (Fig. 6).

Every point in W will have distance $c + k\sqrt{3}$ from p, where c is a positive constant and k ranges over the nonnegative integers. Hereafter we will use $\langle x \rangle$ to denote the fractional part of a real number x, i.e. $\langle x \rangle = x - \lfloor x \rfloor$. Since $\sqrt{3}$ is irrational, the set $\{\langle c + k\sqrt{3} \rangle\}$, where k ranges over the non-negative integers, is dense in $[0, 1]$. Thus, for every $\varepsilon > 0$ there exist infinitely many $w \in W$ such that $\langle d(w, p) \rangle < \varepsilon$, and so there exist such w arbitrarily far from p, p_v, and $p_{f(v)}$. For each $w \in W$ it is clearly true that $d(w, p_v) < d(w, v)$ and $d(w, p_{f(v)}) < d(w, f(v))$.

Consider any fixed $\varepsilon > 0$. For each $w \in W$ which is sufficiently far from p_v and $p_{f(v)}$, we have $d(w, v) - d(w, p_v) < \varepsilon$. Thus, there is a $w \in W$ such that $\langle d(w, p) \rangle < \varepsilon$, $|d(w, v) - d(w, p_v)| < \varepsilon$ and $|d(w, f(v)) - d(w, p_{f(v)})| < \varepsilon$. We now take $\varepsilon = r/2$, and so there exists a $w \in W$ such that

$$
\begin{aligned}
d(w, v) &< d(w, p_v) + r/2 = d(w, p) - r/2 \\
&< d(w, p) - [d(w, p) - \lfloor d(w, p) \rfloor] = \lfloor d(w, p) \rfloor
\end{aligned}
$$

and

$$d(w, f(v)) > d(w, p_{f(v)}) > d(w, p) > \lfloor d(w, p) \rfloor.$$

Thus, if we set $k = \lfloor d(w, p) \rfloor$ we have $d(w, v) < k$ and $d(w, f(v)) > k$, and so our claim is proved.

Now since $d(w, v) < k$ there is a directed path of length k from w to v in D. Thus, there must be a directed walk of length k from $f(w)$ to $f(v)$ in D. But this is impossible, as $f(w) = w$ and $d(w, f(v)) > k$.

We conclude that f is the identity mapping, and so any endomorphism of D is a rigid transformation of the plane. Obviously a rigid transformation of the plane is an automorphism of D, and so D is a core. ∎

We may construct higher-dimensional analogous of D in a natural way. We simply let the vertex-set be $\mathfrak{R}^n$, and define the edge-set exactly as we did for D. The proof of the preceding theorem will also generalize to these digraphs, the major difference being that the triangles in the graph in Fig. 4 will be replaced by copies of K_{n+1}.

In general the digraphs constructed using the theorems in this section need not be cores, although they certainly must have cores. The preceding example is particularly interesting because D is one of the few examples we have of a compact core with an uncountable vertex-set.

7. CONCLUSIONS

All of the results in this paper also hold for undirected graphs, and the proofs are identical. Simple modifications to the proofs will also show that these results hold for hypergraphs and more general relational structures.

Our notion of homomorphic compactness has found other applications in the study of chromatic properties of infinite graphs. One such application will be given in [3, 21]. We believe the notion of homomorphic compactness will prove to be a valuable tool in the study of infinite graphs.

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