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RATIONAL FUNCTION APPROXIMATIONS IN THE NUMERICAL SOLUTION OF CAUCHY-TYPE SINGULAR INTEGRAL EQUATIONS

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Abstract—Cauchy-type singular integral equations of the second kind with constant coefficients are solved via rational and polynomial approximations. Rational functions, similar to that of polynomials, have the property that for $r(t)$ rational and for many of the weight functions $w(t)$ encountered in practice,

$$R(x) \equiv \int_{-1}^{1} \frac{w(t)r(t)}{t-x}\,dt$$

is also rational. Hence, approximations by rational functions is feasible. Rational function approximations in the solution of the dominant equation results in a linear algebraic system which possesses block-diagonal structure. It is further shown that the determinant of the coefficient matrix is bounded below away from zero and stability is ensured under fairly non-restrictive conditions. For the complete Cauchy-type singular integral equation, i.e. the equation with both the principal and regular parts, gaussian quadrature in conjunction with the rational function method is synthesized in the construction of a "hybrid" scheme. Error estimates and convergence are established. A variety of problems from Aerodynamics and Fracture Mechanics are solved and presented as a basis of comparison to polynomial-based schemes.

1. INTRODUCTION

Cauchy-type singular integral equations of the form:

$$ag(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{g(t)dt}{t-x} + \lambda \int_{-1}^{1} k(x, t)g(t)dt = f(x), \quad -1 < x < 1, \tag{1.1}$$

are useful for the solution of many problems of Fracture Mechanics, Fluid Dynamics, Statistical Mechanics, and Electromagnetism. The functions a, b, k, and f are Hölder continuous in each independent variable on $[-1, 1]$; and, the singular integral in (1.1) is a principal-valued integral, i.e.

$$\int_{-1}^{1} \frac{g(t)dt}{t-x} = \lim_{\delta \to 0^-} \left\{ \int_{-1}^{x-\delta} + \int_{x+\delta}^{1} \right\} \frac{g(t)dt}{t-x}. \tag{1.2}$$

The solution g is sought in the class of functions that are Hölder continuous on any closed interval interior to $(-1, 1)$, and integrable at the endpoints -1 and $+1$. A complete treatment of the analytical methods for the solution of these equations is given in the monographs by Gakhov[11] and Muskhelishvilli[18]. A closed form solution to a Cauchy-type singular integral equation is not always attainable. Carleman–Vekua regularization does transform the equation to a Fredholm integral equation of the second kind; but, due to the complexity of the resulting kernel, researchers, most notably in the past decade, have been investigating the alternatives, namely direct numerical methods.

Direct methods of solution are of comparatively recent origin. These methods explicitly incorporate the "correct" singular behavior of the solution into a sequence of approximations. The solution g is expressed as the product of a weight function w and a function ϕ which is continuous and bounded on $[-1, 1]$, i.e.

$$g(x) = w(x)\phi(x) \tag{1.3}$$

and

$$w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \quad -1 < \mathrm{Re}\,(\alpha), \mathrm{Re}\,(\beta) < 1. \tag{1.4}$$

Attention is then directed to the approximation of ϕ. F. Erdogan[7] represents ϕ as a linear combination of those Jacobi polynomials which are orthogonal with respect to the weight function w. A direct consequence of this are the Gauss and Lobatto-Chebyshev quadrature-collocation schemes developed by Erdogan and Gupta[8], Erdogan, Gupta and Cook[9], and by Theocaris and Ioakimidis[27]. Dow and Elliott[5] further generalize these schemes based upon orthogonal polynomials and demonstrate convergence of their algorithm. Convergence is elaborated upon by Elliott[6], Ioakimidis and Theocaris[15], and by Gerasoulis and Srivastav[14]. Collocation and quadrature schemes require the evaluation of the zeros and corresponding weights of the Jacobi polynomials. If a high degree polynomial is required to obtain a "good approximation", there will be a loss of accuracy and computational costs will increase.

For equations in which the kernel and right-hand side are sufficiently smooth and non-oscillatory, the aforementioned Gaussian schemes are suitable. Situations not so well-behaved may be troublesome for these methods; however, piecewise approximations have proven to be quite favorable. Gerasoulis and Srivastav[12] and Gerasoulis[13] demonstrate that piecewise linear and quadratic approximations will increase flexibility and the rate of convergence for problems in which ϕ' has discontinuities within the interval $(-1, 1)$. A higher order of accuracy is obtained by Jen and Srivastav[16], who developed a cubic spline method which obtained $O(h^{7/2})$ accuracy. Although these schemes have obtained an acceptable order of precision, it is always desirable to have higher accuracy methods which require less computational effort.

Polynomial approximations in the solution of Cauchy-type singular integral equations are the core of those numerical techniques developed thus far. It is interesting to note that given a rational function $r(t)$,

$$R(t) = \int_{-1}^{1} \frac{w(x)r(x)\mathrm{d}x}{x - t} \tag{1.5}$$

is also a rational function, whenever $w(x)$ is such that a polynomial $p(x)$ will result into a polynomial $P(x)$. Moreover, functions $r(t)$ and $R(t)$ are of the same order and will possess the same set of singularities. Previously, Gabdulhaev[10] incorporated rational functions by approximating g by a truncated Laurent's series. Physicists as well as mathematicians have utilized similar concepts in the solution of Fredholm equations arising from problems of scattering. In light of these developments, this paper will present a rational function method for the direct solution of these Cauchy-type singular integral equations. Emphasis will be placed upon the direct solution of the dominant equation. It is shown that the resulting discretized equation may be formulated as a linear algebraic system which possesses block-diagonal structure. Recognizing the effectiveness and simplicity of the Gauss–Chebyshev scheme for the solution of the complete singular integral equation, a "hybrid" technique, which synthesizes the aforementioned rational function method with gaussian quadrature is introduced. Error estimates and convergence are considered and established. Various numerical examples arising from mixed boundary value problems of Aerodynamics and Elasticity are solved to illustrate the technique.

The paper is organized as follows: Section 2 contains some of the preliminary results pertaining to the analytic theory of Cauchy-type singular integral equations; in addition, it serves as a reference for a few elementary integrals. Section 3 outlines the basic strategy of the rational function method for the dominant equation and brings out the desirable block-diagonal structure of the linear system of equations. In Sec. 4, the synthesis of the rational function method with gaussian quadrature for the development of the "hybrid" method is presented for the problem of a cruciform crack. This is followed by the analysis of stability and existence as well as of error and convergence in Secs. 5 and 6 respectively. Numerical examples for the dominant equation are given in Sec. 7. Section 8 then ends the paper with numerical results for the cruciform crack problem obtained via the "hybrid" technique.

2. PRELIMINARY MATHEMATICS

2.1 *Singular integral equations*

Basic results from the theory of singular integral equations are given for the convenience of the reader. For the sake of clarity and presentation, only the dominant equation

$$ag(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{g(t)\mathrm{d}t}{t - x} = f(x), \quad -1 < x < 1, \tag{2.1}$$

shall be considered. The general form of the solution to (2.1) is

$$g(x) = X^{+}(x)\left\{2P_{N-1}(x) + \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)\mathrm{d}t}{X^{+}(t)(t - x)}\right\}, \tag{2.2}$$

$X(x)$ is the fundamental solution, i.e. the solution to the homogeneous Riemann boundary value problem. It is defined by

$$X(x) = (x - 1)^{\alpha}(x + 1)^{\beta}, \tag{2.3}$$

where

$$\alpha = \frac{1}{2\pi i}\ln\left(\frac{a + ib}{a - ib}\right) + N, \quad -1 < \mathrm{Re}\,(\alpha) < 1, \tag{2.4}$$

and

$$\beta = \frac{1}{2\pi i}\ln\left(\frac{a + ib}{a - ib}\right) + M, \quad -1 < \mathrm{Re}\,(\beta) < 1. \tag{2.5}$$

The quantity $\kappa = -(N + M)$ is the "index" of the equation and is determined by

$$\kappa = \ln\left(\frac{a + ib}{a - ib}\right), \tag{2.6}$$

where the principal branch is chosen for the logarithm. The restrictions placed on α and β reflect the assumption that g is either bounded or has integrable singularities at the endpoints. The index, which may take on any number of values, but those of interest being -1, 0, 1, determines the additional conditions necessary to ensure a unique non-trivial solution of (2.1). When $\kappa < 0$, g is bounded at both endpoints. The equation possesses a solution if and only if the function f satisfies the orthogonality conditions

$$\int_{-1}^{1}\frac{t^{k-1}f(t)\mathrm{d}t}{X^{+}(t)} = 0, \quad k = 1, 2, \ldots, -\kappa. \tag{2.7}$$

For the case of $\kappa = 0$, g is bounded at one endpoint and the solution is determined uniquely without the specification of any further conditions. If $\kappa = 1$, the solution is unbounded at both endpoints and uniqueness is ensured by requiring that the additional condition

$$\int_{-1}^{1} g(t)\mathrm{d}t = M \tag{2.8}$$

is satisfied.

2.2 *Integral identities*

These integrals are for $|x| < 1$, unless specified differently.

$$\frac{1}{\pi}\int_{-1}^{1} \frac{(1-t)^{\alpha-1}(1+t)^{\beta-1}}{t-x}\,dt = \operatorname{ctg}(\alpha\pi)(x+1)^{\beta-1}(1-x)^{\alpha-1} - \frac{\Gamma(\beta)\Gamma(\alpha-1)}{\pi\Gamma(\alpha+\beta-1)} 2^{\alpha+\beta-2} + {}_2F_1(2-\alpha-\beta, 1; 2-\alpha; 1-x/2) \quad (2.9)$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta}}{(t-z)^{N+1}}\,dt = \left(\frac{a}{b}\right) \sum_{k=0}^{n} \left(\prod_{m=1}^{k} (\alpha-m+1)(\beta-m+1)\right) \frac{(-1)^{2k-N}}{\Gamma(N+1)} (1-z)^{\alpha-k}(1+z)^{\beta-k} \quad (2.10)$$

$$\hat{F}^{j}(\alpha_k) = (-1)^{j}\Gamma(j) \int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta}}{(t-\alpha_k)^{j}}\,dt, \quad |\alpha_k| \geq 1 \quad (2.11)$$

$$\hat{G}^{j}(w_k) = \int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta}}{(t-w_k)^{j}}\,dt, \quad \operatorname{Re}(w_k) \neq 0,\ |w_k| < 1 \quad (2.12)$$

$$\hat{G}^{j}(w_k) = (-1)^{j}\Gamma(j+1) \int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta}}{(t-w_k)^{j+1}}\,dt, \quad |w_k| \geq 1 \quad (2.13)$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{t\,dt}{\sqrt{1-t^2}(t-x)(t^2+a^2)} = \frac{|a|}{(x^2+a^2)\sqrt{1+a^2}} \quad (2.14)$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{t\,dt}{\sqrt{1-t^2}(t-x)(t^2+a^2)^2} = \frac{|a|}{(x^2+a^2)^2\sqrt{1+a^2}} - \frac{1}{2|a|(x^2+a^2)(1+a^2)^{3/2}} \quad (2.15)$$

$$\frac{1}{\pi}\int_{1}^{1} \frac{\sqrt{1-t^2}\,dt}{(x-t)(t^2+a^2)} = \frac{x\sqrt{1+a^2}}{|a|(x^2+a^2)} \quad (2.16)$$

$$\frac{1}{\pi}\int_{1}^{1} \frac{\sqrt{1-t^2}\,dt}{(x-t)(t^2+a^2)^2} = \frac{-x}{|a|^3(a^2+x^2)\sqrt{1+a^2}} + \frac{x\sqrt{1+a^2}}{(a^2+x^2)^2|a|} \quad (2.17)$$

$$\frac{1}{\pi}\int_{1}^{1} \frac{\sqrt{1-t^2}\,t^2\,dt}{(x-t)(t^2+a^2)^2} = \frac{x(\frac{1}{2}+a^2)}{|a|(x^2+a^2)} - \frac{x|a|\sqrt{1+a^2}}{(x^2+a^2)^2} \quad (2.18)$$

3. A RATIONAL FUNCTION SCHEME

Direct methods for the solution of singular integral equations in general consist of initially discretizing the original equation by approximating ϕ, collocating at a set of nodes and then solving the resulting system of algebraic equations for the values of the unknown function at these nodes. The accuracy of the solution is dependent upon the choice of collocation nodes, the quadrature formula used, and the manner in which the data is extrapolated. The realm for the propagation of computational error is vast. The approach here is to first approximate the right-hand side and then compute the solution directly for this approximate equation. A global approximation ensues. Collocation nodes and the extrapolation of data are no longer needed.

For the dominant equation, i.e.

$$ag(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{g(t)\,dt}{t-x} = f(x), \quad -1 < x < 1, \quad (3.1)$$

where $a \pm ib \neq 0$ and $a, b \in \mathbf{R}$, g may be represented by

$$g(x) = w(x)\phi(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad (3.2)$$

where $-1 < \mathrm{Re}(\alpha), \mathrm{Re}(\beta) < 1$ and α, β are determined by the index κ of Eq. (3.1). This takes care of the singularities at the endpoints. On substituting (3.2) into (3.1) and using identity (2.9), the dominant equation is reduced to

$$\frac{b}{\pi}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\left(\frac{\phi(t)-\phi(x)}{t-x}\right)dt = f(x), \quad -1 < x < 1. \tag{3.3}$$

A rational function is constructed which approximates f and consequently transforms the solution space of Eq. (3.1) to the space of rational functions. The right-hand side may be approximated by

$$f(x) \simeq f_R(x) = \sum_{i=1}^{p'}\sum_{j=1}^{\rho_i'}\frac{\hat{A}_{ij}}{(x-\alpha_i)^j} + \sum_{i=1}^{q'}\sum_{j=1}^{\xi_i'}\frac{\hat{B}_{ij}}{(x-w_i)^j} + \sum_{i=1}^{q'}\sum_{j=1}^{\xi_i'}\frac{\hat{C}_{ij}}{(x-\overline{w}_i)^j}, \tag{3.4}$$

where the $\{\alpha_i\}_{i=1}^{p'}$ and $\{w_i\}_{i=1}^{q'}$ are respectively the real and complex poles of the approximation to f. The complex poles are ordered such that $|w_i| < 1$, $i = 1, 2, \ldots, s'$ and $|w_i| \geq 1$, $i = s'+1, s'+2, \ldots, q'$. ρ_i' and ξ_i' represent the order of multiplicity of the α_i and w_i. The $\hat{A}_{ij}$, $\hat{B}_{ij}$, and $\hat{C}_{ij}$ are the partial fraction coefficients of the expansion of the approximation to f. The solution has been projected onto the space of rational functions and may be represented by

$$\phi(x) \simeq \phi_R(x) = \sum_{i=1}^{p}\sum_{j=1}^{\rho_i}\frac{A_{ij}}{(x-r_i)^j} + \sum_{i=1}^{q}\sum_{j=1}^{\xi_i}\frac{B_{ij}}{(x-z_i)^j} + \sum_{i=1}^{q}\sum_{j=1}^{\xi_i}\frac{C_{ij}}{(x-\overline{z}_i)^j}. \tag{3.5}$$

Here, the A_{ij}, B_{ij}, C_{ij}, r_i, and z_i are to be determined. On substituting (3.4) and (3.5) into (3.3) and further expansion in terms of partial fractions, the subtracted term will cancel leaving the following discretized equation.

$$\begin{aligned}
&\sum_{i=1}^{p}\left[\sum_{m=1}^{\rho_i}\frac{\Gamma(m)}{(r_i-x)^m}\sum_{j=m}^{\rho_i}\frac{A_{ij}\hat{F}^{j-m}(r_i)}{\Gamma(j-m+1)}\right] + \sum_{i=1}^{s}\left[\sum_{m=1}^{\xi_i}\frac{\Gamma(m)}{(z_i-x)^m}\sum_{j=1}^{\xi_i}B_{ij}(-1)^{(j-m)}\hat{G}^{j-m}(z_i)\right]\\
&\quad + \sum_{i=s+1}^{q}\left[\sum_{m=1}^{\xi_i}\frac{\Gamma(m)}{(z_i-x)^m}\sum_{j=m}^{\xi_i}\frac{B_{ij}\hat{G}^{j-m}(z_i)}{\Gamma(j-m+1)}\right]\\
&\quad + \sum_{i=1}^{s}\left[\sum_{m=1}^{\xi_i}\frac{\Gamma(m)}{(\overline{z}_i-x)^m}\sum_{j=m}^{\xi_i}C_{ij}(-1)^{(j-m)}\hat{G}^{j-m}(z_i)\right]\\
&\quad + \sum_{i=s+1}^{q}\left[\sum_{m=1}^{\xi_i}\frac{\Gamma(m)}{(\overline{z}_i-x)^m}\sum_{j=m}^{\xi_i}\frac{C_{ij}\hat{G}^{j-m}(\overline{z}_i)}{\Gamma(j-m+1)}\right]\\
&\quad = \sum_{i=1}^{p'}\sum_{j=1}^{\rho_i'}\frac{\hat{A}_{ij}}{(x-\alpha_i)^j} + \sum_{i=1}^{s'}\sum_{j=1}^{\xi_i'}\frac{\hat{B}_{ij}}{(x-w_i)^j}\\
&\quad + \sum_{i=s'+1}^{q'}\sum_{j=1}^{\xi_i'}\frac{\hat{B}_{ij}}{(x-w_i)^j} + \sum_{i=1}^{s'}\sum_{j=1}^{\xi_i'}\frac{\hat{C}_{ij}}{(x-\overline{w}_i)^j}\\
&\quad + \sum_{i=s'+1}^{q'}\sum_{j=1}^{\xi_i'}\frac{\hat{C}_{ij}}{(x-\overline{w}_i)^j}, \quad -1 < x < 1.
\end{aligned} \tag{3.6}$$

From Eq. (3.6), it is evident that $p = p'$, $\rho_i = \rho_i'$, $s = s'$, $q = q'$, $\xi_i = \xi_i'$, $r_i = \alpha_i$, $z_i = w_i$, and $\overline{z}_i = \overline{w}_i$. Consequently, ϕ_R will have the same set of singularities as f_R. This relationship follows immediately from (2.2). After replacing r_i by α_i, z_i by w_i, and $\overline{z}_i$ by $\overline{w}_i$, (3.6) will result in a block-diagonal system of equations

$$A\mathbf{x} = \mathbf{b}. \tag{3.7}$$

Here, A is given by:

$$\langle a_{ij} \rangle \equiv \begin{cases} A[\alpha_k] & i = j = 1, \ldots, p'; k = i \\ B[w_k] & i = j = p' + 1, \ldots, p' + s'; k = i - p' \\ B[w_k] & i = j = p' + s' + 1, \ldots, p' + q'; k = i - p' \\ C[\overline{w}_k] & i = j = p' + q' + 1, \ldots, p' + q' + s'; k = i - (p' + q') \\ C[\overline{w}_k] & i = j = p' + q' + s' + 1, \ldots, p' + 2q' + 1; k = i - (p' + q'). \end{cases} \tag{3.8}$$

where, the $A[\cdot]$, $B[\cdot]$, and $C[\cdot]$ are upper triangular matrices. These matrices are defined as follows:

$$a_{ij}[\alpha_k] = \begin{cases} \hat{F}^{j-1}(\alpha_k)/j! & i \le j \\ 0 & i > j \end{cases} \tag{3.9}$$

for $k = 1, 2, \ldots, p'$, and

$$b_{ij}[\cdot_k] = c_{ij}[\cdot_k] = \begin{cases} G^j(\cdot k)(-1)^{j-1} & i \le j \\ 0 & i > j \end{cases} \tag{3.10}$$

for $k = 1, 2, \ldots, s'$, and

$$b_{ij}[\cdot_k] = c_{ij}[\cdot_k] = \begin{cases} \hat{G}(\cdot k)/(j-1)! & i \le j \\ 0 & i > j \end{cases} \tag{3.11}$$

for $k = s' + 1, s' + 2, \ldots, q'$. The functionals $\hat{F}^j$ and $\hat{G}^j$ are the integrals which arise from the earlier substitution and are defined in (2.11), (2.12), and (2.13) respectively. A has this structure:

$$\left[\begin{array}{c|c|c|c} \begin{array}{ccccccc} x&x&x&x&x&x&x\\ &x&x&x&x&x&x\\ &&x&x&x&x&x\\ &&&x&x&x&x\\ &&&&x&x&x\\ &&&&&x&x\\ &&&&&&x \end{array} & & & \\ \hline & \begin{array}{ccccccc} x&x&x&x&x&x&x\\ &x&x&x&x&x&x\\ &&x&x&x&x&x\\ &&&x&x&x&x\\ &&&&x&x&x\\ &&&&&x&x\\ &&&&&&x \end{array} & & \\ \hline & & \ddots & \\ \hline & & & \begin{array}{ccccccc} x&x&x&x&x&x&x\\ &x&x&x&x&x&x\\ &&x&x&x&x&x\\ &&&x&x&x&x\\ &&&&x&x&x\\ &&&&&x&x\\ &&&&&&x \end{array} \end{array}\right]$$

The vectors $\mathbf{x}$ and $\mathbf{b}$ represent the unknown coefficients and the known partial fraction coefficients, respectively. More explicitly,

$$\mathbf{x} = [A_{ij}, B_{ij}, C_{ij}]^{\mathrm{T}} \tag{3.12}$$

and

$$\mathbf{b} = \frac{(-1)^j}{\pi\Gamma(j)} [\hat{A}_{ij}, \hat{B}_{ij}, \hat{C}_{ij}]^{\mathrm{T}}. \tag{3.13}$$

In addition to being block-diagonal with each block being upper triangular, the submatrices of A are independent. It is further noted that only back substitution is required to solve the system of equations. For the case in which all the singularities are simple poles, the matrix A reduces to an $N \times N$ diagonal matrix, where N is the number of poles of f. Thus the major computational effort is in the construction of rational function approximations.

4. A HYBRID METHOD

The "hybrid" method, similar to the rational function scheme, will construct a rational function which is a global approximation to the unknown. The "hybrid" technique will be developed for the complete equation

$$\frac{b}{\pi}\int_{-1}^{1} \frac{g(t)\mathrm{d}t}{t - x} + \lambda \int_{-1}^{1} k(x, t)g(t)\mathrm{d}t = f(x), \quad -1 < x < 1. \tag{4.1}$$

In particular, the equation arising from the static problem of a cruciform crack opened by internal pressure is considered. The extension of the technique for arbitrary kernels is indicated.

4.1 *Equation for dislocation density*

The problem of a cruciform crack may be reduced to the singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1} \frac{g(t)\mathrm{d}t}{t - x} + \frac{1}{\pi}\int_{-1}^{1} \frac{(t(t^2 - x^2))}{(t^2 + x^2)^2} g(t)\mathrm{d}t = 1. \tag{4.1.1}$$

The function g is proportional to the dislocation density along the crack branches and is an odd function of t. $g(t)$ has reciprocal square-root singularities at both endpoints $+1$ and -1. By setting

$$g(t) = (1 - t^2)^{-1/2}\phi(t) \tag{4.1.2}$$

$\phi(t)$ becomes an odd function of t. Its value at $t = 1$ corresponds to the stress intensity factor, a quantity of prime importance in Fracture Mechanics. If the Gauss–Chebyshev formula is used for the second integral, Eq. (8.1.1) becomes

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\phi_R(t)\mathrm{d}t}{\sqrt{1 - t^2}(t - x)} + \frac{1}{N}\sum_{j=1}^{N} \frac{\phi(t_j)t_j(t_j^2 - x^2)}{(t_j^2 + x^2)^2} = 1, \tag{4.1.3}$$

where (t_i) are the zeros of the Chebyshev polynomial $T_n(x)$. Equation (8.1.3) may now be solved in closed form by simply representing ϕ by:

$$\phi(t) = t + \sum_{k=1}^{N} \frac{a_k t}{t^2 + t_k^2} + \sum_{k=1}^{N} \frac{b_k t}{(t^2 + t_k^2)^2}. \tag{4.1.4}$$

In this particular case, the right-hand side is a polynomial. Its pre-image is adjusted for in the expression for ϕ by the additional term t. On substituting (8.1.4) into (8.1.3), using the relations

(2.14) and (2.15), and comparing the coefficients of $(t_j^2 + x^2)^{-1}$ and $(t_j^2 + x^2)^{-2}$, the resulting system of equations will be given by:

$$\frac{a_m|t_m|}{\sqrt{1 + t_m^2}} - \frac{b_m}{2|t_m|(1 + t_m^2)^{3/2}} - \frac{t_m\phi(t_m)}{N} = 0 \tag{4.1.5}$$

$$\frac{b_m|t_m|}{\sqrt{1 + t_m^2}} + \frac{2t_m^3\phi(t_m)}{N} = 0 \tag{4.1.6}$$

for $m = 1, 2, \ldots, N$. Multiplying (8.1.5) by $2t_m^2$ and adding the product to (8.1.6), the following relation results

$$b_m = -2a_m(1 + t_m^2) \quad m = 1, 2, \ldots, N. \tag{4.1.7}$$

Moreover, (cf. (8.1.4))

$$\phi(t_m) = t_m + \sum_{k=1}^{N} \frac{a_k t_m}{(t_m^2 + t_k^2)} + \sum_{k=1}^{N} \frac{b_k t_m}{(t_m^2 + t_k^2)^2}. \tag{4.1.8}$$

Substituting for b_k in (4.1.8) and by further using the expression (4.1.6), the former system of algebraic equations is reduced to

$$\frac{a_m\sqrt{1 + t_m^2}}{t_m} + \frac{1}{N}\sum_{k=1}^{N} \frac{t_m^2(a_k(t_k^2 + 2 - t_m^2))}{(t_m^2 + t_k^2)} = \frac{t_m^2}{n} \tag{4.1.9}$$

On calculation of the a_m, the b_m may be determined easily by (4.1.7).

4.2 *Lobatto–Chebyshev vs Gauss–Chebyshev*

Lobatto–Chebyshev as well as Gauss–Chebyshev may be employed in the approximation of the regular part. This leads to the equation

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)\mathrm{d}t}{\sqrt{1 - t^2}(t - x)} + \frac{1}{N}\left\{\frac{\phi(1)(1 - x^2)}{(1 + x^2)^2}\right\} \sum_{j=1}^{N-1} \frac{\phi(s_j)s_j(s_j^2 - x^2)}{(s_j^2 + x^2)^2} \tag{4.2.1}$$

where $\{s_j\}$, $j = 1, 2, \ldots, N - 1$ are the zeros of $U_{N-1}(x)$, and $s_0 = 1$. Then, the solution of the above equation is of the form

$$\phi(t) = t + \sum_{j=0}^{N-1} \frac{A_j t}{s_j^2 + t^2} + \sum_{j=0}^{N-1} \frac{B_j t}{(s_j^2 + t^2)^2}. \tag{4.2.2}$$

Following the steps outlined in the previous section, the resulting system

$$\frac{B_j|s_j|}{(1 + s_j^2)} + \frac{1}{N}\sum_{k=0}^{N-1} B_k\left\{\frac{2(1 + s_k^2) - (s_k^2 + s_j^2)}{(s_k^2 + s_j^2)^2(1 + s_k^2)}\right\}s_j^4 = \frac{-2s_j}{N} \tag{4.2.3}$$

may be solved for the B_j, and the A_j may be computed via

$$B_j = -2(1 + s_j^2)A_j. \tag{4.2.4}$$

The solution of (4.2.3) with (4.2.4) will yield the global solution $\phi(t)$.

4.3 *Equation for the normal component of displacement*

Solution of the equations of elastic equilibrium for a cruciform crack opened by internal pressure[21] may be expressed in terms of an auxiliary function $A(\xi)$ which satisfies the dual integral equations

$$\int_0^x \sigma_{yy}(x, 0)\mathrm{d}x = f_s[\xi^{-1}A(\xi); \xi \longrightarrow x] + \sqrt{\frac{2}{\pi}}\int_0^\infty A(\xi)e^{-\xi x}\mathrm{d}\xi = p_0\int_0^x f(t)\mathrm{d}t, \quad 0 \le x < 1 \quad (4.3.1)$$

$$U_y(x, 0) = \frac{2(1-\eta)}{2\mu} f_c[\xi^{-1}A(\xi)j; \xi \longrightarrow x] = 0, \quad x > 1, \quad (4.3.2)$$

where $p_0 f(x)$ is the specified internal pressure. For simplicity, the ensuing discussion will be restricted to the case when $f(x) = 1$.

If we let $U_y(x, 0) = g(|x|)$, for $|x| \le 1$, the Fourier inversion formula gives us an integral representation for $A(\xi)$, which when substituted in Eq. (4.3.1) yields the singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{g(t)\mathrm{d}t}{x-t} + \frac{x}{\pi}\int_{-1}^{1}\frac{g(t)(x^2-t^2)\mathrm{d}t}{(x^2+t^2)^2} = \frac{2(1-\eta)p_0 x}{2\mu}, \quad -1 < x < 1. \quad (4.3.3)$$

This equation satisfies the compatability condition for a bounded solution. $g(t)$ is an even function of t and vanishes at $t = -1, 1$. Following standard procedures[7], let

$$g(t) = \frac{2(1-\eta)}{2\mu}(1-t^2)^{1/2}\phi(t). \quad (4.3.4)$$

Gaussian quadrature applied to the regular part of (4.3.3) yields the approximation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}\phi(t)\mathrm{d}t}{x-t} + \frac{x}{N+1}\sum_{j=1}^{N}(1-s_j^2)\frac{\phi(s_j)(x^2-s_j^2)}{(x^2+s_j^2)^2} = x, \quad (4.3.5)$$

where $\{s_j\}_1^N$ are the zeros of $U_N(x)$. Solution of (4.3.5) is a rational function of the form

$$\phi(t) = 1 + \sum_{j=1}^{N}\frac{A_j}{t^2+s_j^2} + \sum_{j=1}^{N}\frac{C_j}{(t^2+s_j^2)^2}. \quad (4.3.6)$$

By substituting this expression in (4.3.5), evaluating the integrals using relations (2.14) and (2.15), and by comparing the coefficients of $x/(x^2+s_j^2)$ and $x/(x^2+s_j^2)^2$, the following system of equations results

$$\frac{A_j\sqrt{1+s_j^2}}{s_j} + \frac{(1-s_j^2)\phi(s_j)}{N+1} + \frac{C_j}{2|s_j|^3\sqrt{1+s_j^2}} = 0 \quad (4.3.7)$$

$$\frac{C_j\sqrt{1+s_j^2}}{s_j} - \frac{2s_j^2(1-s_j^2)\phi(s_j)}{N+1} = 0. \quad (4.3.8)$$

Multiplying (4.3.7) by $2s_j^2$ and adding the product to (4.3.8), the A_j may be related to the C_j by

$$A_j = \frac{-(2+s_j^2)C_j}{2s_j^2(1+s_j^2)}. \quad (4.3.9)$$

Finally, using (4.3.6) and (4.3.7), (4.3.8) and (4.3.9) may be condensed to

$$\frac{C_j\sqrt{1 + s_j^2}}{|s_j|} - \frac{2s_j^2(1 - s_j^2)}{N + 1}\sum_{k=1}^{N}\frac{C_k(s_k^4 - s_j^2 s_k^2 - 2s_j^2)}{2s_k^2(1 + s_k^2)(s_j^2 + s_k^2)^2} = \frac{2s_j^2(1 - s_j^2)}{N + 1} \quad j = 1, 2, \ldots, N. \tag{4.3.10}$$

Although the "hybrid" technique was only presented for a specific problem of Fracture Mechanics, the extension for general singular integral equations is obvious. The only added difficulty is when the kernel is not a rational function. For this case, the kernel would be approximated by a rational function. The complexity of the resulting discretized system of equations will be a function of the order of approximation to the kernel. Consequently, the adaptation of the technique for the solution of mixed boundary value problems seems to be a viable alternative.

5. EXISTENCE AND STABILITY

As in the analytic theory of Cauchy-type singular integral equations, the singular behavior of the solution is determined by the dominant term. Because of its many applications in Mathematical Physics, the ensuing analyses given in the following two sections shall focus upon the dominant equation with index one.

5.1 *Existence and Uniqueness*

The block-diagonal structure of the resulting discretized equation facilitates the analysis of both existence and uniqueness of the numerical solution. The eigenvalues of A are simply the diagonal elements, which consist of the functionals $\hat{F}$ and $\hat{G}$. The determinant will be the product of these elements, i.e.

$$\det(A) = \left(\prod_k \prod_j \frac{\hat{F}^{j-1}(\alpha_k)}{\Gamma(j + 1)}\right)\left(\prod_k \prod_j (-1)^{j-1}\hat{G}^j(w_k)\right). \tag{5.1}$$

The original assumption that ϕ be Hölder continuous on $(-1, 1)$ ensures that f and ϕ do not have singularities within this interval. Hence, the values of the functionals $\hat{F}$, $\hat{G}$ and consequently the a_{ii} will not vanish. A is thus nonsingular and a unique, non-trivial solution is guaranteed by standard theorems in Linear Algebra.

5.2 *Stability*

The stability of this numerical method is measured in terms of the conditioning of A. The spectral norm of A is strictly dependent upon the location of the poles in the complex plane, i.e.

$$\|A\|_2 = \sqrt{r_\sigma(A*A)} \leq r_\sigma(A) = \frac{\pi}{\sqrt{m^2 - 1}} \tag{5.2}$$

where the $\{z_i\}_{i=1}^N$ consists of all the singularities of ϕ and $m = \min_j \|z_j^2 - 1\|$. Similarly, for A^{-1} it follows that

$$\|A^{-1}\|_2 \leq \frac{\sqrt{M^2 - 1}}{\pi} \tag{5.3}$$

where $M = \max_j \|z_j^2 - 1\|$. From (5.2) and (5.3), the condition number of A is given by:

$$K(A) = \|A\|_2\|A^{-1}\|_2 \leq \left(\frac{M^2 - 1}{m^2 - 1}\right). \tag{5.4}$$

It is evident from (5.4) that A will be well-conditioned for all z_i outside ϵ − neighborhoods of -1 and $+1$.

6. ERROR PROPAGATION AND CONVERGENCE

Error is propagated through a number of stages beginning with the initial rational function approximation and ending with the round-off error contributed by the calculation of the poles and the evaluation of the partial fraction coefficients. To establish the order of error in the rational function scheme, some results from G. A. Baker[1] concerning the order of error incurred by approximating by a rational function of Padé type and the convergence of Padé approximants shall be referred to later. No claims to the sharpness of the estimates are made; however, the following analyses should prove adequate.

The analysis will proceed by first making the following definitions:

(i) ϕ is the *analytic solution.*

(ii) ϕ_R is the *rational function approximation to* ϕ which is evaluated on the exact values of the poles.

(iii) ϕ_R^ϵ is the *rational function approximation to* ϕ which is evaluated on the computed values of the poles.

From the above definitions, it is clear that the quantity of interest is $\|\phi - \phi_R^\epsilon\|$, which is bounded:

$$\|\phi - \phi_R^\epsilon\| \leq \|\phi - \phi_R\| + \|\phi_R - \phi_R^\epsilon\| \tag{6.1}$$

6.1 *Round-off Error*: $\|\phi_R - \phi_R^\epsilon\|$

As before, A, $\mathbf{x}$, $\mathbf{b}$ are the exact expressions for the matrix, the vector containing the partial fraction coefficients of the solution ϕ_R, and the vector containing the partial fraction coefficients of the right-hand side, respectively; and A_ϵ, $\mathbf{x}_\epsilon$, $\mathbf{b}_\epsilon$ are the computed expressions. In the following analyses, the error in the evaluation of the poles of ϕ_R is ϵ, i.e.

$$z_i = z_i^\epsilon + \epsilon \tag{6.1.1}$$

where $\{z_i\}_1^M$ and $\{z_i^\epsilon\}_1^M$ is the collection of exact and computed poles. (Without a loss of generality, the real and complex poles are taken as simple and are given by the one collection $\{z_i\}_1^M$). It is clear from the above definitions that:

$$A\mathbf{x} = \mathbf{b} \tag{6.1.2}$$

and

$$A_\epsilon \mathbf{x}^\epsilon = \mathbf{b}^\epsilon. \tag{6.1.3}$$

From which, it follows that:

$$A(\mathbf{x} - \mathbf{x}^\epsilon) = (\mathbf{b} - \mathbf{b}^\epsilon) - \mathscr{E}\mathbf{x}^\epsilon, \tag{6.1.4}$$

where $A = A_\epsilon + \mathscr{E}$ and $\mathscr{E}$ is the error matrix resulting from the evaluation of the poles.

LEMMA 6.1.1

Given a rational function approximation $R(z) = P_L(z)/Q_M(z)$ to a function $f(z)$, where $R(x) \in C^1[-1, 1]$ then the error in the computed partial fraction coefficients as a result of the error in the z_i is of order ϵ, i.e.

$$\|\mathbf{b} - \mathbf{b}^\epsilon\|_2 \simeq O(\epsilon).$$

Proof. $R(z)$ possesses simple poles. Thus,

$$b_i^\epsilon = \frac{P_L(z_i^\epsilon)}{Q_M'(z_i)}.$$

By considering the numerator, it is clear that:

$$P_L(z_i^\epsilon) = P_L(z_i - \epsilon) = P_L(z_i) + R_{L-1}(z_i, \epsilon),$$

where

$$R_{L-1}(z_i, \epsilon) = \sum_{m=1}^{L} \left(\sum_{k=1}^{L} a_k(z_i)^k \right) \left(\frac{-\epsilon}{z_i} \right)^m$$

and so the lemma follows.

LEMMA 6.1.2

$\|\mathcal{E}\|_2 \leqq c(\epsilon)\|A\|_2\epsilon$, where c is a constant dependent on ϵ.

LEMMA 6.1.3

Given z_i^ϵ such that $z_i^\epsilon \to z_i$ as $M \to \infty$ then:

$$\|\mathbf{x} - \mathbf{x}^\epsilon\|_2 \leq \|A^{-1}\|_2(c(\epsilon) - c_1 K_A\|\mathbf{b}\|_2)\epsilon$$

and so $x_i^\epsilon \to x_i$ as $M \to \infty$.

Proof. From (6.1.4) and the fact that A is nonsingular, it follows that

$$\|\mathbf{x} - \mathbf{x}^\epsilon\|_2 \leq \|A^{-1}\|_2\{\|\mathbf{b} - \mathbf{b}^\epsilon\|_2 + \|\mathcal{E}\|_2\|\mathbf{x}^\epsilon\|_2\}$$

Since $x^\epsilon = \mathbf{x} + \boldsymbol{\epsilon}$ then by the triangle inequality $\|\mathbf{x}^\epsilon\| \leq \|\mathbf{x}\| + \|\boldsymbol{\epsilon}\|$. ϵ is small and $\|\mathcal{E}\|_2 \simeq O(\epsilon)$. By neglecting $O(\epsilon^2)$ terms and by applying lemmas (6.1.1) and (6.1.2), the result follows.

LEMMA 6.1.4

$$\|\phi_R - \phi_R^\epsilon\|_2 \leq c_2 M\|A^{-1}\|_2\epsilon$$

Here, the previous lemma follows by considering the difference of the M individual terms $x_i/t - z_i$ and $x_i^\epsilon/t - z_i^\epsilon$.

6.2 *Error of rational approximation*: $\|\phi - \phi_R\|$

The error in the approximation to ϕ by a rational function ϕ_R is strictly dependent on the image of the error incurred on the right-hand side under the inverse finite Hilbert transform. (*Note*: The following analyses have been carried out for Padé approximants and may be generalized easily for other types of rational function approximations).

LEMMA 6.2.1

Let $f \in C^N[-1, 1]$ then

$$\|\phi - \phi_R\|_2 \leq \frac{1}{2}\frac{\|\Omega_M\|_2}{\|Q_M\|_2} + \frac{1}{2}\left\|\frac{Q_M}{Q_M}\right\|_2 \|f - [L/M]\|_2,$$

where

$$\Omega(x) = \sum_{k=n+1}^{\infty} k\left(\sum_{i=0}^{M} a_{k-i}q_i\right)x^{k-1}.$$

Proof. The inversion of the dominant equation yields

$$\phi(t) - \sigma_R(t) = \frac{1}{\pi}\int_1^1 \sqrt{1-x^2}\left\{\frac{\epsilon(x)-\epsilon(t)}{x-t}\right\}\mathrm{d}x + \frac{\epsilon(t)}{\pi}\int_1^1 \frac{\sqrt{1-x^2}}{x-t}\,\mathrm{d}x,$$

where $\epsilon(x) = f(x) - f_R(x)$ and f_R is the $[L/M]$ Pade approximant to f. After some elementary integration and an application of the Mean Value Theorem, it follows that

$$\|\phi(t) - \phi_R(t)\|_2 \leq \tfrac{1}{2}\|\epsilon'(t)\|_2 + \|\epsilon(t)\|_2.$$

From Cheney[4] it follows that

$$\epsilon(x) = \sum_{k=n+1}^{\infty} \frac{c_k x^k}{Q_M(x)},$$

where $c_k = \Sigma_{i=0}^{M} q_{k-i}q_i$ and thus,

$$\epsilon'(x) = \frac{1}{Q(x)}\sum_{k=n+1}^{\infty} kc_k x^{k-1} - \frac{Q_M'(x)}{Q_M(x)}\frac{\sum_{k=n+1}^{\infty} c_k x^k}{Q_M(x)}$$

and so,

$$\|\epsilon'(x)\|_2 \leq \frac{\|\Omega\|_2}{\|Q_M\|_2} + \left\|\frac{Q_M'}{Q_M}\right\| \|\epsilon(x)\|_2.$$

THEOREM 6.2

Let f be as in Lemma 6.2.1 then as $N \rightarrow \infty$, $\phi_R^{\epsilon} \rightarrow \phi$, where $N = L + M + 1$.

Proof. This follows from Lemata 6.1.4 and 6.2.1 and the standard convergence theorems for Padé approximants (C.f. Baker[1]).

7. NUMERICAL RESULTS FOR THE AIRFOIL EQUATION

The problem of determining the velocity field of a two-dimensional plane parallel stream over a thin smooth airfoil has been considered by many. It is formulated as a dominant singular integral equation of the first kind, i.e.

$$\frac{1}{\pi}\int_{-1}^{1} \frac{g(t)\mathrm{d}t}{t-x} = f(x), \quad -1 < x < 1. \tag{7.1}$$

Here, g is the velocity potential and f is the velocity distribution of the stream at infinity. To ensure single-valuedness, a normalization condition of the form

$$\int_{-1}^{1} g(t)\mathrm{d}t = 0 \tag{7.2}$$

is specified. In the following numerical examples, the airfoil equation is solved when $f(x) = e^x$ and for when $f(x) = e^x/(x^2 + a^2)$, where $a = 0.01$. Tables (7.1) and (7.3) compare the results obtained from the rational function scheme with those obtained from the Gauss and Lobatto–Chebyshev schemes. Table (7.2) presents the computational time required to obtain a prescribed order of accuracy by the methods. Here, a saturation point has been reached for $N = 10$. This phenomena is the result of truncation and round-off error resulting from the Jenkins–Traub algorithm for the determination of the roots of a polynomial. This problem may be avoided by updating the values of the poles using a newton iterate. All computations were performed on

Table 7.1. Evaluation of g at $+1$: $f(x) = e^x$

N	Gauss-Chebyshev	Lobatto-Chebyshev	Rational Functions
3	1.648721	1.803313	1.814535
4	1.803313	1.831223	1.833861
5	1.827922	1.831225	1.831358
6	1.803907	1.831225	1.831212
7	1.831199	1.831225	1.831224
8	1.831223	1.831225	1.831224
9	1.831225	1.831225	1.831224
10	1.831225	1.831225	1.831224
11	1.831225	1.831225	1.831224
12	1.831225	1.831225	1.831224
13	1.831225*	1.831225	1.831224

*These values were obtained by extrapolation

Table 7.2. Computational time vs accuracy

Scheme	N = 4	6	8	10
Gauss Chebyshev	Time: 0.16 Error: 2.79E(-2)	0.28 2.73E(-2)	0.47 1.92E(-6)	0.65 8.0E(-8)
Rational Functions	Time: 0.10 Error: 2.63E(-2)	0.15 1.29E(-5)	0.24 9.2E(-7)	0.29 9.2E(-7)

Table 7.3. Evaluation of g at 0: $f(x) = e^x/(x^2 + a^2)$ $(a = 0.01)$

N	Gauss-Chebyshev	Lobatto-Chebyshev
9	-0.8060332 +001	-0.1106093 +004
10	-0.9960251 +003	-0.9051370 +001
15	-0.1397315 +002	-0.6664834 +003
16	-0.4500369 +003	-0.1494952 +002
24	-0.4210579 +003	-0.2263439 +002
25	-0.2357669 +002	-0.4048272 +003
49	-0.4450691 +002	-0.2179399 +003
50	-0.2142019 +003	-0.4529700 +002
99	-0.7482246 +002	-0.1304767 +003
100	-0.1297495 +003	-0.7524566 +002
511	-0.9908045 +002	-0.9896899 +002
512	-0.9348854 +002	-0.9408059 +002
1023	-0.9908774 +002	-0.9902478 +002
1024	-0.9628456 +002	-0.9908774 +002

N	RATFNS
3	-0.1000851 +003
5	-0.1000880 +003
7	-0.1000877 +003
9	-0.1000879 +003
10	-0.1000874 +003
11	-0.1000874 +003
12	-0.1000874 +003
13	-0.1000874 +003

Table 8.1. Dislocation density at equally spaced points $t \in [0, 1]$. N is the number of quadrature nodes

N	t / ALG.	0.0000	0.2500	0.5000	0.7500	1.0000
8	(a)	0.0000	-7.1629E-03	-0.1961	-0.5240	-0.8644
8	(b)	0.0000	-9.5386E-03	-0.1946	-0.5229	-0.8634
9	(c)	0.0000	-0.0260	-0.1847	-0.5147	-0.8561
16	(a)	0.0000	1.3257E-02	-0.1959	-0.5239	-0.8640
16	(b)	0.0000	1.4656E-02	-0.1952	-0.5233	-0.8635
17	(c)	0.0000	0.0178	-0.1925	-0.5211	-0.8615
32	(a)	0.0000	0.0137	-0.1953	-0.5234	-0.8636
32	(b)	0.0000	1.3936E-02	-0.1952	-0.5233	-0.8635
33	(c)	0.0000	0.0148	-0.1945	-0.5228	-0.8630
64	(a)	0.0000	1.3916E-02	-0.1952	-0.5233	-0.8636
64	(b)	0.0000	1.3931E-02	-0.1952	-0.5233	-0.8635
65	(c)	0.0000	0.0142	-0.1950	-0.5234	-0.8634

(a) Gauss-Chebyshev quadrature & collocation with Nystrom interpolation.
(b) Hybrid scheme.
(c) Piecewise linear approximation.

Table 8.2. Stress intensity factors (normalized)

ALG. / N	Gauss-Chebyshev	Hybrid-Rational G.C.	Lobatto-Chebyshev	Hybrid-Rational L.C.
4	0.86436	0.86565	0.87102	0.86350
8	0.86401	0.86342	0.86410	0.86354
12	0.86370	0.86354	0.86351	0.86353
16	0.86350	0.86354	0.86318	0.86354
20	0.80356	0.86354	0.86350	0.86354
24	0.86370	0.86354	0.86347	0.86354

Table 8.3. Dislocation density at Chebyshev nodes

	Nodes	Gauss-Chebyshev	Rational Hybrid	Difference
N=4	0.9808	0.8388	0.8379	9.0×10^{-4}
	0.8315	0.6363	0.6353	1.0×10^{-3}
	0.5556	0.2638	0.2624	1.4×10^{-3}
	0.1951	-1.8606E-02	-2.1080E-02	2.47×10^{-3}
N=6	0.9914	0.8530	0.8522	8.0×10^{-4}
	0.9239	0.7624	0.7616	8.0×10^{-4}
	0.7934	0.5839	0.5831	8.0×10^{-4}
	0.6088	0.3327	0.3316	1.1×10^{-3}
	0.3827	7.3292E-02	7.1728E-02	1.6×10^{-3}
	0.1305	-2.084E-02	-2.3636E-02	2.8×10^{-3}
N=8	0.9952	0.8576	0.8572	4.0×10^{-4}
	0.9569	0.8066	0.8061	5.0×10^{-4}
	0.8819	0.7052	0.7046	6.0×10^{-4}
	0.7730	0.5556	0.5550	6.0×10^{-4}
	0.6344	0.3664	0.3657	7.0×10^{-4}
	0.4714	0.1630	0.1623	7.0×10^{-4}
	0.2903	5.6356E-03	4.4178E-03	1.2×10^{-3}
	9.8017E-02	-1.6248E-02	-1.8438E-02	2.2×10^{-3}
N=12	0.9979	0.8607	0.8609	2.0×10^{-4}
	0.9469	0.7927	0.7929	2.0×10^{-4}
	0.8315	0.6354	0.6358	2.0×10^{-4}
	0.6593	0.3993	0.3995	2.0×10^{-4}
	0.4423	0.1306	0.1310	4.0×10^{-4}
	0.3214	2.4123E-02	2.4544E-02	4.0×10^{-4}
	0.1951	-2.8113E-02	-2.7420E-02	6.9×10^{-4}
	6.5403E-02	-1.0350E-02	-9.1191E-03	1.23×10^{-3}

NOTE: The difference is larger near zero and is approximately halved when the number of nodes are doubled. For N=12, we have omitted certain nodes but the difference is of the same order of magnitude.

Table 8.4. Dislocation density and normal component of displacement at Chebyshev nodes. The former is an odd function and the latter is even. Only half of the nodes. Extrema of $T_{16}(x)$, are shown

Nodes	Dislocation Density	Displacement
1	-0.86354	0
0.99547	-0.85754	7.79690E-03
0.95949	-0.80957	6.80392E-02
0.88883	-0.71408	0.17748
0.78605	-0.57302	0.31561
0.65486	-0.39325	0.45579
0.50000	-0.19522	0.57087
0.32707	-2.80234E-02	0.64196
0.14231	+2.83568E-02	0.66229
4.75821E-02	+5.65705E-03	0.62860
-4.75821E-02	-5.65705E-03	0.62860

a UNIVAC 1100 using FORTRAN (ASC II) level 9RI. The computational time is in standard units of central processing time (in thousands of units) computed by the Fortran library subroutine SUPTIM.

8. NUMERICAL RESULTS FOR THE CRUCIFORM CRACK

In this section, the numerical results obtained from solving the cruciform crack problem using the "hybrid" technique are presented. Computing was done on an IBM PC using single precision. The systems of linear equations were solved using Gaussian elimination without pivoting, and without exploiting the symmetry which halves the size of the system. Since the computed solution is required to be odd, the difference between the values indicated the round-off errors of the order $\mp$5E-6. Table 8.1 gives the dislocation density at equally spaced nodes calculated by Nystrom interpolation for product quadrature techniques. The rational function approximation gives a global representation. Table 8.2 contains the values of $\phi(1)$, the numerical constant in the stress intensity factor obtained by pure gaussian quadrature techniques and by the "hybrid" technique proposed in this paper. Finally in Table 8.4, the values of the dislocation density and the normal component of displacements are given. These results were obtained by solving (4.1.1) and (4.3.3) by the hybrid technique of 4. The dislocation density changes sign (cf. Table 8.3) and the normal component of the displacement vector is not a monotone function. From these tables, it is clear that the aforementioned methods converge for this problem, and the difference between the techniques becomes smaller as N gets larger. Since no analytical solution is available, a claim of greater accuracy for the same computational effort is resisted. However, it is reasonable to compare the "hybrid" values for small N with those of other methods for much larger N.

REFERENCES

1. G. A. Baker, *Essentials of Padé Approximants*. Academic Press, New York (1975).
2. W. Barrett, On Best Rational approximantions to certain functions defined as integrals. *J. Inst. Maths Applics.* **13**, 107–116 (1974).
3. L. T. Boiko and L. N. Karpenko, On some numerical methods for the solution of the plane elasticity problem for bodies with cracks by means of singular integral equations. *Int. J. Fracture* **17**, 381–388 (1981).
4. E. W. Cheney *Introduction to Approximation Theory*. McGraw-Hill.
5. M. L. Dow and D. Elliott, The numerical solution of singular integral equations over (−1, 1). *SIAM J. of Numer. Anal.*, 115–134 (1979).
6. D. Elliott, A convergence theorem for singular integral equations. *J. Austral. Math. Soc.* **22**, 539–552 (1981).
7. F. Erdogan, Approximate solutions of systems of singular integral equations. *SIAM J. Appl. Math.*, 1041–1059 (1969).
8. F. Erdogan and G. Gupta, On the numerical solution of singular integral equations. 525–534 (1972).
9. F. Erdogan, G. Gupta and T. S. Cook, Numerical solution of singular integral equations, *Mechanics of Fracture* (edited by G. Sih), **3**, 368–425 (1973).
10. B. G. Gabdulhaev, Approximate solution of singular integral equations by the method of mechanical quadratures. *Soviet Math. Dokl.*, 329–332 (1968).
11. F. G. Gakhov, *Boundary Value Problems*. Addison-Wesley, New York (1966).
12. A. Gerasoulis and R. P. Srivastav, A Method for the Numerical Solution of Singular Integral Equations with a Principal-Valued Integral. *Int. J. Engng. Sci.* **19**, 1293–1298 (1981).

13. A. Gerasoulis, On the use of Piecewise Quadratic Polynomials for the Solution of Singular Integral Equations of Cauchy-type, *Comp. and Math. with Appls.* **8**, 15–22 (1982).
14. A. Gerasoulis and R. P. Srivastav, On the solvability of singular integral equations. *Inter. J. Comp. Math.* 59–75 (1982).
15. N. I. Ioakimidis and P. S. Theocaris, A comparison between the direct and classical methods for the solution of Cauchy-type singular integral equations. *SIAM J. Numer. Anal.* **17**, 115–118 (1980).
16. E. Jen and R. P. Srivastav, Cubic splines and approximate solution of singular integral equations. *Math. of Comp.* **37**, 417–423 (1981).
17. P. Linz, An analysis of a method for solving singular integral equations *BIT*, 17–22 (1977).
18. N. I. Muskhelishvilli, *Singular Integral Equations*. Noordhoff (1953).
19. V. A. Popov, On the connection between rational uniform approximation and polynomial L_p approximation of functions. *Quantitative Analysis*, 267–277 (1980).
20. T. Rivlin, *An Introduction to the Approximation of Functions*. Dover (1969).
21. D. P. Rooke and I. N. Sneddon, The crack energy and the stress intensity factor for a cruciform crack deformed by internal pressure. *Int. J. Engng. Sci.* **7**, 1079–1090 (1969).
22. I. N. Sneddon, *Mechanics of Fracture*, (edited by George C. Sih) Vol. 1. Noordhoff (1973).
23. R. P. Srivastav, Numerical solution of singular integral equations using Gauss-type formulae. *IMA J. Numer. Anal.* **3**, 305–318 (1983).
24. R. P. Srivastav and E. Jen, Lobatto–Chebyshev quadrature and collocation on Chebyshev nodes, *IMA J. Numer. Anal.* **3**, 319–325 (1983).
25. M. P. Stallybrass, A Pressurized Crack in the Form of a Cross. *Q. J. Mech. Appl. Math.* **23**, 35 (1970).
26. C. E. Stewart, On the numerical evaluation of singular integrals of Cauchy-type. *SIAM Rev.* 342–353 (1960).
27. P. S. Theocaris and N. I. Ioakimidis, Numerical integral methods for the solution of singular integral equations. *Q. J. of Applied Math.* **35**, 173–183 (1977).