ADVANCES IN APPLIED MATHEMATICS 11, 109–163 (1990)

On the Theory of Equivalent Operators and Application to the Numerical Solution of Uniformly Elliptic Partial Differential Equations*

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This work is motivated by the preconditioned iterative solution of linear systems that arise from the discretization of uniformly elliptic partial differential equations. Iterative methods with bounds independent of the discretization are possible only if the preconditioning strategy is based upon equivalent operators. The operators $A, B\colon \mathbf{W} \to \mathbf{V}$ are said to be $\mathbf{V}$ norm equivalent if $\|Au\|_{\mathbf{V}}/\|Bu\|_{\mathbf{V}}$ is bounded above and below by positive constants for $u \in \mathbf{D}$, where $\mathbf{D}$ is "sufficiently dense." If A is $\mathbf{V}$ norm equivalent to B, then for certain discretization strategies one can use B to construct a preconditioned iterative scheme for the approximate solution of the problem $Au = f$. The iteration will require an amount of work that is at most a constant times the work required to approximately solve the problem $B\hat{u} = \hat{f}$ to reduce the $\mathbf{V}$ norm of the error by a fixed factor. This paper develops the theory of equivalent operators on Hilbert spaces. Then, the theory is applied to uniformly elliptic operators. Both the strong and weak forms are considered. Finally, finite element and finite difference discretizations are examined.

*This work was performed under the auspices of the U.S. Department of Energy and the Air Force under Grants AFOSR-86-0061 and AFOSR-82-0275 and National Science Foundation Grant DMS-8704169.

0196-8858/90 $7.50

1. Introduction

1.1. *Motivation*

In the development of iterative methods for the solution of large, sparse linear systems arising from the discretization of elliptic boundary-value problems, it has become important to produce work estimates that are asymptotically competitive with those of the multigrid methods. A multigrid convergence theorem yields results of the form

$$\|e^i\| \le (r)^i \|e^0\|, \tag{1.1}$$

where e^0 and e^i denote the initial error and the error after i iterations while $0 \le r < 1$ is a constant *that is independent of the mesh spacing h*. (The norm in (1.1) is generally a discrete form of the energy norm, equivalent in the limit to an H_1 norm.)

Almost all iterative methods, including the multigrid methods (cf. McCormick [27]), can be cast in the framework of a preconditioning followed by iterative improvement. Given the discrete *matrix* problem

$$\mathbf{A}_h u_h = f_h, \tag{1.2}$$

preconditioned polynomial iterative methods (cf. Faber and Manteuffel [15]) can be expressed in the form

$$r_h^i = f_h - \mathbf{A}_h u_h^i, \tag{1.3a}$$

$$s_h^i = \mathbf{C}_h r_h^i, \tag{1.3b}$$

$$u_h^{i+1} = u_h^i + \alpha_i p_h^i, \tag{1.3c}$$

$$p_h^i \in sp\left\{ s_h^0, \ldots, s_h^i \right\}, \tag{1.3d}$$

where $\mathbf{C}_h$ is a preconditioning and h is a discretization parameter.

Given the symmetric positive definite matrix $\mathbf{M}$, we denote

$$\|u_h\|_{\mathbf{M}}^2 = \langle \mathbf{M} u_h, u_h \rangle_{l_2}, \tag{1.4}$$

where $\langle \cdot, \cdot \rangle_{l_2}$ is the l_2 inner product. If $\mathbf{A}_h$ and $\mathbf{B}_h$ are symmetric positive definite (spd), we may define the spectral condition number of $\mathbf{A}_h$ with respect to $\mathbf{B}_h$ as

$$\begin{aligned} c_s = c_s(\mathbf{A}_h, \mathbf{B}_h) &= C_{l_2}\left(\mathbf{B}_h^{-1/2} \mathbf{A}_h \mathbf{B}_h^{-1/2}\right) \\ &= \|\mathbf{B}_h^{-1/2} \mathbf{A}_h \mathbf{B}_h^{-1/2}\|_{l_2} \|\mathbf{B}_h^{1/2} \mathbf{A}_h^{-1} \mathbf{B}_h^{1/2}\|_{l_2}. \end{aligned} \tag{1.5a}$$

In general, we define the left and right l_2 norm condition numbers of $\mathbf{A}_h$

with respect to $\mathbf{B}_h$ as

$$c_l = c_l(\mathbf{A}_h, \mathbf{B}_h) = C_{l_2}(\mathbf{B}_h^{-1}\mathbf{A}_h) = \|\mathbf{B}_h^{-1}\mathbf{A}_h\|_{l_2}\|\mathbf{A}_h^{-1}\mathbf{B}_h\|_{l_2}, \quad (1.5b)$$

$$c_r = c_r(\mathbf{A}_h, \mathbf{B}_h) = C_{l_2}(\mathbf{A}_h\mathbf{B}_h^{-1}) = \|\mathbf{A}_h\mathbf{B}_h^{-1}\|_{l_2}\|\mathbf{B}_h\mathbf{A}_h^{-1}\|_{l_2}. \quad (1.5c)$$

Note that each condition number is symmetric in $\mathbf{A}_h$ and $\mathbf{B}_h$. If $\mathbf{A}_h$ and $\mathbf{B}_h$ are symmetric positive definite, then we may set $\mathbf{C}_h = \mathbf{B}_h^{-1}$ in (1.3b) and use either a conjugate gradient iteration or a Chebychev iteration to yield estimates of the form

$$\|e_h^i\|_{\mathbf{M}} \le 2\left(\frac{c_s^{1/2} - 1}{c_s^{1/2} + 1}\right)^i \|e_h^0\|_{\mathbf{M}}, \quad (1.6)$$

where $\mathbf{M} = \mathbf{A}_h$ for the conjugate gradient iteration and $\mathbf{M} = \mathbf{B}_h$ for the Chebychev iteration.

For general $\mathbf{A}_h$ and $\mathbf{B}_h$, we may proceed in two ways (cf. Ashby, Manteuffel, and Saylor [1]). Setting $\mathbf{C}_h = \mathbf{A}_h^*\mathbf{B}_h^{-*}\mathbf{B}_h^{-1}$ in (1.3b) corresponds to solving the system

$$\mathbf{A}_h^*\mathbf{B}_h^{-*}\mathbf{B}_h^{-1}\mathbf{A}_h u_h = \mathbf{A}_h^*\mathbf{B}_h^{-*}\mathbf{B}_h^{-1}f \quad (1.7)$$

and yields bounds

$$\|e_h^i\|_{\mathbf{M}} \le 2\left(\frac{c_l - 1}{c_l + 1}\right)^i \|e_h^0\|_{\mathbf{M}}, \quad (1.8)$$

where $\mathbf{M} = I$ (or l_2 norm) or $\mathbf{M} = \mathbf{A}_h^*\mathbf{B}_h^{-*}\mathbf{B}_h^{-1}\mathbf{A}_h$, depending upon the implementation.

On the other hand, setting $\mathbf{C}_h = (\mathbf{B}_h^*\mathbf{B}_h)^{-1}\mathbf{A}_h^*$ in (1.3b) corresponds to solving the system

$$(\mathbf{B}_h^*\mathbf{B}_h)^{-1}(\mathbf{A}_h^*\mathbf{A}_h)x = (\mathbf{B}_h^*\mathbf{B}_h)^{-1}\mathbf{A}_h^*f \quad (1.9)$$

and leads to bounds

$$\|e_h^i\|_{\mathbf{M}} \le 2\left(\frac{c_r - 1}{c_r + 1}\right)^i \|e_h^0\|_{\mathbf{M}}, \quad (1.10)$$

where $\mathbf{M} = \mathbf{A}_h^*\mathbf{A}_h$ or $\mathbf{M} = \mathbf{B}_h^*\mathbf{B}_h$ depending upon the implementation.

In each case, we see that if the relevant condition number is bounded independent of h, then an iteration can be constructed that will produce bounds like (1.1).

If $\mathbf{A}_h$ arises from the discretization of a second-order elliptic partial differential equation in two-space dimensions, then generally $C_{l_2}(\mathbf{A}_h) =$

$O(h^{-2})$. Matrix splittings, such as Gauss–Seidel, SOR, and ADI, and preconditionings, such as incomplete LU factorizations, at best give $C_{l_2}(\mathbf{B}_h^{-1}\mathbf{A}_h) = O(h^{-1})$. In these cases, the bound (1.8) can be rearranged to yield

$$\|e_h^i\|_I \leq K(1 - \Lambda h)^i \|e_h^i\|_I, \tag{1.11}$$

where K and Λ are generic constants independent of h (cf. Young [36]).

In the early 1960s, D'Yakanov [12, 13] and Gunn [19, 20] introduced a preconditioned iterative method with $c_s = C_{l_2}(\mathbf{B}_h^{-1/2}\mathbf{A}_h\mathbf{B}_h^{-1/2})$ bounded independent of h. D'Yakanov used the concept of spectrally equivalent operators to motivate this preconditioning. The positive definite, self-adjoint matrices $\mathbf{A}_h$ and $\mathbf{B}_h$, parameterized by the grid indicator h, were said to be spectrally equivalent if there exists $0 < \alpha,\ \beta < \infty$ such that

$$\alpha \leq \frac{\langle \mathbf{A}_h u_h, u_h \rangle}{\langle \mathbf{B}_h u_h, u_h \rangle} \leq \beta \qquad \text{for } every\ u_h, \tag{1.12}$$

independent of h. Clearly,

$$C_{l_2}\left(\mathbf{B}_h^{-1/2}\mathbf{A}_h\mathbf{B}_h^{-1/2}\right) \leq \beta/\alpha, \tag{1.13}$$

which yields estimates of the form (1.6) with c_s independent of h. However, one must solve a system involving $\mathbf{B}_h$ at every step in the iteration.

D'Yakanov went on to show that if $\mathbf{A}_h$ is the centered finite difference approximation to a positive definite, self-adjoint, uniformly elliptic partial differential operator on a rectangle in two-space dimensions, with homogeneous Dirichlet boundary conditions, then $\mathbf{B}_h$ could be constructed as the discrete approximation to the Laplace operator. Since then, several authors [3, 10–14, 19, 20, 28] have extended these results to a variety of special classes of elliptic operators, discretizations, and domains (see below for details).

Gunn [19] originally suggested using ADI to solve the equations involving $\mathbf{B}_h$. D'Yakanov [13] suggested a Chebychev iteration. Others [3, 7–9, 14, 34–35] have suggested choosing $\mathbf{B}_h$ to be the discrete approximation to a separable self-adjoint, elliptic operator and using fast direct methods (cf. Swarztrauber [30, 31]) when the domain is a rectangle.

In each of the above cases the bounds (1.6) hold with c_s independent of h. If Eqs. (1.3b) with $\mathbf{C}_h = \mathbf{B}_h$ can be solved by fast direct methods, these bounds lead to asymptotic work counts of the form

$$O\left(N^m \ln N \log \varepsilon^{-1}\right), \qquad N = 1/h, \tag{1.14}$$

where ε is the relative accuracy of the solution and m is the number of space dimensions. These bounds appear competitive with multigrid methods. In practice, however, one finds these methods perform poorly unless the operator and the preconditioner are very "close." In general, it is seldom useful to precondition a non-self-adjoint operator with the inverse of the self-adjoint part of that operator (cf. van der Vorst [28, 29]).

There is no flaw in the analysis, only a flaw in the conclusions drawn from the analysis. These bounds are only asymptotic. For reasonable values of h, for example, one may find that SOR yields

$$(1 - \Lambda h) < \frac{c_s^{1/2} - 1}{c_s^{1/2} + 1}$$

for any separable $\mathbf{B}_h$. Further, even if small values of h are to be employed, asymptotic estimates ignore the constant multiplier. Methods with similar asymptotic work estimates may behave quite differently in practice.

In reviewing the development in this area, we are led to several immediate observations. First, as defined by D'Yakanov, equivalence in spectrum must be reproved for each discretization scheme. In this paper we examine the concept in the context of operators on Hilbert spaces. We show that if discrete approximations are equivalent in the sense of D'Yakanov and if they converge pointwise to limit operators, then the limit operators are equivalent with the same bounds. Thus, unless the limits are equivalent, the discrete approximations cannot be. The converse, however, is not true (see the example following Theorem 2.15).

At this point, we would like to call attention to the difference between the matrix and the operator it represents. If A_h is an operator on the finite dimensional Hilbert space $\mathbf{V}_h$, then the basis $\{\phi_i\}$ determines a matrix $\mathbf{A}_h$. If we let $\mathbf{M}_h$,

$$(\mathbf{M}_h)_{ij} = \langle \phi_i, \phi_j \rangle_{\mathbf{V}_h},$$

be the "mass matrix" associated with this basis, then

$$\|A_h\|_{\mathbf{V}_h} = \|\mathbf{M}_h^{1/2} \mathbf{A}_h \mathbf{M}_h^{-1/2}\|_{l_2}.$$

Thus, the condition of A_h in $\mathbf{V}_h$ satisfies the bounds

$$\frac{C_{l_2}(\mathbf{A}_h)}{C_{l_2}(\mathbf{M}_h)} \leq C_{\mathbf{V}_h}(A_h) \leq C_{l_2}(\mathbf{M}_h) C_{l_2}(\mathbf{A}_h). \tag{1.15}$$

The condition of the mass matrix will play an important role in relating the continuous operators to their associated matrices.

Second, we observe that equivalence in spectrum is not the appropriate tool to examine non-self-adjoint operators. We introduce the concept of equivalence in norm. Suppose A and B are operators from Hilbert space **W** to Hilbert space **V**. We say that A is equivalent in **V** norm to B on the set $\mathbf{D} \subseteq \mathbf{W}$ and write $A \sim_{\mathbf{V}} B$ on **D**, if there exist $0 < \alpha, \beta < \infty$ such that

$$\alpha \le \frac{\|Au\|_{\mathbf{V}}}{\|Bu\|_{\mathbf{V}}} \le \beta \tag{1.16}$$

for $u \in \mathbf{D}$ such that the ratio is defined. If the set **D** is "sufficiently dense," (see Section 2 for details), then the **V** condition number of AB^{-1} is bounded; that is,

$$C_{\mathbf{V}}(AB^{-1}) = \|AB^{-1}\|_{\mathbf{V}} \|BA^{-1}\|_{\mathbf{V}} \le \beta/\alpha. \tag{1.17}$$

This corresponds to the right condition number (1.5c). If A and B are one-to-one, then we say A^{-1} is **W** norm equivalent to B^{-1} on the set $\mathbf{D} \subseteq \mathbf{V}$, and write $A^{-1} \sim_{\mathbf{W}} B^{-1}$ on **D**, if there exist $0 < \alpha, \beta < \infty$ such that

$$\alpha \le \frac{\|A^{-1}u\|_{\mathbf{W}}}{\|B^{-1}u\|_{\mathbf{W}}} \le \beta \tag{1.18}$$

for $u \in \mathbf{D}$ such that the ratio is defined. Again, if **D** is "sufficiently dense," the **W** condition number of $B^{-1}A$ is bounded; that is,

$$C_{\mathbf{W}}(B^{-1}A) = \|B^{-1}A\|_{\mathbf{W}} \|A^{-1}B\|_{\mathbf{W}} \le \beta/\alpha. \tag{1.19}$$

This corresponds to the left condition number (1.5b). In Section 2, we show that (1.17) is essentially necessary to achieve bounds of type (1.10) independent of h, and (1.19) is essentially necessary to achieve bounds of type (1.8) independent of h. However, in general, (1.17) and (1.19) do not hold simultaneously. We also show that for positive self-adjoint, compact operators norm equivalence implies spectral equivalence. For these reasons, it is important to study the concept of norm equivalence in the Hilbert space setting.

Finally, we observe that the property of equivalence is transitive, reflexive, and symmetric. In the proper context one can divide elliptic operators (or their inverses) into equivalence classes. In Section 3 we show that boundary conditions determine these classes. These classes are very large. For any operator, it is possible to find an equivalent operator for which the condition (1.17) is arbitrarily large (condition (1.19) for inverse operators). The observations above lead to the conclusions:

(i) For a fixed h, using a preconditioning strategy based upon an equivalent operator may not be superior to classical methods.

(ii) If bounds of type (1.6), (1.8), or (1.10) are desired, then it is necessary to use a preconditioning strategy based upon operators that are equivalent in the infinite-dimensional space.

(iii) Equivalence alone is not sufficient for a good preconditioning strategy. One must also choose an equivalent operator for which the bound (1.17) (or (1.19) for inverse operators) is small.

The above observations indicate that a more precise measure of the "closeness" of two operators is required to evaluate preconditioning strategies. In Section 4, we suggest a semi-metric on a set of equivalent operators. Unfortunately, this metric is difficult to evaluate. Our hope is that this work will provide a sound mathematical framework for the study of preconditioned iterative methods.

1.2. *Summary of Results*

In Section 2, we study the concept of equivalence of operators on Hilbert spaces. Let $A, B: \mathbf{W} \to \mathbf{V}$. We say $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$ if (1.16) is satisfied. Of course, there is always some $\mathbf{D}$ for which (1.16) holds. We say $\mathbf{D}$ is sufficiently dense if $\mathbf{D}$ is dense in both the domain of A, $\mathbf{D}_A$, and the domain of B, $\mathbf{D}_B$, and $A\mathbf{D}$ is dense in the range of A, $\mathbf{R}_A$, and $B\mathbf{D}$ is dense in the range of B, $\mathbf{R}_B$. If $\mathbf{D}$ is sufficiently dense, we write $A \dot{\sim}_{\mathbf{V}} B$. Theorems 2.2 and 2.3 show that if either A and B are bounded or A^{-1} and B^{-1} are bounded, then $A \dot{\sim}_{\mathbf{V}} B$ if and only if $\mathbf{D}_A = \mathbf{D}_B$ and $A \sim_{\mathbf{V}} B$ on $\mathbf{D}_A = \mathbf{D}_B$.

Next we examine inverses and adjoints. Examples are provided to show that $A \dot{\sim}_{\mathbf{V}} B$ does not imply $A^{-1} \sim_{\mathbf{W}} B^{-1}$ or $A^* \sim_{\mathbf{W}} B^*$ on any reasonable set $\mathbf{D}$, even for compact operators. Theorem 2.5 shows, however, that $A \dot{\sim}_{\mathbf{V}} B$ implies $(A^{-1})^* \sim_{\mathbf{V}} (B^{-1})^*$ on the intersection of their domains. This result is used to establish Theorem 3.3.

Spectral equivalence of positive, self-adjoint operators is introduced. We say the positive, self-adjoint operators A, B: $\mathbf{H} \to \mathbf{H}$ are spectrally equivalent on $\mathbf{D}$, and write $A \approx_{\mathbf{H}} B$ on $\mathbf{D}$, if there exist $0 < \alpha, B < \infty$ such that

$$\alpha \leq \frac{\langle Ax, x\rangle_{\mathbf{H}}}{\langle Bx, x\rangle_{\mathbf{H}}} \leq \beta, \qquad \forall x \in \mathbf{D}, \langle Ax, x\rangle_{\mathbf{H}}\langle Bx, x\rangle_{\mathbf{H}} \neq 0. \quad (1.20)$$

Theorem 2.9 shows that if $A \approx_{\mathbf{H}} B$ on a "sufficiently dense" set $\mathbf{D}$, then $A^{-1} \approx_{\mathbf{H}} B^{-1}$ on the intersection of their domains. A major result of Section 2 is Theorem 2.10, which states that if A and B are self-adjoint, positive, one-to-one, and compact on $\mathbf{H}$ and $A \dot{\sim}_{\mathbf{H}} B$ on $\mathbf{H}$, then $A \approx_{\mathbf{H}} B$ on $\mathbf{H}$. Corollary 2.11 yields a similar result on the inverses. The converse is not true and an example is provided. Examples of differential operators that are spectrally equivalent but not norm equivalent are presented in [26]. Equivalence in norm is a stronger condition than spectral equivalence. Moreover, it is essential for constructing bounds of the type (1.8) or (1.10) for elliptic operators. With equivalence in spectrum, we obtain the bound (1.6), which yields bounds for $\|e_h\|_{\mathbb{M}}$ with $\mathbb{M} = \mathbb{A}_h$ or $\mathbb{M} = \mathbb{B}_h$. For the standard choices of bases, this is equivalent to the discrete analogue of the $\mathbf{H}_1$ norm. However, with equivalence in $\mathbf{L}_2$ norm, one may obtain bounds with $\mathbb{M} = I_h$ (the l_2 norm), which is the discrete analogue to the $\mathbf{L}_2$ norm, or with $\mathbb{M} = \mathbb{A}_h^* \mathbb{A}_h$ or $\mathbb{M} = \mathbb{B}_h^* \mathbb{B}_h$, which is the discrete analog to the $\mathbf{H}_2$ norm.

Next, we introduce the concept of uniformly $\mathbf{V}$ norm equivalent families of operators. We say $\{A_n\} \sim_{\mathbf{V}} \{B_n\}$ on $\mathbf{D}$ if $A_n \sim_{\mathbf{V}} B_n$ on $\mathbf{D}$ for every n, and the bounds do not depend upon n. Uniformly spectrally equivalent families are defined similarly. Theorem 2.12 shows that if $\{A_n\} \sim_{\mathbf{V}} \{B_n\}$ on $\mathbf{D}$ and $A_n \to A$, $B_n \to B$ pointwise on $\mathbf{D}$ then $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$. Thus, if B_n is a uniformly $\mathbf{V}$ norm equivalent preconditioning strategy for A_n and if B_n converges pointwise to some operator, B, then B must be $\mathbf{V}$ norm equivalent to A. A similar result holds for spectral equivalence.

Finally, we examine sequence of operators that are derived by the restricting and projecting of operators. For example, if P_n, Q_n are projections onto $\mathbf{W}_n$, $\mathbf{V}_n$, respectively, we consider $A_n = Q_n A P_n$. The finite element method can be viewed in this context. For positive definite, self-adjoint operators, we let $Q_n = P_n^*$ and show that $A \approx_{\mathbf{H}} B$ on $\mathbf{D}$ implies $\{A_n\} \approx_{\mathbf{H}} \{B_n\}$ on $\mathbf{D}$. We also show that if A_n is a weak formulation the same result holds. For non-self-adjoint operators, we add the hypotheses $A_n \sim_{\mathbf{V}} A$, $B_n \sim_{\mathbf{V}} B$ on $\mathbf{W}_n$. Then if $A \sim_{\mathbf{V}} B$, we have $\{A_n\} \sim_{\mathbf{V}} \{B_n\}$. These additional hypotheses can be thought of as a restriction on the angle between $\mathbf{V}_n$ and $A\mathbf{W}_n$. These results give uniform equivalence for finite element approximations. We remark that when using elements in $\mathbf{H}_m$ but not in $\mathbf{H}_{2m}$, one must prove results on the weak form of the operator. Equivalence is obtained in the corresponding norms. We will say more about this in Section 3.

In Section 3, we examine the equivalence of linear uniformly elliptic operators on bounded regions and the uniform equivalence of both finite element and finite difference approximations. In the finite element formulation, we consider the general case of elliptic operators of order $2m$. In the finite difference case, we restrict our discussion to the case of second-order operators. Equivalence follows from regularity bounds like those developed in [16, 25, 28]. For example (see Section 3 for a complete presentation), let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain and let A be a linear uniformly elliptic second-order differential operator on Ω. Under appropriate smoothness hypothesis on the coefficients of A and the domain Ω and appropriate hypothesis on the boundary conditions, one knows that

$$A^{-1}\colon \mathbf{L}_2 \to \mathbf{H}_2 \tag{1.21}$$

is bounded and one-to-one. Thus, there exists a $K_1 = K_1(A)$ such that

$$\|u\|_{\mathbf{H}_2} \le K_1 \|Au\|_{\mathbf{L}_2}. \tag{1.22}$$

From the definition of A, there exists a $K_2 = K_2(A)$ such that

$$\|Au\|_{\mathbf{L}_2} \le K_2 \|u\|_{\mathbf{H}_2}. \tag{1.23}$$

Theorem 3.1 and Corollary 3.2 show that in the presence of regularity bounds like (1.22) and (1.23)

(i) $A^{-1} \sim_{\mathbf{H}_{2m}} B^{-1}$ on $\mathbf{L}_2(\Omega)$,
(ii) $A \dot{\sim}_{\mathbf{L}_2} B$ if and only if $\mathbf{D}_A = \mathbf{D}_B$.

Theorem 3.3 shows that

(iii) $A^{-1} \sim_{\mathbf{L}_2} B^{-1}$ if and only if $\mathbf{D}_{A^*} = \mathbf{D}_{B^*}$.

Theorem 3.3 implies that in order to establish a bound of type (1.10), it is necessary to base the preconditioning strategy on an operator B whose $\mathbf{L}_2$ adjoint B^* has the same boundary conditions as A^*.

These results depend upon the $\mathbf{H}_2$ ($\mathbf{H}_{2m}$ in the general case) regularity assumptions (1.22) and (1.23). Examples of boundary conditions for which (1.22) does not hold are given in Grisvard [17]. That does not imply that equivalence is lost, but rather that our proof is inadequate in that case.

Next, the weak form of the operator is examined. Here, equivalence follows from coercivity bounds like those in Babuska and Aziz [2]. For example, let $\Omega = R^2$ and suppose

$$\langle Au, v\rangle_{\mathbf{L}_2} = a(u,v) - \gamma_a(u,v)$$

for $u \in \mathbf{D}_A$, $v \in \mathbf{T}_A$, where $\mathbf{T}_A$ is dense in $\mathbf{L}_2(\Omega)$. Let $\mathbf{D}_a = \overline{\mathbf{D}}_A^{\mathbf{H}_m}$, $\mathbf{T}_a = \overline{\mathbf{T}}_A^{\mathbf{H}_m}$ ($\overline{\mathbf{D}}_A^{\mathbf{H}_m}$ represents closure in the $\mathbf{H}_m$ norm) and assume there exist constants $M_1(a)$, $M_2(a)$ such that

$$|a(u,v) - \gamma_a(u,v)| \le M_2(a)\|u\|_{\mathbf{H}_m}\|v\|_{\mathbf{H}_m} \tag{1.24}$$

for $u \in \mathbf{D}_a$, $v \in \mathbf{T}_a$;

$$\sup_{v \in \mathbf{T}_a} \frac{|a(u,v) - \gamma_a(u,v)|}{\|v\|_{\mathbf{H}_m}} \ge \frac{1}{M_1(a)}\|u\|_{\mathbf{H}_m} \tag{1.25}$$

for $u \in \mathbf{D}_a$; and

$$\sup_{u \in \mathbf{D}_a} |a(u,v) - \gamma_a(u,v)| > 0 \tag{1.26}$$

for $v \in \mathbf{T}_a$. Then we may consider

$$a(u) = a(u,\cdot) - \gamma_a(u,\cdot) \tag{1.27}$$

to be a map from $\mathbf{D}_a$ to the set of bounded linear functionals on $\mathbf{T}_a$. We write

$$a : \mathbf{D}_a \to \mathbf{T}_a^* \tag{1.28}$$

and note that (1.24), (1.25), and (1.26) imply that a is bounded, one-to-one, and onto with bounded inverse. In the presence of bounds (1.24), (1.25), and (1.26), Theorem 3.5 yields

(iv) $a^{-1} \dot{\sim}_{\mathbf{H}_m} b^{-1}$ if $\mathbf{T}_a = \mathbf{T}_b$.

It is often the case in practice that $\mathbf{T}_A = \mathbf{D}_{A^*}$ and $\mathbf{T}_a = \mathbf{D}_a$. With these extra hypotheses, Corollary 3.7 yields

(v) $a^{-1} \approx_{\mathbf{L}_2} b^{-1}$ on $\mathbf{L}_2$ if $\mathbf{D}_a = \mathbf{D}_b$.

The converse of these results is proven in [26] for $m = 1$ and $\Omega \subseteq R^2$. Notice that these results do not depend upon $\mathbf{H}_{2m}$ regularity but rather the weaker $\mathbf{H}_m$ regularity.

Next, we treat finite element approximations. The results of Section 2 show that the finite element approximation using sufficiently smooth elements yields uniform equivalent families in the same norms as the continuous operator. Of primary interest, however, is the "stiffness" matrix $\mathbf{A}_h$, where

$$(\mathbf{A}_h)_{ij} = \langle \phi_i, A\phi_j \rangle \tag{1.29}$$

and $\{\phi_i\}_{i=1}^n$ are the finite element basis elements. If the moment matrix $\mathbf{M}_h$, where

$$(\mathbf{M}_h)_{ij} = \langle \phi_i, \phi_j \rangle, \tag{1.30}$$

has bounded l_2 condition, then $\{A_h^{-1}\} \sim_{\mathbf{L}_2} \{B_h^{-1}\}$ yields $C_{l_2}(\mathbf{B}_h^{-1}\mathbf{A}_h) < K$ independent of h.

The weak formulation presents some difficulties. Theorem 3.11 yields $\{A_h^{-1}\} \sim_{\mathbf{H}_m} \{B_h^{-1}\}$ on $\mathbf{L}_2(\Omega)$. However, uniform l_2 equivalence of the matrices does not follow. In general, the condition of the mass matrix (1.30) is not bounded in any discrete equivalent to the $\mathbf{H}_m$ norm. In the case of second-order problems Bramble and Pasciak [6] add the hypothesis of optimal-order convergence and an inverse bound to show $\{A_h^{-1}\} \sim_{\mathbf{L}_2} \{B_h^{-1}\}$ on $\mathbf{L}_2$. This argument is developed in Theorem 3.12 for problems at general order. Again, this leads to uniform l_2 equivalence of the corresponding stiffness matrices. However, the work in [6] is flawed by the fact that the authors failed to note that the correct condition for $\{A_h^{-1}\} \sim_{\mathbf{L}_2} \{B_h^{-1}\}$ is $D(A^*) = D(B^*)$ rather than $D(A) = D(B)$. A complete discussion of these matters is found in [26]. We also note that the hypothesis of convergence of order h^{2m} implies that $\mathbf{H}_{2m}$ estimates of the form (1.22) exist.

Finite difference approximations appear to require a case-by-case examination. Estimates of the type (1.12) were obtained by D'Yakanov [12] for the solution of positive definite, self-adjoint, elliptic problems in two dimensions on a rectangle discretized by central differences. He used the inverse of the Laplace operator as a preconditioning coupled with a stationary one-step iteration. In [19], Gunn extended these results to a preconditioning by a partial solution of the Laplace equation obtained by a number of steps of the Peaceman–Rachford ADI iteration. These results are further extended to positive definite, self-adjoint, elliptic operators plus first-order terms in Gunn [20]. There he suggests the use of more sophisticated iterative techniques such as a nonstationary one-step Richardson's iteration and the Chebychev iteration. In [13], D'Yakanov considered positive definite, uniformly elliptic systems on n-dimensional rectangles with Dirichlet boundary conditions. It is shown that the discrete analogue using centered differences is spectrally equivalent to the discrete analogue of a certain positive definite, Helmholz-type operator using centered finite differences and that the bounds are independent of the mesh.

Nitsche and Nitsche [28] obtain the discrete analogue of the bounds (1.22) and (1.23) for linear second-order elliptic operators with possibly mixed derivatives but no first-order terms on rectangles in two dimensions and with Dirichlet boundary conditions using centered finite differences. In [11], Drya establishes similar bounds for uniformly elliptic equations of second order with positive definite symmetric part without mixed derivatives and Dirichlet boundary conditions using centered differences on a convex polyhedron with a nonuniform grid. In the case of two dimensions and a uniform grid, he obtains these bounds in the more general case involving mixed derivatives. Because these bounds are independent of the mesh space, bounds of the type (1.10) hold independently of h.

In Section 3, we show that in the important special case where Ω is a rectangle, the only case where fast direct methods can be conveniently used, and where the boundary conditions are either Dirichlet or Neumann on each edge, three varieties of finite difference discretizations yield uniformly equivalent families of matrices. However, our discussion of the uniform equivalence of $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$ is limited to the case of Dirichlet boundary conditions. We assume only that the second-order elliptic differential operators are uniformly elliptic. This extends the results of [3, 11–14, 20, 28]. It also emphasizes the fact that it is essential to undertake further research to find a B for which $\mathbf{B}_h$ is easily invertible and $C_{l_2}(\mathbf{B}_h^{-1}\mathbf{A}_h)$ will be relatively small.

The results of the first three sections show that equivalence norm is a ubiquitous property. It is clear that a more precise measure of the distance between two equivalent operators must be developed. In Section 4, we present such a measure. This measure has been used by Bank [4] to motivate an algorithm for finding a separable operator that is close to a self-adjoint, elliptic operator on a rectangle. It is hoped that this concept of distance between operators will be helpful in determining appropriate preconditionings in a more general setting.

2. Equivalence of Operators

2.1. *Definitions*

In this section we develop the concept of equivalence of operators. Given two operators A and B from Hilbert space $\mathbf{W}$ to Hilbert space $\mathbf{V}$, we say A is $\mathbf{V}$ norm equivalent to B on set $\mathbf{D} \subseteq \mathbf{D}_A \cap \mathbf{D}_B$, and we write $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$ if there exist $0 < \alpha, \beta < \infty$ such that

$$\alpha \le \frac{\|Ax\|_{\mathbf{V}}}{\|Bx\|_{\mathbf{V}}} \le \beta, \qquad \forall x \in \mathbf{D},\ \|Ax\|_{\mathbf{V}},\ \|Bx\|_{\mathbf{V}} \ne 0. \tag{2.1}$$

Suppose A and B are one-to-one on $\mathbf{D}$. Then we can define

$$\begin{aligned} Q &= AB^{-1} \qquad \text{on } B\mathbf{D}, \\ Q^{-1} &= BA^{-1} \qquad \text{on } A\mathbf{D}. \end{aligned} \tag{2.2}$$

If $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$, then from (2.1), we have for $y \in B\mathbf{D}$, $y \ne 0$,

$$\frac{\|Qy\|_{\mathbf{V}}}{\|y\|_{\mathbf{V}}} = \frac{\|AB^{-1}y\|_{\mathbf{V}}}{\|y\|_{\mathbf{V}}} = \frac{\|Ax\|_{\mathbf{V}}}{\|Bx\|_{\mathbf{V}}} \le \beta \tag{2.3}$$

for some $x \in \mathbf{D}$. Likewise for $y \in A\mathbf{D}$, $y \ne 0$,

$$\frac{\|Q^{-1}y\|_{\mathbf{V}}}{\|y\|_{\mathbf{V}}} \le \frac{1}{\alpha}. \tag{2.4}$$

We see that Q is a bounded invertible operator from $B\mathbf{D}$ to $A\mathbf{D}$.

Thus, Q can be uniquely extended to a bounded invertible operator

$$\hat{Q}:\overline{B\mathbf{D}}^{\mathbf{V}} \to \overline{A\mathbf{D}}^{\mathbf{V}}. \tag{2.5}$$

(The notation $\overline{B\mathbf{D}}^{\mathbf{V}}$ denotes closure in $\mathbf{V}$.) Since we are interested in solving problems of the form $Ax = f$, we would like $\mathbf{D}$ to be large enough so that $A\mathbf{D}$ is dense in $\mathbf{R}_A$, the range of A. This motivates the following definition. We say that A is $\mathbf{V}$ norm equivalent to B and write $A \dot{\sim}_{\mathbf{V}} B$ if $A \sim_{\mathbf{V}} B$ on $\mathbf{D} \subseteq \mathbf{D}_A \cap \mathbf{D}_B$ such that

(i) $$\mathbf{D}_A \subseteq \overline{\mathbf{D}}^{\mathbf{W}}, \qquad \mathbf{D}_B \subseteq \overline{\mathbf{D}}^{\mathbf{W}} \tag{2.6a}$$

and

(ii) $$\mathbf{R}_A \subseteq \overline{A\mathbf{D}}^{\mathbf{V}}, \qquad \mathbf{R}_B \subseteq \overline{B\mathbf{D}}^{\mathbf{V}}. \tag{2.6b}$$

Let us examine the second criteria. Suppose A and B are one-to-one and (2.6b) holds; then, Q can be extended to a bounded invertible operator

$$\hat{Q}:\overline{\mathbf{R}}_B^V \to \overline{\mathbf{R}}_A^V \tag{2.7}$$

and the $\mathbf{V}$ condition of Q is given by

$$C_{\mathbf{V}}(Q) = \|Q\|_{\mathbf{V}}\|Q^{-1}\|_{\mathbf{V}} \leq \beta/\alpha. \tag{2.8}$$

In this case, we can also extend $\mathbf{D}$ to $\hat{\mathbf{D}} = \mathbf{D}_A \subseteq \mathbf{D}_B$.

LEMMA 2.1. *Suppose A and B are one-to-one on $\mathbf{D}_A$ and $\mathbf{D}_B$, respectively, and $A \sim_{\mathbf{V}} B$ on $\mathbf{D} \subset \mathbf{D}_A \cap \mathbf{D}_B$ such that* (2.6b) *holds. Then $A \sim_{\mathbf{V}} B$ on $\hat{\mathbf{D}} = \mathbf{D}_A \cap \mathbf{D}_B$.*

Proof. By the above discussion, there exists a bounded invertible Q on $\mathbf{V}$ such that $Q = AB^{-1}$ on $B\mathbf{D}$. Since the extension of a bounded operator to the closure of its domain is unique, $Q = AB^{-1}$ on $B\hat{\mathbf{D}}$. Thus, the estimate (2.3) holds for every $x \in \hat{\mathbf{D}}$. The lower bound in (2.1) is achieved in a similar manner. □

The next result shows that if A^{-1} and B^{-1} are bounded and $A \dot{\sim}_{\mathbf{V}} B$, then $\mathbf{D}_A = \mathbf{D}_B$.

THEOREM 2.2. *Let $A, B : \mathbf{W} \to \mathbf{V}$ be one-to-one and assume that $\|A^{-1}\|_{\mathbf{W}}$, $\|B^{-1}\|_{\mathbf{W}} < \infty$ and that $\mathbf{R}_A$ and $\mathbf{R}_B$ are closed in V. Then $A \dot{\sim}_{\mathbf{V}} B$ if and only if $A \sim_{\mathbf{V}} B$ on $\mathbf{D} = \mathbf{D}_A = \mathbf{D}_B$.*

Proof. If $A \sim_{\mathbf{V}} B$ on $\mathbf{D} = \mathbf{D}_A = \mathbf{D}_B$, then clearly $A \dot{\sim}_{\mathbf{V}} B$. Now assume $A \dot{\sim}_{\mathbf{V}} B$. By Lemma 2.1, Eq. (2.1) holds for $\mathbf{D} = \mathbf{D}_A \cap \mathbf{D}_B$ and $\mathbf{R}_A = \overline{A\mathbf{D}}^{\mathbf{V}}$,

$\mathbf{R}_B = \overline{B\mathbf{D}}^{\mathbf{V}}$. Let $Q = AB^{-1}$ and let the bounded invertible

$$\hat{Q}: \mathbf{R}_B \to \mathbf{R}_A$$

be the unique extension.

Now let $y \in \mathbf{D}_B$, $By = g \in \mathbf{R}_B$. We will now show that $y \in \mathbf{D}_A$. Since $B\mathbf{D}$ is dense in $\mathbf{R}_B$, there exists $\{g_n\} \subseteq B\mathbf{D}$ such that

$$\lim_{n \to \infty} \|g_n - g\|_{\mathbf{V}} = 0.$$

If $y_n = B^{-1}g_n \in \mathbf{D}$, then

$$\|y_n - y\|_{\mathbf{W}} = \|B^{-1}(g_n - g)\|_{\mathbf{W}} \leq \|B^{-1}\|_{\mathbf{W}} \|g_n - g\|_{\mathbf{V}}.$$

Thus,

$$\lim_{n \to \infty} \|y_n - y\|_{\mathbf{W}} = 0.$$

Now let $\hat{Q}g = f \in \mathbf{R}_A$ and $Qg_n = AB^{-1}g_n = f_n \in A\mathbf{D}$. We have

$$\|f_n - f\|_{\mathbf{V}} \leq \|\hat{Q}\|_{\mathbf{V}} \|g_n - g\|_{\mathbf{V}}.$$

Finally, let $A^{-1}f = x \in \mathbf{D}_A$, $A^{-1}f_n = x_n \in \mathbf{D}$. This yields

$$\|x_n - x\|_{\mathbf{W}} \leq \|A^{-1}\|_{\mathbf{W}} \|f_n - f\|_{\mathbf{V}}$$

and

$$\lim_{n \to \infty} \|x_n - x\|_{\mathbf{W}} = 0.$$

Notice that $x_n = A^{-1}(AB^{-1})By_n$; that is, $x_n = y_n$. Thus, $x = y \in \mathbf{D}_A$.

We have shown that $\mathbf{D}_B \subseteq \mathbf{D}_A$. Reversing the roles of A and B yields $\mathbf{D}_A \subseteq \mathbf{D}_B$. Finally, Lemma 2.1 yields the result. □

We remark that although Condition (2.6a) is implied by the result $\mathbf{D}_A = \mathbf{D}_B$, it was never used in the proof of Theorem 2.2. The result still holds if (2.6a) is removed from the definition. However, (2.6a) is essential to prove the following result, which shows that if A and B are bounded and $A \dot{\sim}_{\mathbf{V}} B$, then $\mathbf{D}_A = \mathbf{D}_B$.

THEOREM 2.3. *Let $A, B: \mathbf{W} \to \mathbf{V}$ be one-to-one and assume $\|A\|_{\mathbf{V}}, \|B\|_{\mathbf{V}} < \infty$ and that $\mathbf{D}_A$ and $\mathbf{D}_B$ are closed in $\mathbf{W}$. Then $A \dot{\sim}_{\mathbf{V}} B$ if and only if $A \sim_{\mathbf{V}} B$ on $\mathbf{D} = \mathbf{D}_A = \mathbf{D}_B$.*

Proof. If $A \sim_{\mathbf{V}} B$ on $\mathbf{D} = \mathbf{D}_A = \mathbf{D}_B$, then clearly $A \dot{\sim}_{\mathbf{V}} B$. Now assume $A \dot{\sim}_{\mathbf{V}} B$. Since $\mathbf{D}_A$ and $\mathbf{D}_B$ are closed, $\mathbf{D}_A \cap \mathbf{D}_B$ is also closed. Now,

$\mathbf{D} \subseteq \mathbf{D}_A \cap \mathbf{D}_B$. By (2.6a)

$$\mathbf{D}_A \subseteq \overline{\mathbf{D}}^{\mathbf{W}} \subseteq \overline{\mathbf{D}_A \cap \mathbf{D}_B}^{\mathbf{W}} = \mathbf{D}_A \cap \mathbf{D}_B.$$

Likewise $\mathbf{D}_B \subseteq \mathbf{D}_A \cap \mathbf{D}_B$. Thus, $\mathbf{D}_A = \mathbf{D}_B$. Finally, since A and B are bounded, the inequalities in (2.1) can be extended to $\overline{\mathbf{D}}^{\mathbf{W}}$. □

In Theorem 2.3, it is the condition (2.6b) that is implied by the result $\mathbf{D}_A = \mathbf{D}_B$ but never required in the proof.

The next result is rather peculiar by itself but is fundamental to Theorem 3.3.

LEMMA 2.4. *Let $A, B : \mathbf{W} \to \mathbf{V}$ be one-to-one and onto. Let $\overline{\mathbf{D}_A \cap \mathbf{D}_B}^{\mathbf{W}} = \mathbf{W}$ and $Ax = Bx$ for $x \in \mathbf{D}_A \cap \mathbf{D}_B$. Then $A^{-1} \sim_{\mathbf{W}} B^{-1}$ on $\mathbf{V}$ if and only if $\mathbf{D}_A = \mathbf{D}_B$ $(A = B)$.*

Proof. It is easy to show that $A^{-1} \sim_{\mathbf{W}} B^{-1}$ on $\mathbf{V}$ if and only if $A^{-1}B : \mathbf{D}_B \to \mathbf{D}_A$ is bounded with bounded inverse. If $\mathbf{D}_A = \mathbf{D}_B$, then $A^{-1}B : \mathbf{D}_B \to \mathbf{D}_A$ is the identity map. Now suppose $\mathbf{D}_A \neq \mathbf{D}_B$. Let $v \in \mathbf{D}_B$, $v \notin \mathbf{D}_A$ and let $z = A^{-1}Bv$. Since $\mathbf{D}_A \cap \mathbf{D}_B$ is dense in $\mathbf{W}$, there exists a sequence $v_n \in \mathbf{D}_A \cap \mathbf{D}_B$ such that

$$\lim_{n \to \infty} \|v - v_n\|_{\mathbf{W}} = 0.$$

Now $A^{-1}Bv_n = v_n$ and so

$$\lim_{n \to \infty} \|v - A^{-1}Bv_n\|_{\mathbf{W}} = 0.$$

Now consider

$$\lim_{n \to \infty} \|A^{-1}B(v - v_n)\|_{\mathbf{W}} = \|z - v\|_{\mathbf{W}} \neq 0.$$

Thus, $A^{-1}B$ is not bounded on the sequence

$$q_n = \frac{v - v_n}{\|v - v_n\|_{\mathbf{W}}} \in \mathbf{D}_B.$$

This completes the proof. □

2.2. *Equivalence of Adjoints and Inverses*

The equivalence of A and B does not guarantee the equivalence of adjoints or inverses as the following example will demonstrate. Here, the adjoint of A: $\mathbf{W} \to \mathbf{V}$ is the unique operator A^*: $\mathbf{V} \to \mathbf{W}$ such that

$$\langle Ax, g \rangle_{\mathbf{V}} = \langle x, A^*g \rangle_{\mathbf{W}} \tag{2.9}$$

for every $x \in \mathbf{D}_A$, $g \in \mathbf{D}_{A^*}$. We remark that A^* and B^* are well defined only if $\overline{\mathbf{D}}_A^{\mathbf{W}} = \overline{\mathbf{D}}_B^{\mathbf{W}} = \mathbf{W}$ (cf. Helmberg [21, p. 117]).

The following is an example in which A and B are one-to-one and compact on $\mathbf{W} = \mathbf{V} = \mathbf{H}$, $A \sim_{\mathbf{H}} B$ on $\mathbf{D} = \mathbf{D}_A = \mathbf{D}_B = \mathbf{H}$, but $A^* \nsim_{\mathbf{H}} B^*$ on $\mathbf{D}_{A^*} \cap \mathbf{D}_{B^*} = \mathbf{H}$, and $A^{-1} \nsim_{\mathbf{H}} B^{-1}$ on $\mathbf{D}_{A^{-1}} \cap \mathbf{D}_{B^{-1}}$. Let A and B have singular value decomposition in an orthonormal basis given by the infinite matrices

$$A = U\Sigma V^*, \tag{2.10a}$$

$$B = \hat{U}\Sigma V^*, \tag{2.10b}$$

where U, $\hat{U}$, and V are unitary operators, and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_i, \ldots)$. Here we assume that A and B are one-to-one and compact, so we require $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_i > 0$. For every $x \in \mathbf{H}$, $x \neq 0$, we have

$$\frac{\|Ax\|_{\mathbf{H}}}{\|Bx\|_{\mathbf{H}}} = \frac{\|U\Sigma V^* x\|_{\mathbf{H}}}{\|\hat{U}\Sigma V^* x\|_{\mathbf{H}}} = \frac{\|\Sigma V^* x\|_{\mathbf{H}}}{\|\Sigma V^* x\|_{\mathbf{H}}} = 1. \tag{2.11}$$

Thus $A \sim_{\mathbf{H}} B$ on $\mathbf{H}$. Now A^* and B^* are given by

$$A^* = V\Sigma U^*, \tag{2.12a}$$

$$B^* = V\Sigma\hat{U}^*. \tag{2.12b}$$

Suppose, we let $\sigma_i = 2^{-i}$, $U = I_{\mathbf{H}}$ (the identity on $\mathbf{H}$), and $\hat{U}$ be represented by the block diagonal matrix

$$\hat{U} = \begin{pmatrix} P_1 & & & & & & \\ & P_2 & & & & & \\ & & P_4 & & & & \\ & & & P_8 & & & \\ & & & & \ddots & & \\ & & & & & P_{2^j} & \\ & & & & & & \ddots \end{pmatrix}, \tag{2.13}$$

where P_k is the $k \times k$ permutation matrix

$$P_k = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix}.$$

Notice that $\hat{U}^* = \hat{U}$. Let e_i represent the ith basis element. Suppose we write

$$\frac{\|A^*e_i\|_{\mathbf{H}}}{\|B^*e_i\|_{\mathbf{H}}} = \frac{\|\Sigma e_i\|_{\mathbf{H}}}{\|\Sigma\hat{U}^*e_i\|_{\mathbf{H}}} = \gamma_i. \tag{2.14a}$$

For $i = 2^k$, we have

$$\gamma_i = \frac{2^{-(2^k)}}{2^{-(2^{k-1}+1)}} = 2^{-(2^{k-1}-1)}. \tag{2.14b}$$

For $i = 2^k + 1$, we have

$$\gamma_i = \frac{2^{-(2^k+1)}}{2^{-2^{k+1}}} = 2^{(2^k-1)}. \tag{2.14c}$$

We see that (2.14a) is neither bounded above nor below, and thus $\|A^*x\|_{\mathbf{H}}/\|B^*x\|_{\mathbf{H}}$ is neither bounded above nor below for any set containing the basis elements.

Likewise, we have

$$A^{-1} = V\Sigma^{-1}U^*, \tag{2.15a}$$

$$B^{-1} = V\Sigma^{-1}\hat{U}^*. \tag{2.15b}$$

Using the same U and $\hat{U}$ yields

$$\frac{\|A^{-1}e_i\|_{\mathbf{H}}}{\|B^{-1}e_i\|_{\mathbf{H}}} = \frac{\|B^*e_i\|_{\mathbf{H}}}{\|A^*e_i\|_{\mathbf{H}}} = \frac{1}{\gamma_i}, \tag{2.16}$$

and again (2.16) is neither bounded above nor below, and thus $\|A^{-1}x\|_{\mathbf{H}}/\|B^{-1}x\|_{\mathbf{H}}$ is neither bounded above nor below for any set containing the basis.

In this example, however,

$$(A^{-1})^* = U\Sigma^{-1}V^*, \tag{2.17a}$$

$$(B^{-1})^* = \hat{U}\Sigma^{-1}V^*. \tag{2.17b}$$

As in (2.11), we have

$$\frac{\|(A^{-1})^*x\|_{\mathbf{H}}}{\|(B^{-1})^*x\|_{\mathbf{H}}} = \frac{\|U\Sigma^{-1}V^*x\|_{\mathbf{H}}}{\|\hat{U}\Sigma^{-1}V^*x\|_{\mathbf{H}}} = \frac{\|\Sigma^{-1}V^*x\|_{\mathbf{H}}}{\|\Sigma^{-1}V^*x\|_{\mathbf{H}}} = 1 \tag{2.18}$$

for every $x \in \mathbf{D}_{(A^{-1})^*} \cap \mathbf{D}_{(B^{-1})^*}$, $x \neq 0$. Thus $A \sim_{\mathbf{H}} B$ on $\mathbf{H}$ yields $(A^{-1})^* \sim_{\mathbf{H}} (B^{-1})^*$ on $\mathbf{D} = \mathbf{D}_{(A^{-1})^*} \cap \mathbf{D}_{(B^{-1})^*}$. This result extends to a wider class of operators.

THEOREM 2.5. *Let $A, B : \mathbf{W} \to \mathbf{V}$ be one-to-one such that $\overline{\mathbf{D}}_A^{\mathbf{W}} = \overline{\mathbf{D}}_B^{\mathbf{W}} = \mathbf{W}$, $\overline{\mathbf{R}}_A^{\mathbf{V}} = \overline{\mathbf{R}}_B^{\mathbf{V}} = \mathbf{V}$. Suppose $A \stackrel{.}{\sim}_{\mathbf{V}} B$; then, $(A^{-1})^* \sim_{\mathbf{V}} (B^{-1})^*$ on $\mathbf{D} = \mathbf{D}_{(A^{-1})^*} \cap \mathbf{D}_{(B^{-1})^*}$.*

Proof. Since $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$ such that $\overline{A\mathbf{D}}^{\mathbf{V}} = \overline{B\mathbf{D}}^{\mathbf{V}} = \mathbf{V}$, the operator $Q = AB^{-1}$ is bounded and can be extended to $\mathbf{V}$. Likewise $Q^{-1} = BA^{-1}$ is bounded and extendable to $\mathbf{V}$. Thus, Q^* and $(Q^{-1})^*$ are also bounded invertible operators such that

$$\|Q^*\|_{\mathbf{V}} = \|Q\|_{\mathbf{V}} \leq \beta,$$
$$\|(Q^{-1})^*\|_{\mathbf{V}} = \|Q^{-1}\|_{\mathbf{V}} \leq 1/\alpha.$$

Since $\overline{\mathbf{D}}_A^{\mathbf{W}} = \overline{\mathbf{D}}_B^{\mathbf{W}} = \mathbf{W}$, $\overline{\mathbf{R}}_A^{\mathbf{V}} = \overline{\mathbf{R}}_B^{\mathbf{V}} = \mathbf{V}$, we have

$$Q^* = (B^{-1})^* A^* \qquad \text{on } (A^{-1})^*\mathbf{D},$$
$$(Q^{-1})^* = (A^{-1})^* B^* \qquad \text{on } (B^{-1})^*\mathbf{D}.$$

Thus, for $x \in \mathbf{D}$, $x \neq 0$, there exists $y \in (B^{-1})^*\mathbf{D}$, $y \neq 0$, such that

$$\frac{\|(A^{-1})^* x\|_{\mathbf{V}}}{\|(B^{-1})^* x\|_{\mathbf{V}}} = \frac{\|(A^{-1})^* B^* y\|_{\mathbf{V}}}{\|y\|_{\mathbf{V}}} = \frac{\|Q^* y\|_{\mathbf{V}}}{\|y\|_{\mathbf{V}}} \leq \beta.$$

Similarly, we obtain

$$\alpha \leq \frac{\|(A^{-1})^* x\|_{\mathbf{V}}}{\|(B^{-1})^* x\|_{\mathbf{V}}}$$

for $x \in \mathbf{D}$, $x \neq 0$. □

If A and B are self-adjoint, then some of the above difficulties disappear. Recall that if A and B are self-adjoint, then $\mathbf{W} = \mathbf{V} = \mathbf{H}$ and $\overline{\mathbf{D}}_A = \overline{\mathbf{D}}_B = \mathbf{H}$ (cf. Helmberg, [21, p. 117]).

COROLLARY 2.6. *Let A and B be self-adjoint and one-to-one. Suppose $A \stackrel{.}{\sim}_{\mathbf{H}} B$, then $A^{-1} \sim_{\mathbf{H}} B^{-1}$ on $\mathbf{D} = \mathbf{D}_{A^{-1}} \cap \mathbf{D}_{B^{-1}}$.*

Proof. If A and B are self-adjoint and one-to-one, then A^{-1} and B^{-1} are self-adjoint and $\overline{\mathbf{D}}_{A^{-1}} = \overline{\mathbf{D}}_{B^{-1}} = \mathbf{H}$ (cf. Helmberg [21, p. 121]). The result now follows from Theorem 2.5. □

2.3. *Spectral Equivalence*

We now introduce the concept of equivalence in spectrum. The treatment here is similar to that of D'Yakanov [13]. If A and B are positive, self-adjoint operators on $\mathbf{H}$, we say that A is equivalent in spectrum to B

on the set $\mathbf{D} \subseteq \mathbf{H}$ and write $A \approx_{\mathbf{H}} B$ on $\mathbf{D}$, if there exist $0 < \alpha$, $\beta < \infty$ such that

$$\alpha \leq \frac{\langle Ax, x\rangle_{\mathbf{H}}}{\langle Bx, x\rangle_{\mathbf{H}}} \leq \beta, \qquad \forall x \in \mathbf{D}, \langle Ax, x\rangle_{\mathbf{H}}, \langle Bx, x\rangle_{\mathbf{H}} \neq 0. \quad (2.19)$$

Let $A, B: \mathbf{W} \to \mathbf{V}$. Then there is an obvious relationship between $A \sim_{\mathbf{V}} B$ and $A^*A \approx_{\mathbf{W}} B^*B$. We have the following result.

LEMMA 2.7. *Let A and B be such that $\overline{\mathbf{D}}_A^{\mathbf{W}} = \overline{\mathbf{D}}_B^{\mathbf{W}} = \mathbf{W}$. If $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$ and if A^*A and B^*B are self-adjoint, then $A^*A \approx_{\mathbf{W}} B^*B$ on $\hat{\mathbf{D}} = \mathbf{D} \cap \mathbf{D}_{A^*A} \cap \mathbf{D}_{B^*B}$. Conversely, if $A^*A \approx_{\mathbf{W}} B^*B$ on $\mathbf{D}$, then $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$.*

Proof. The proof is obvious once it is noted that $\mathbf{D}_{A^*A} \subseteq \mathbf{D}_A$, $\mathbf{D}_{B^*B} \subseteq \mathbf{D}_B$. We have

$$\frac{\|Ax\|_{\mathbf{V}}^2}{\|Bx\|_{\mathbf{V}}^2} = \frac{\langle Ax, Ax\rangle_{\mathbf{V}}}{\langle Bx, Bx\rangle_{\mathbf{V}}} = \frac{\langle A^*Ax, x\rangle_{\mathbf{W}}}{\langle B^*Bx, x\rangle_{\mathbf{W}}},$$
$$\forall x \in \mathbf{D}_{A^*A} \cap \mathbf{D}_{B^*B}, \|Ax\|_{\mathbf{V}}, \|Bx\|_{\mathbf{V}} \neq 0. \quad (2.20)$$

If $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$ and $x \in \hat{\mathbf{D}} = \mathbf{D} \cap \mathbf{D}_{A^*A} \cap \mathbf{D}_{B^*B}$, then there exist $0 < \alpha$, $\beta < \infty$ such that the bounds

$$\alpha^2 \leq \frac{\langle A^*Ax, x\rangle_{\mathbf{W}}}{\langle B^*Bx, x\rangle_{\mathbf{W}}} \leq \beta^2$$

apply. Conversely, if $A^*A \approx_{\mathbf{W}} B^*B$ on $\mathbf{D}$, then $\mathbf{D} \subseteq \mathbf{D}_A \cap \mathbf{D}_B$, and (2.20) yields $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$. □

Suppose $A: \mathbf{H} \to \mathbf{H}$ is self-adjoint, positive, and bounded. Then $A^{1/2}$ exists and is self-adjoint, positive, and bounded on $\mathbf{H}$. If A is also one-to-one, then $A^{1/2}$ is one-to-one and

$$\mathbf{D}_{A^{-1/2}} = A^{1/2}\mathbf{H}, \qquad \mathbf{D}_{A^{-1}} = A\mathbf{H}$$

and, further,

$$\mathbf{D}_{A^{-1}} \subseteq \mathbf{D}_{A^{-1/2}}. \quad (2.21)$$

This can easily be seen from the relation

$$A^{-1/2} = A^{1/2}A^{-1} \quad (2.22)$$

on $\mathbf{D}_{A^{-1}}$. We use this to build several lemmas.

LEMMA 2.8. *Suppose A and B are self-adjoint, positive, and bounded on $\mathbf{H}$, then $A \approx_{\mathbf{H}} B$ on $\mathbf{D}$ if and only if $A^{1/2} \sim_{\mathbf{H}} B^{1/2}$ on $\mathbf{D}$. Further, if A and B*

are one-to-one, $A^{-1} \approx_{\mathbf{H}} B^{-1}$ *on* $\mathbf{D}$ *implies* $A^{-1/2} \sim_{\mathbf{H}} B^{-1/2}$ *on* $\mathbf{D}$, *and* $A^{-1/2} \sim_{\mathbf{H}} B^{-1/2}$ *on* $\mathbf{D}$ *implies* $A^{-1} \approx_{\mathbf{H}} B^{-1}$ *on* $\mathbf{D} \cap \mathbf{D}_{A^{-1}} \cap \mathbf{D}_{B^{-1}}$.

Proof. The first assertion follows from

$$\frac{\langle Ax, x\rangle_{\mathbf{H}}}{\langle Bx, x\rangle_{\mathbf{H}}} = \frac{\langle A^{1/2}x, A^{1/2}x\rangle_{\mathbf{H}}}{\langle B^{1/2}x, B^{1/2}x\rangle_{\mathbf{H}}}, \qquad \forall x \in \mathbf{D}, \langle Ax, x\rangle_{\mathbf{H}}, \langle Bx, x\rangle_{\mathbf{H}} \neq 0.$$

The second assertion follows similarly once we notice that $\mathbf{D}_{A^{-1}} \subseteq \mathbf{D}_{A^{-1/2}}$ and $\mathbf{D}_{B^{-1}} \subseteq \mathbf{D}_{B^{-1/2}}$. The possibilities of proper inclusion forces the added restriction in the third assertion. □

The next result shows that if A and B are equivalent in spectrum on $\mathbf{H}$, then so are their inverses.

THEOREM 2.9. *Suppose* A *and* B *are self-adjoint, positive, one-to-one, and bounded on* $\mathbf{H}$. *If* $A \approx_{\mathbf{H}} B$ *on* $\mathbf{D}$ *such that* $\overline{A^{1/2}\mathbf{D}} = \overline{B^{1/2}\mathbf{D}} = \mathbf{H}$, *then* $A^{-1} \approx_{\mathbf{H}} B^{-1}$ *on* $\hat{\mathbf{D}} = \mathbf{D}_{A^{-1}} \cap \mathbf{D}_{B^{-1}}$. *Conversely, if* $A^{-1} \approx_{\mathbf{H}} B^{-1}$ *on* $\mathbf{D}$ *such that* $\overline{A^{-1/2}\mathbf{D}} = \overline{B^{-1/2}\mathbf{D}} = \mathbf{H}$, *then* $A \approx_{\mathbf{H}} B$ *on* $\mathbf{H}$.

Proof. Assume that $A \approx_{\mathbf{H}} B$ on $\mathbf{D}$ such that $\overline{A^{1/2}\mathbf{D}} = \overline{B^{1/2}\mathbf{D}} = \mathbf{H}$. Then, by Lemma 2.8, $A^{1/2} \sim_{\mathbf{H}} B^{1/2}$ on $\mathbf{D}$. Since $\overline{A^{1/2}\mathbf{D}} = \overline{B^{1/2}\mathbf{D}} = \mathbf{H}$, Corollary 2.6 yields $A^{-1/2} \sim_{\mathbf{H}} B^{-1/2}$ on $\tilde{\mathbf{D}} = \mathbf{D}_{A^{-1/2}} \cap \mathbf{D}_{B^{-1/2}}$. Finally, another application of Lemma 2.8 yields $A^{-1} \approx_{\mathbf{H}} B^{-1}$ on $\hat{\mathbf{D}} = \tilde{\mathbf{D}} \cap \mathbf{D}_{A^{-1}} \cap \mathbf{D}_{B^{-1}} = \mathbf{D}_{A^{-1}} \cap \mathbf{D}_{B^{-1}}$.

In the other direction, assume $A^{-1} \approx_{\mathbf{H}} B^{-1}$ on $\mathbf{D}$ such that $\overline{A^{-1/2}\mathbf{D}} = \overline{B^{-1/2}\mathbf{D}} = \mathbf{H}$. Lemma 2.8 yields $A^{-1/2} \sim_{\mathbf{H}} B^{-1/2}$ on $\mathbf{D}$. Corollary 2.6 now gives $A^{1/2} \sim_{\mathbf{H}} B^{1/2}$ on $\mathbf{D}_{A^{1/2}} \cap \mathbf{D}_{B^{1/2}} = \mathbf{H}$, and a final application of Lemma 2.8 yields $A \approx_{\mathbf{H}} B$ on $\mathbf{H}$. □

These tools lead to the following important result. Here we require compactness. The result actually applies to a slightly larger class of operators, but that proof is difficult. Moreover, the following will suffice for our needs later in this paper.

THEOREM 2.10. *Let* A *and* B *be self-adjoint, positive, one-to-one, and compact operators on* $\mathbf{H}$. *If* $A \sim_{\mathbf{H}} B$ *on* $\mathbf{H}$, *then* $A \approx_{\mathbf{H}} B$ *on* $\mathbf{H}$.

Proof. By assumption there exist $0 < \alpha, \beta < \infty$ such that

$$\alpha \leq \frac{\|Ax\|_{\mathbf{H}}}{\|Bx\|_{\mathbf{H}}} \leq \beta, \qquad \forall x \in \mathbf{H}, x \neq 0.$$

Then $Q = AB^{-1}$ is bounded on $D_{B^{-1}}$ and

$$\|Q\|_{\mathbf{H}} \leq \beta, \qquad \|Q^{-1}\|_{\mathbf{H}} \leq 1/\alpha.$$

Since $\overline{\mathbf{D}}_{A^{-1}} = \overline{\mathbf{D}}_{B^{-1}} = \mathbf{H}$, Q and Q^{-1} can be extended to all of $\mathbf{H}$.

Since A and B are self-adjoint and compact, they both have a complete set of mutually orthogonal eigenvectors. Assume the eigenvalues are ordered in descending order and let $\mathbf{W}_n$ be the subspace spanned by the first n eigenvectors of B. Let P_n be the orthogonal projection onto $\mathbf{W}_n$ and let

$$B_n = P_n B P_n = P_n B = B P_n.$$

The key to this proof is to notice that if B^+ is the Moore-Penrose pseudo-inverse of B (cf. Stewart [29]), then

$$B_n^{1/2} = P_n B^{1/2} P_n = P_n B^{1/2} = B^{1/2} P_n,$$

$$B_n^{+} = P_n B^{-1} P_n = P_n B^{-1} = B^{-1} P_n,$$

$$\left(B_n^{1/2}\right)^{+} = P_n B^{-1/2} P_n = P_n B^{-1/2} = B^{-1/2} P_n.$$

Let $A_n = P_n A P_n$. Since these are operators of finite rank, we have

$$\|\left(B_n^{1/2}\right)^{+} A_n \left(B_n^{1/2}\right)^{+}\|_{\mathbf{H}} = \rho\left(\left(B_n^{1/2}\right)^{+} A_n \left(B_n^{1/2}\right)^{+}\right) = \rho\left(A_n B_n^{+}\right)$$
$$\leq \|A_n B_n^{+}\|_{\mathbf{H}} = \|P_n A B^{-1} P_n\|_{\mathbf{H}}.$$

(Here $\rho(A)$ is the spectral radius of A.) Finally, we have

$$\|P_n A B^{-1} P_n\|_{\mathbf{H}} \leq \|A B^{-1}\|_{\mathbf{H}} \leq \beta.$$

Thus, $\|(B_n^{1/2})^{+} A_n (B_n^{1/2})^{+}\|_{\mathbf{H}} \leq \beta$ for every n. Let

$$\mathbf{F} = \{ f \in \mathbf{H} : P_n f = f \text{ for some } n \}.$$

Now for any $f \in \mathbf{F}$, $f \neq 0$, there exists $g \in \mathbf{F}$, $g \neq 0$, and an n, such that $B^{1/2} f = B_n^{1/2} f = g$. This yields

$$\frac{\langle Af, f\rangle_{\mathbf{H}}}{\langle Bf, f\rangle_{\mathbf{H}}} = \frac{\langle AB^{-1/2}g, B^{-1/2}g\rangle_{\mathbf{H}}}{\langle g, g\rangle_{\mathbf{H}}} = \frac{\langle B^{-1/2}AB^{-1/2}g, g\rangle_{\mathbf{H}}}{\langle g, g\rangle_{\mathbf{H}}}$$
$$= \frac{\langle B_n^{-1/2} A_n B_n^{-1/2} g, g\rangle_{\mathbf{H}}}{\langle g, g\rangle_{\mathbf{H}}} \leq \beta.$$

Since $\mathbf{F}$ is dense in $\mathbf{H}$, the bound can be extended to all of $\mathbf{H}$. The lower bound is proved in the same fashion. □

Theorem 2.10 immediately leads to a result that is more appropriate for our later needs.

COROLLARY 2.11. *Suppose A^{-1} and B^{-1} are self-adjoint, positive, one-to-one, and compact on $\mathbf{H}$. If $A \stackrel{.}{-}_{\mathbf{H}} B$ then $A \approx_{\mathbf{H}} B$ on $\mathbf{D} = \mathbf{D}_A = \mathbf{D}_B$.*

Proof. Since A^{-1} and B^{-1} are bounded, Theorem 2.2 yields $\mathbf{D}_A = \mathbf{D}_B$. Corollary 2.6 yields $A^{-1} \sim_{\mathbf{H}} B^{-1}$ on **H**. Theorem 2.10 then implies that $A^{-1} \approx_{\mathbf{H}} B^{-1}$ on **H**, and Theorem 2.9 completes the result. □

The converse of Theorem 2.10 is not true as the following example will show. Define the 2×2 matrices

$$A_i = \begin{pmatrix} 1 & \sqrt{\delta_i/2} \\ \sqrt{\delta_i/2} & \delta_i \end{pmatrix}, \qquad B_i = \begin{pmatrix} 1 & 0 \\ 0 & \delta_i/2 \end{pmatrix}. \tag{2.23}$$

The eigenvalues of

$$A_i B_i^{-1} = \begin{pmatrix} 1 & \sqrt{2/\delta_i} \\ \sqrt{\delta_i/2} & 2 \end{pmatrix}$$

are

$$\mu_{1,2}^i = \frac{3 \pm \sqrt{5}}{2}.$$

The eigenvalues of $B_i^{-1} A_i^2 B_i^{-1}$ are

$$\lambda_{1,2}^i = \left(\left(5 + \frac{\delta_i}{2} + \frac{2}{\delta_i}\right) \pm \sqrt{\left(5 + \frac{\delta_i}{2} + \frac{2}{\delta_i}\right)^2 - 4} \right) \Big/ 2.$$

(Notice that $\lambda_{1,2}^i$ are the squares of the singular values of $A_i B_i^{-1}$.) Let A and B be the compact operators on **H** that have infinite matrix representation

$$A = \begin{pmatrix} \delta_1 A_1 & & & & \\ & \delta_2 A_2 & & & \\ & & \ddots & & \\ & & & \delta_i A_i & \\ & & & & \ddots \end{pmatrix},$$

$$B = \begin{pmatrix} \delta_1 B_1 & & & & \\ & \delta_2 B_2 & & & \\ & & \ddots & & \\ & & & \delta_i B_i & \\ & & & & \ddots \end{pmatrix}, \tag{2.24}$$

where $\delta_i > 0$, $\lim_{i\to\infty} \delta_i = 0$. Here δ_i plays a dual role. We multiply each block by δ_i to make A and B compact. This has no effect on AB^{-1}. Also, δ_i influences the condition of each block of AB^{-1}. We see that AB^{-1} is unbounded because

$$\|AB^{-1}\| \geq \|A_i B_i^{-1}\| = \sqrt{\lambda_1^i}, \qquad \forall i.$$

Thus, A is not $\mathbf{H}$ norm equivalent to B (see the discussion surrounding (2.2)). However,

$$\|B^{-1/2}AB^{-1/2}\|_{\mathbf{H}} = \rho(B^{-1/2}AB^{-1/2}) = \rho(AB^{-1}) = \frac{3+\sqrt{5}}{2},$$

$$\|A^{-1/2}BA^{-1/2}\|_{\mathbf{H}} = \rho(A^{-1/2}BA^{-1/2}) = \rho(BA^{-1}) = \frac{2}{3-\sqrt{5}}.$$

If we let

$$\mathbf{F} = \{ f \in \mathbf{H} : f \text{ is a finite linear combination of basis elements}\},$$

then for $f \in \mathbf{F}$, $f \neq 0$, there exists $g \in \mathbf{F}$, $g \neq 0$, such that

$$\frac{\langle Af, f\rangle_{\mathbf{H}}}{\langle Bf, f\rangle_{\mathbf{H}}} = \frac{\langle B^{-1/2}AB^{-1/2}g, g\rangle_{\mathbf{H}}}{\langle g, g\rangle_{\mathbf{H}}} \leq \|B^{-1/2}AB^{-1/2}\|_{\mathbf{H}},$$

$$\frac{\langle Bf, f\rangle_{\mathbf{H}}}{\langle Af, f\rangle_{\mathbf{H}}} = \frac{\langle A^{-1/2}BA^{-1/2}g, g\rangle_{\mathbf{H}}}{\langle g, g\rangle_{\mathbf{H}}} \leq \|A^{-1/2}BA^{-1/2}\|_{\mathbf{H}}.$$

Since $\mathbf{F}$ is dense in $\mathbf{H}$, then $A \approx_{\mathbf{H}} B$ on $\mathbf{H}$.

2.4. *Uniform Equivalence*

The next set of results is intended to shed light upon the equivalence of discrete approximations to continuous operators, even though these results do not require finite rank. Let A_n and B_n be two sequences of operators from $\mathbf{W} \to \mathbf{V}$. We say the families are $\mathbf{V}$ norm uniformly equivalent on $\mathbf{D}$ and write $\{A_n\} \sim_{\mathbf{V}} \{B_n\}$ on $\mathbf{D} \subseteq \mathbf{D}_{A_n} \cap \mathbf{D}_{B_n}$ if there exist $0 < \alpha, \beta < \infty$ such that

$$\alpha \leq \frac{\|A_n x\|_{\mathbf{V}}}{\|B_n x\|_{\mathbf{V}}} \leq \beta; \qquad \forall x \in \mathbf{D},\ \|A_n x\|_{\mathbf{V}}, \|B_n x\|_{\mathbf{V}} \neq 0 \tag{2.25}$$

for every n.

Similarly, we say the families are uniformly V norm equivalent and write $\{A_n\} \dot{\sim}_{\mathbf{V}} \{B_n\}$ if (2.25) holds independent of n and, further,

$$\text{(i)} \qquad \mathbf{D}_{A_n} \subseteq \overline{\mathbf{D}}^{\mathbf{W}}, \qquad \mathbf{D}_{B_n} \subseteq \overline{\mathbf{D}}^{\mathbf{W}}, \tag{2.26a}$$

$$\text{(ii)} \qquad \mathbf{R}_{A_n} \subseteq \overline{A_n \mathbf{D}}^{\mathbf{V}}, \qquad \mathbf{R}_{B_n} \subseteq \overline{B_n \mathbf{D}}^{\mathbf{V}}. \tag{2.26b}$$

Finally, if $\{A_n\}$ and $\{B_n\}$ are positive and self-adjoint on $\mathbf{H}$, we say the families are uniformly equivalent in spectrum on $\mathbf{D}$ and write $\{A_n\} \approx_{\mathbf{H}} \{B_n\}$ on $\mathbf{D}$ if there exist $0 < \alpha, \beta < \infty$ such that

$$\alpha \leq \frac{\langle A_n x, x \rangle_{\mathbf{H}}}{\langle B_n x, x \rangle_{\mathbf{H}}} \leq \beta, \qquad \forall x \in \mathbf{D}, \langle A_n x, x \rangle_{\mathbf{H}}, \langle B_n x, x \rangle_{\mathbf{H}} \neq 0 \tag{2.27}$$

for every n.

If the two sequences both converge pointwise and if they are uniformly equivalent, then their limits are also equivalent.

THEOREM 2.12. *Let A, B: $\mathbf{W} \to \mathbf{V}$ be one-to-one and A_n and B_n be defined on $\hat{\mathbf{D}}_A \subseteq \mathbf{D}_A$ and $\hat{\mathbf{D}}_B \subseteq \mathbf{D}_B$, respectively, such that for every $x \in \mathbf{D} \subseteq \hat{\mathbf{D}}_A \cap \hat{\mathbf{D}}_B$,*

$$\lim_{n \to \infty} \|A_n x - Ax\|_{\mathbf{V}} = \lim_{n \to \infty} \|B_n x - Bx\|_{\mathbf{V}} = 0. \tag{2.28}$$

If $\{A_n\} \sim_{\mathbf{V}} \{B_n\}$ on $\mathbf{D}$, then $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$. Further, if all operators are positive and self-adjoint on $\mathbf{H}$ and if $\{A_n\} \approx_{\mathbf{H}} \{B_n\}$ uniformly on $\mathbf{D}$, then $A \approx_{\mathbf{H}} B$ on $\mathbf{D}$.

Proof. Since A and B are one-to-one, if $x \in \mathbf{D}$, $x \neq 0$, then $\|Ax\|_{\mathbf{V}}, \|Bx\|_{\mathbf{V}} \neq 0$. Given $x \in \mathbf{D}$ and $\varepsilon > 0$, choose N large enough so that

$$\|(A_n - A)x\|_{\mathbf{V}} \leq \varepsilon \|Ax\|_{\mathbf{V}},$$
$$\|(B_n - B)x\|_{\mathbf{V}} \leq \varepsilon \|Bx\|_{\mathbf{V}}$$

for $n \geq N$. Then, we have

$$\|Ax\|_{\mathbf{V}} \leq \frac{1}{1-\varepsilon} \|A_n x\|_{\mathbf{V}},$$
$$\|Bx\|_{\mathbf{V}} \leq \frac{1}{1-\varepsilon} \|B_n x\|_{\mathbf{V}}$$

for $n \geq N$. Since $\{A_n\} \sim_{\mathbf{V}} \{B_n\}$ on $\mathbf{D}$, there exist $0 < \alpha, \beta < \infty$ such that for $x \in \mathbf{D}$,

$$\alpha \le \frac{\|A_n x\|_{\mathbf{V}}}{\|B_n x\|_{\mathbf{V}}} \le \beta$$

for all n. Now for $n > N$,

$$\frac{\|Ax\|_{\mathbf{V}}}{\|Bx\|_{\mathbf{V}}} \le \frac{\|A_n x\|_{\mathbf{V}} + \|(A - A_n)x\|_{\mathbf{V}}}{\|B_n x\|_{\mathbf{V}} - \|(B - B_n)x\|_{\mathbf{V}}} \le \frac{1 + \varepsilon/(1-\varepsilon)}{1 - \varepsilon/(1-\varepsilon)} \frac{\|A_n x\|_{\mathbf{V}}}{\|B_n x\|_{\mathbf{V}}}$$
$$\le \frac{\beta}{1 - 2\varepsilon}.$$

Since this bound holds for all sufficiently small $\varepsilon > 0$, we have

$$\frac{\|Ax\|_{\mathbf{V}}}{\|Bx\|_{\mathbf{V}}} \le \beta.$$

The lower bound is proved similarly, as are the bounds for equivalence in spectrum. □

COROLLARY 2.13. *Assume the hypotheses of Theorem* 2.12. *Suppose, in addition, that* (2.26a), (2.26b) *hold. If* $\{A_n\} \dot{\sim}_{\mathbf{V}} \{B_n\}$, *then* $A \dot{\sim}_{\mathbf{V}} B$.

Proof. The proof follows from Theorem 2.12 and the definition of $A \dot{\sim}_{\mathbf{V}} B$. □

Theorem 2.12 and Corollary 2.13 imply that discrete approximations $\{A_n\}$ and $\{B_n\}$ are uniformly equivalent in norm (in spectrum) only if their limits are equivalent in a norm (in spectrum). In general, while the converse of Theorem 2.12 does not hold, we have the following results that pertain to finite element discretizations of sufficiently high order.

THEOREM 2.14. *Let A and B be one-to-one, positive, and self-adjoint on* $\mathbf{H}$, *and let P_n be a projection operator onto* $\mathbf{W}_n \subseteq \mathbf{D}_A \cap \mathbf{D}_B$. *Let*

$$A_n = P_n^* A P_n,$$
$$B_n = P_n^* B P_n.$$

If $A \approx_{\mathbf{H}} B$ on $\mathbf{D}$, *then* $\{A_n\} \approx_{\mathbf{H}} \{B_n\}$ *on* $\mathbf{D}$.

Proof. Suppose there exist $0 < \alpha, \beta < \infty$ such that

$$\alpha \le \frac{\langle Ax, x\rangle_{\mathbf{H}}}{\langle Bx, x\rangle_{\mathbf{H}}} \le \beta, \qquad \forall x \in \mathbf{D}, \langle Ax, x\rangle, \langle Bx, x\rangle \ne 0.$$

Then if $\langle A_n x, x\rangle_{\mathbf{H}}, \langle B_n x, x\rangle_{\mathbf{H}} \neq 0$, we have

$$\alpha \leq \frac{\langle A_n x, x\rangle_{\mathbf{H}}}{\langle B_n x, x\rangle_{\mathbf{H}}} = \frac{\langle AP_n x, P_n x\rangle_{\mathbf{H}}}{\langle BP_n x, P_n x\rangle_{\mathbf{H}}} \leq \beta. \square$$

The comparable result for equivalence in norm requires an additional hypothesis.

THEOREM 2.15. *Let A, B: $\mathbf{W} \to \mathbf{V}$ and let P_n, Q_n be projections onto subspaces $\mathbf{W}_n$, $\mathbf{V}_n$, respectively, such that $\|Q_n\| \leq \gamma$ and $\mathbf{W}_n \subseteq \mathbf{D}_A \cap \mathbf{D}_B$ for every n. Let*

$$A_n = Q_n A P_n,$$
$$B_n = Q_n B P_n,$$

and suppose there exists $\varepsilon_A, \varepsilon_B > 0$ such that

$$\varepsilon_A \|Ax\|_{\mathbf{V}} \leq \|A_n x\|_{\mathbf{V}}, \tag{2.29a}$$
$$\varepsilon_B \|Bx\|_{\mathbf{V}} \leq \|B_n x\|_{\mathbf{V}} \tag{2.29b}$$

for $x \in \mathbf{W}_n$, for every n. If $A \sim_{\mathbf{V}} B$ on $\mathbf{D} = \mathbf{D}_A \cap \mathbf{D}_B$, then $\{A_n\} \sim_{\mathbf{V}} \{B_n\}$ on $\mathbf{D}$.

Proof. Let $x \in \mathbf{W}_n$ such that $\|A_n x\|_{\mathbf{V}}, \|B_n x\|_{\mathbf{V}} \neq 0$. We have

$$\|A_n x\|_{\mathbf{V}} = \|Q_n A P_n x\|_{\mathbf{V}} \leq \|Q_n\|_{\mathbf{V}} \|A P_n x\|_{\mathbf{V}} = \|Q_n\|_{\mathbf{V}} \|Ax\|_{\mathbf{V}},$$
$$\|B_n x\|_{\mathbf{V}} = \|Q_n B P_n x\|_{\mathbf{V}} \leq \|Q_n\|_{\mathbf{V}} \|B P_n x\|_{\mathbf{V}} = \|Q_n\|_{\mathbf{V}} \|Bx\|_{\mathbf{V}}.$$

Thus,

$$\frac{\|A_n x\|_{\mathbf{V}}}{\|B_n x\|_{\mathbf{V}}} \leq \frac{\gamma}{\varepsilon_B} \frac{\|Ax\|_{\mathbf{V}}}{\|Bx\|_{\mathbf{V}}},$$

which is bounded above. The lower bound is proved in a similar manner. $\square$

Another interpretation of the preceding theorem is to say that $\{A_n\} \sim_{\mathbf{V}} \{B_n\}$ on $\mathbf{D}$ if $A \sim_{\mathbf{V}} B$ on $\mathbf{D}$ and

$$\{A_n\} \sim_{\mathbf{V}} \{AP_n\} \quad \text{on } \mathbf{D},$$
$$\{B_n\} \sim_{\mathbf{V}} \{BP_n\} \quad \text{on } \mathbf{D}.$$

This explains why the hypotheses of Theorem 2.15 require a bound on $\|Q_n\|_{\mathbf{V}}$ but not on $\|P_n\|_{\mathbf{V}}$.

We also remark that without the extra hypothesis (2.29a), (2.29b), uniform norm equivalence is in doubt. The following example illustrates this point. Let

$$A = B = \operatorname{diag}(\sigma_i)$$

be compact on **H**. That is, let $\sigma_1 \geq \sigma_2 \geq \sigma_3 \cdots > 0$ and $\lim_{i \to \infty} \sigma_i = 0$. Let

$$A_n = \operatorname{diag}(\sigma_1, \ldots, \sigma_{n-1}^2, \sigma_n, 0, 0, \ldots),$$
$$B_n = \operatorname{diag}(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2, 0, 0, \ldots).$$

Then

$$\lim_{n \to \infty} \frac{\|A_n e_{n-1}\|_{\mathbf{H}}}{\|B_n e_{n-1}\|_{\mathbf{H}}} = 0, \qquad \lim_{n \to \infty} \frac{\|A_n e_n\|_{\mathbf{H}}}{\|B_n e_n\|_{\mathbf{H}}} = \infty,$$

which shows that even though $A \sim_{\mathbf{H}} B$ on **H**, $\{A_n\} \nsim_{\mathbf{H}} \{B_n\}$ on **H**. Both (2.29a), (2.29b) are violated. Now, let

$$A_n = B_n = \operatorname{diag}(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2, 0, 0, \ldots).$$

Again, both (2.25a), (2.25b) are violated, but $\{A_n\} \sim_{\mathbf{H}} \{B_n\}$ on **H**.

In many finite element discretizations, the elements are of lower order than required by the domain of the operator. If the operators are self-adjoint, we have the following result.

THEOREM 2.16. *Let A, B be positive and self-adjoint on* **H** *such that* $\overline{\mathbf{D}_A \cap \mathbf{D}_B} = \mathbf{H}$ *and $A^{1/2}$, $B^{1/2}$ exist. Let P_n be a projection onto $\mathbf{W}_n \subseteq \mathbf{D}_{A^{1/2}} \cap \mathbf{D}_{B^{1/2}}$ and let*

$$A_n = P_n^* \hat{A}_n P_n,$$
$$B_n = P_n^* \hat{B}_n P_n,$$

where $\hat{A}_n$, $\hat{B}_n$ are operators on W_n such that for every $x, y \in \mathbf{W}_n$,

$$\langle \hat{A}_n x, y \rangle_{\mathbf{H}} = \langle A^{1/2} x, A^{1/2} y \rangle_{\mathbf{H}},$$
$$\langle \hat{B}_n x, y \rangle_{\mathbf{H}} = \langle B^{1/2} x, B^{1/2} y \rangle_{\mathbf{H}}.$$

If $A \approx_{\mathbf{H}} B$ on $\mathbf{D}_A \cap \mathbf{D}_B$, then $\{A_n\} \approx_{\mathbf{H}} \{B_n\}$ on $\mathbf{D}_{A^{1/2}} \cap \mathbf{D}_{B^{1/2}}$.

Proof. From previous results $A^{1/2} \sim_{\mathbf{H}} B^{1/2}$ on $\mathbf{D}_{A^{1/2}} \cap \mathbf{D}_{B^{1/2}}$. For every $x \in \mathbf{W}_n$, we have

$$\frac{\langle A_n x, x \rangle_{\mathbf{H}}}{\langle B_n x, x \rangle_{\mathbf{H}}} = \frac{\langle \hat{A}_n x, x \rangle_{\mathbf{H}}}{\langle \hat{B}_n x, x \rangle_{\mathbf{H}}} = \frac{\langle A^{1/2} x, A^{1/2} x \rangle_{\mathbf{H}}}{\langle B^{1/2} x, B^{1/2} x \rangle_{\mathbf{H}}}.$$

The result follows. □

2.5. *Equivalence Classes*

We end this section by pointing out that if

$$\mathbf{G} = \{\text{all compact operators on } \mathbf{H}\},$$

$$\mathbf{G}_s = \{\text{all positive, self-adjoint, compact operators on } \mathbf{H}\},$$

then $\mathbf{H}$ norm equivalence on $\mathbf{H}$ determines an equivalence relation on $\mathbf{G}$ and equivalence in spectrum on $\mathbf{H}$ determines an equivalence relation on $\mathbf{G}_s$. It is easy to see that both equivalences are reflexive, symmetric, and transitive.

Since $\mathbf{G}_s \subseteq \mathbf{G}$, equivalence in norm also yields an equivalence relation on $\mathbf{G}_s$. From Theorem 2.10, we see that if $\mathbf{E} \subseteq \mathbf{G}_s$ is an equivalence class under $\sim$, then $\mathbf{E} \subseteq \mathbf{E}_s$, where $\mathbf{E}_s$ is an equivalence class under $\approx$. The example following Theorem 2.10 shows that this inclusion may be proper.

The theorems of this section provide rules for determining the equivalence of adjoints and inverses of elements of $\mathbf{G}$. We are primarily interested in elliptic boundary-value problems that are inverses of compact operators. In the next section, we show that an elliptic boundary-value operator belongs to a very large equivalence class.

3. Elliptic Operators and Their Discretizations

In this section, we discuss the equivalence of uniformly elliptic operators on bounded regions. Then, we examine the uniform equivalence of finite element approximations of the elliptic operators. Finally, we consider certain finite difference discretizations on rectangular regions.

3.1. *The Continuous Problem*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let A be a uniformly elliptic operator defined on $\mathbf{H}_{2m}(\Omega)$. That is, using the notation of Laurent Schwartz (cf. John [23; Chap. 3]),

$$Au = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u, \qquad x \in \Omega, \tag{3.1a}$$

$$\Gamma_j(x) u(x) = 0, \qquad x \in \partial\Omega,\ j = 1, 2, \ldots, m, \tag{3.1b}$$

and there is a constant $\lambda > 0$ such that for all real $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$, we have

$$\sum_{|\alpha| = 2m} a_\alpha \xi^\alpha \geq \lambda |\xi|^{2m}. \tag{3.2}$$

The boundary operators Γ_j are linearly independent linear differential operators of order $\leq 2m - 1$.

Under appropriate smoothness hypothesis on the domain Ω and the coefficients $a_\alpha(x)$ and appropriate hypothesis on the boundary conditions (3.1b), we know that

$$A^{-1}\colon \mathbf{L}_2 \to \mathbf{H}_{2m} \tag{3.3}$$

is bounded and one-to-one. That is, there is a constant $K_1 = K_1(A)$, and for every $f \in \mathbf{L}_2(\Omega)$ there exists a unique $u = u(x) \in \mathbf{H}_{2m}(\Omega)$ such that

$$Au = f$$

and

$$\|u\|_{\mathbf{H}_{2m}} \leq K_1(A)\|f\|_{\mathbf{L}_2}. \tag{3.4}$$

For example, if (3.1b) corresponds to the generalized Neumann boundary conditions, this result is found in Theorem 9.1 of Lions [25]. If (3.1b) corresponds to Dirichlet boundary conditions, this result is found in Theorem 17.2 of Friedman [16]. Of course, the very definition of A and $\mathbf{H}_{2m}(\Omega)$ implies that there is a constant $K_2 = K_2(A)$ such that

$$\|Au\|_{\mathbf{L}_2} \leq K_2(A)\|u\|_{\mathbf{H}_{2m}}. \tag{3.5}$$

We formalize the above discussion in the following theorem.

THEOREM 3.1. *Let A and B be uniformly elliptic operators on $\mathbf{H}_{2m}(\Omega)$ such that the estimates* (3.4) *and* (3.5) *hold. Then*

$$A \sim_{\mathbf{L}_2} B \qquad \text{on } \mathbf{D} = \mathbf{D}_A \cap \mathbf{D}_B,$$

$$A^{-1} \sim_{\mathbf{H}_{2m}} B^{-1} \qquad \text{on } \mathbf{L}_2(\Omega).$$

Proof. The proof follows from the discussion above. Since (3.4) and (3.5) hold, we have

$$\frac{1}{K_2(B)K_1(A)} \leq \frac{\|Au\|_{\mathbf{L}_2}}{\|Bu\|_{\mathbf{L}_2}} \leq K_2(A)K_1(B) \qquad \text{for } u \in \mathbf{D}$$

and

$$\frac{1}{K_1(B)K_2(A)} \leq \frac{\|A^{-1}f\|_{\mathbf{H}_{2m}}}{\|B^{-1}f\|_{\mathbf{H}_{2m}}} \leq K_1(A)K_2(B) \qquad \text{for } f \in \mathbf{L}_2(\Omega). \ \square$$

The results of Section 2 yield the following corollaries.

COROLLARY 3.2. *Under the hypotheses of Theorem* 3.1, $A \dot{\sim}_{\mathbf{L}_2} B$ *if and only if* $\mathbf{D}_A = \mathbf{D}_B$.

Proof. If $\mathbf{D}_A = \mathbf{D}_B$, then $\mathbf{D} = \mathbf{D}_A = \mathbf{D}_B$ satisfies $\mathbf{R}_A = A\mathbf{D} = \mathbf{L}_2(\Omega)$, $\mathbf{R}_B = B\mathbf{D} = \mathbf{L}_2(\Omega)$ and so $A \dot{\sim}_{\mathbf{L}_2} B$. Conversely, bounds (3.4) imply that $\|A^{-1}\|_{\mathbf{H}_{2m}}, \|B^{-1}\|_{\mathbf{H}_{2m}} < \infty$. Theorem 2.2 yields $\mathbf{D}_A = \mathbf{D}_B$. □

For the next result, let us define the operator $\hat{B}$ that has the same coefficients as A in (3.1a) but boundary conditions (3.1b) chosen so that $\mathbf{D}_{\hat{B}^*} = \mathbf{D}_{B^*}$ ($\mathbf{L}_2$ adjoint).

THEOREM 3.3. *Assume the hypothesis of Theorem* 3.1 *holds for* A^*, B^*, *and* $\hat{B}^*$ ($\mathbf{L}_2$ *adjoint*). *Then* $A^{-1} \dot{\sim}_{\mathbf{L}_2} B^{-1}$ *if and only if* $\mathbf{D}_{A^*} = \mathbf{D}_{B^*}$.

Proof. Suppose $\mathbf{D}_{A^*} = \mathbf{D}_{B^*}$. By Corollary 3.2, $A^* \dot{\sim}_{\mathbf{L}_2} B^*$. Now, consider A^*, B^*: $\mathbf{L}_2(\Omega) \to \mathbf{L}_2(\Omega)$ and apply Theorem 2.5. Since $\mathbf{D}_{(A^{-1})} = \mathbf{D}_{(B^{-1})} = \mathbf{L}_2(\Omega)$, we have $A^{-1} \sim_{\mathbf{L}_2} B^{-1}$ on $\mathbf{L}_2(\Omega)$; that is, $A^{-1} \dot{\sim}_{\mathbf{L}_2} B^{-1}$. Now consider $\hat{B}$. We have just shown that $B^{-1} \sim_{\mathbf{L}_2} \hat{B}^{-1}$ on $\mathbf{L}_2(\Omega)$. By transitivity $A^{-1} \sim_{\mathbf{L}_2} B^{-1}$ on $\mathbf{L}_2(\Omega)$ if and only if $A^{-1} \sim_{\mathbf{L}_2} \hat{B}^{-1}$ on $\mathbf{L}_2(\Omega)$. Lemma 2.4 yields $A^{-1} \sim_{\mathbf{L}_2} \hat{B}^{-1}$ on $\mathbf{L}_2(\Omega)$ if and only if $\mathbf{D}_A = \mathbf{D}_{\hat{B}}$; that is, $A = \hat{B}$. This implies that $\mathbf{D}_{A^*} = \mathbf{D}_{\hat{B}^*} = \mathbf{D}_{B^*}$, which completes the proof. □

Corollary 3.2 and Theorem 3.3 were established in [26] for the case $m = 1$ and $\Omega = R^2$. There the proof is more constructive in nature. The previous results also yield results on spectral equivalence.

COROLLARY 3.4. *Assume the hypothesis of Theorem* 3.1. *Let A and B be positive and self-adjoint. If* $\mathbf{D}_A = \mathbf{D}_B$, *then* $A \approx_{\mathbf{L}_2} B$ *on* $\mathbf{D}_A = \mathbf{D}_B$ *and* $A^{-1} \approx_{\mathbf{L}_2} B^{-1}$ *on* $\mathbf{L}_2(\Omega)$.

Proof. Since A^{-1} and B^{-1} are compact on $\mathbf{L}_2(\Omega)$, Theorem 2.10 and Theorem 3.3 yield $A^{-1} \approx_{\mathbf{L}_2} B^{-1}$ on $\mathbf{L}_2(\Omega)$. Corollary 2.11 yields $A \approx_{\mathbf{L}_2} B$ on $\mathbf{D}_A = \mathbf{D}_B$. □

In general, neither $\mathbf{D}_A = \mathbf{D}_B$ nor $\mathbf{D}_{A^*} = \mathbf{D}_{B^*}$ is necessary to establish $A^{-1} \approx_{\mathbf{L}_2} B^{-1}$ on $\mathbf{L}_2(\Omega)$. As the example after Corollary 2.11 shows, spectral equivalence does not imply norm equivalence. In [26] it is shown that for $m = 1$, $\Omega \subseteq R^2$ the necessary and sufficient condition is $\overline{\mathbf{D}}_A^{\mathbf{H}_1} = \overline{\mathbf{D}}_B^{\mathbf{H}_1}$. A similar result is given in Corollary 3.7.

Similar results hold for the weak form of the problem. Suppose we seek u such that

$$Au = f \tag{3.6}$$

for $f \in \mathbf{L}_2(\Omega)$. This is equivalent to finding u such that

$$\langle Au, v \rangle_{\mathbf{L}_2} = \langle f, v \rangle_{\mathbf{L}_2}, \qquad \forall v \in \mathbf{T}_A, \tag{3.7}$$

where $\mathbf{T}_A$ is dense in $\mathbf{L}_2(\Omega)$. Using generalized versions of Green's identities, we write

$$\langle Au, v\rangle_{\mathbf{L}_2} = a(u, v) - \gamma_a(u, v), \tag{3.8}$$

where $a(u, v)$ is a bilinear form defined for $u, v \in \mathbf{H}_m(\Omega)$ and $\gamma_a(u, v)$ is a bilinear form on $\mathbf{H}_m(\partial\Omega)$. (Here u and v and their derivatives are evaluated using trace operators (cf. Lions [25]).

For a careful choice of $\mathbf{T}_A$ the following bounds can often be established:

$$|a(u, v) - \gamma_a(u, v)| \leq M_2(A)\|u\|_{\mathbf{H}_m}\|v\|_{\mathbf{H}_m} \tag{3.9}$$

for $u \in \mathbf{D}_A$, $v \in \mathbf{T}_A$;

$$\sup_{v \in \mathbf{T}_A} \frac{|a(u, v) - \gamma_a(u, v)|}{\|v\|_{\mathbf{H}_m}} \geq \frac{1}{M_1(A)}\|u\|_{\mathbf{H}_m} \tag{3.10}$$

for $u \in \mathbf{D}_A$; and

$$\sup_{u \in \mathbf{D}_A} |a(u, v) - \gamma_a(u, v)| > 0 \tag{3.11}$$

for $v \in \mathbf{T}_A$. The choice of $\mathbf{T}_A$ is critical here. The bounds (3.9) and (3.11) are more easily obtained with $\mathbf{T}_A$ small, while (3.10) requires $\mathbf{T}_A$ to be large. Clearly, (3.9), (3.10), and (3.11) can be extended to the $\mathbf{H}_m$ closure. Let

$$\mathbf{D}_a = \overline{\mathbf{D}}_A^{\mathbf{H}_m}, \qquad \mathbf{T}_a = \overline{\mathbf{T}}_A^{\mathbf{H}_m}. \tag{3.12}$$

Then, (3.9), (3.10), and (3.11) imply

$$|a(u, v) - \gamma_a(u, v)| \leq M_2(A)\|u\|_{\mathbf{H}_m}\|v\|_{\mathbf{H}_m} \tag{3.13}$$

for $u \in \mathbf{D}_a$, $v \in \mathbf{T}_a$; and

$$\sup_{v \in \mathbf{T}_a} \frac{|a(u, v) - \gamma_a(u, v)|}{\|v\|_{\mathbf{H}_m}} \geq \frac{1}{M_1(A)}\|u\|_{\mathbf{H}_m} \tag{3.14}$$

for $u \in \mathbf{D}_a$; and

$$\sup_{u \in \mathbf{D}_a} |a(u, v) - \gamma_a(u, v)| > 0 \tag{3.15}$$

for $v \in \mathbf{T}_a$.

The bound (3.13) implies that for $u \in \mathbf{D}_a$ the functional

$$a(u) = a(u, \cdot) - \gamma_a(u, \cdot) \tag{3.16}$$

is bounded on $\mathbf{T}_a$. Thus, we have defined a map

$$a: \mathbf{D}_a \to \mathbf{T}_a^*, \tag{3.17}$$

where $\mathbf{T}_a^*$ is the space of bounded linear functionals on $\mathbf{T}_a$. Further, the bound (3.13) implies that

$$\|a(u)\|_{\mathbf{T}_a^*} = \sup_{v \in \mathbf{T}_a} \frac{|a(u,v) - \gamma_a(u,v)|}{\|v\|_{\mathbf{H}_m}} \le M_2(A)\|u\|_{\mathbf{H}_m}. \tag{3.18}$$

The bound (3.14) implies that a is one-to-one with bounded inverse; that is,

$$\|u\|_{\mathbf{H}_m} \le M_1(A)\|a(u)\|_{\mathbf{T}_a^*}. \tag{3.19}$$

Finally, (3.15) ensures that a is onto (cf. Babuska and Aziz [2, Chap. 5]).

Let us next define the isomorphism

$$E: \mathbf{L}_2(\Omega) \to \mathbf{L}_2^*(\Omega) \tag{3.20a}$$

by associating f with the bounded linear functional,

$$E(f) = \langle f, \cdot \rangle_{\mathbf{L}_2}, \tag{3.20b}$$

defined on $\mathbf{L}_2(\Omega)$. Since $\mathbf{L}_2^*(\Omega) \subseteq \mathbf{T}_a^*$ the weak form of (3.6),

$$a(u) = E(f), \tag{3.21}$$

is well defined; that is, we seek $u \in \mathbf{D}_a$ such that

$$a(u,v) - \gamma_a(u,v) = \langle f, v \rangle_{\mathbf{L}_2} \tag{3.22}$$

for every $v \in \mathbf{T}_a$. The bounds (3.13), (3.14), and (3.15), together with the requirement that $\mathbf{T}_A$ is dense in $\mathbf{L}_2(\Omega)$, imply that the solution to (3.21) is the solution to (3.6). We remark that it is often the case that $\mathbf{D}_a = \mathbf{T}_a$.

Consider the weak form of the uniformly elliptic operator B with bilinear form

$$b(u,v) - \gamma_b(u,v)$$

defined for $u \in \mathbf{D}_b$, $v \in \mathbf{T}_b$. Suppose that bounds (3.13), (3.14), and (3.15) hold for this choice of $\mathbf{T}_b$. Then, we can define the map

$$b: \mathbf{D}_b \to \mathbf{T}_b^*. \tag{3.23}$$

As before, b is bounded, one-to-one, and onto with bounded inverse.

Notice that $\mathbf{D}_a$ and $\mathbf{D}_b$ are closed subspaces of $\mathbf{H}_m$. Also, notice $\mathbf{L}_2^*(\Omega) \subseteq \mathbf{T}_a^*$ and $\mathbf{L}_2^*(\Omega) \subseteq \mathbf{T}_b^*$. However, in general $\mathbf{T}_a^* \ne \mathbf{T}_b^*$. Moreover, in general,

$$\sup_{v\in\mathbf{T}_a}\frac{|\langle f,v\rangle|}{\|v\|_{\mathbf{H}_m}}=\|E(f)\|_{\mathbf{T}_a^*}\neq\|E(f)\|_{\mathbf{T}_b^*}=\sup_{v\in\mathbf{T}_b}\frac{|\langle f,v\rangle|}{\|v\|_{\mathbf{H}_m}}. \tag{3.24}$$

We have the situation of operators

$$a\colon H_m\to\mathbf{T}_a^*,$$
$$b\colon H_m\to\mathbf{T}_b^*,$$

where $\mathbf{T}_a^*\neq\mathbf{T}_b^*$. However, the concept of equivalence is still valid. The bounds (3.18) and (3.19) imply

$$\frac{1}{M_1(A)M_2(B)}\le\frac{\|a(u)\|_{\mathbf{T}_a^*}}{\|b(u)\|_{\mathbf{T}_b^*}}\le M_2(A)M_1(B) \tag{3.25}$$

for $u\in\mathbf{D}=\mathbf{D}_a\cap\mathbf{D}_b$. Since $\mathbf{D}_a$ and $\mathbf{D}_b$ are closed in $\mathbf{H}_m$, $\mathbf{D}$ is dense in $\mathbf{D}_a$ only if $\mathbf{D}_a=\mathbf{D}_b$. In this case,

$$ab^{-1}\colon\mathbf{T}_b^*\to\mathbf{T}_a^*$$

is bounded, one-to-one, and onto with bounded inverse.

We may also consider the equivalence of $a^{-1}E$ and $b^{-1}E$ on $\mathbf{L}_2(\Omega)$. Bounds (3.18) and (3.19) yield

$$\frac{1}{M_2(A)M_1(B)}\frac{\|E(f)\|_{\mathbf{T}_b^*}}{\|E(f)\|_{\mathbf{T}_a^*}}\le\frac{\|a^{-1}E(f)\|_{\mathbf{H}_m}}{\|b^{-1}E(f)\|_{\mathbf{H}_m}}\le M_1(A)M_2(B)\frac{\|E(f)\|_{\mathbf{T}_a^*}}{\|E(f)\|_{\mathbf{T}_b^*}} \tag{3.26}$$

for $f\in\mathbf{L}_2(\Omega)$. Clearly, $a^{-1}\sim_{\mathbf{H}_m}b^{-1}$ on $\mathbf{L}_2(\Omega)$ if and only if the $\mathbf{T}_a^*$ norm is equivalent to the $\mathbf{T}_b^*$ norm on $\mathbf{L}_2^*(\Omega)$. This is trivially true if $\mathbf{T}_a=\mathbf{T}_b$. In [26] it is shown that for $m=1$ and $\Omega=R^2$ and a certain class of boundary conditions that $a^{-1}\sim_{\mathbf{H}_1}b^{-1}$ on $\mathbf{L}_2(\Omega)$ if and only if $\mathbf{D}_a=\mathbf{T}_a=\mathbf{D}_b=\mathbf{T}_b$. We believe that a generalization of the arguments there will carry over to the more general setting; that is, we conjecture that $a^{-1}\sim_{\mathbf{H}_m}b^{-1}$ if and only if $\mathbf{D}_a=\mathbf{T}_a=\mathbf{D}_b=\mathbf{T}_b$.

We sum up the preceding discussion in the following theorem.

THEOREM 3.5. *Let A and B be uniformly elliptic operators on $\mathbf{H}_{2m}(\Omega)$ that give rise to a and b that satisfy the bounds* (3.13), (3.14), *and* (3.15).

Then,

(i) *Bounds* (3.25) *hold for* $u \in \mathbf{D} = \mathbf{D}_a \cap \mathbf{D}_b$.

(ii) $\mathbf{D}$ *is dense in* $\mathbf{D}_a$ *and* $\mathbf{D}_b$ *if and only if* $\mathbf{D}_a = \mathbf{D}_b$.

(iii) $a^{-1}E \sim_{\mathbf{H}_m} b^{-1}E$ *on* $\mathbf{L}_2(\Omega)$ *if and only if the* $\mathbf{T}_a^*$ *norm is equivalent to the* $\mathbf{T}_b^*$ *norm on* $\mathbf{L}_2^*(\Omega)$. (*For example, if* $\mathbf{T}_a = \mathbf{T}_b$.)

Proof. The proof follows from the discussion above. □

COROLLARY 3.6. *Let A and B be uniformly elliptic operators on* $\mathbf{H}_{2m}(\Omega)$ *that give rise to a and b that satisfy the bounds* (3.13), (3.14), *and* (3.15). *If* $\mathbf{T}_a = \mathbf{T}_b$, *then*

$$A^{-1} \sim_{\mathbf{H}_m} B^{-1} \qquad \text{on } \mathbf{L}_2(\Omega).$$

Proof. Since $A^{-1} = a^{-1}E$, $B^{-1} = b^{-1}E$ on $\mathbf{L}_2(\Omega)$, the result follows from Theorem 3.5. □

The next result assumes that the operator A is positive definite and self-adjoint and that the norm $\langle Ax, x\rangle^{1/2}$ is equivalent to the $\mathbf{H}_m$ norm on $\mathbf{D}_A$; that is, there exist constants $0 < M_3(A), M_4(A) < \infty$ such that

$$\frac{1}{M_3(A)}\|x\|_{\mathbf{H}_m} \le \langle Ax, x\rangle^{1/2} \le M_4(A)\|x\|_{\mathbf{H}_m} \tag{3.27}$$

for $x \in \mathbf{D}_A$.

COROLLARY 3.7. *Let A and B be positive definite, self-adjoint, uniformly elliptic operators on* $\mathbf{H}_{2m}(\Omega)$ *that give rise to a and b that satisfy* (3.13), (3.14), *and* (3.15) *with* $\mathbf{D}_A = \mathbf{T}_A$, $\mathbf{D}_B = \mathbf{T}_B$. *Further, suppose that* $\langle Ax, x\rangle_{\mathbf{L}_2}^{1/2}$ *and* $\langle Bx, x\rangle_{\mathbf{L}_2}^{1/2}$ *are equivalent to the* $\mathbf{H}_m$ *norm on* $\mathbf{D}_A$ *and* $\mathbf{D}_B$, *respectively. If* $\mathbf{D}_a = \mathbf{D}_b$, *then*

$$A^{-1} \approx_{\mathbf{L}_2} B^{-1} \qquad \text{on } \mathbf{L}_2(\Omega).$$

Proof. For $x \in \mathbf{L}_2(\Omega)$, we have

$$\frac{\langle A^{-1}x, x\rangle_{\mathbf{L}_2}}{\langle B^{-1}x, x\rangle_{\mathbf{L}_2}} = \frac{\langle AA^{-1}x, A^{-1}x\rangle_{\mathbf{L}_2}}{\langle BB^{-1}x, B^{-1}x\rangle_{\mathbf{L}_2}} \le M_3(B)M_4(A)\frac{\|A^{-1}x\|_{\mathbf{H}_m}}{\|B^{-1}x\|_{\mathbf{H}_m}}.$$

By Corollary 3.6, the right-hand side is bounded for $x \in \mathbf{L}_2(\Omega)$. The lower bound is established in a similar manner. □

We remark that, in general, $\mathbf{L}_2$ norm equivalence of the self-adjoint operators A^{-1} and B^{-1} requires $\mathbf{D}_{A^*} = \mathbf{D}_{B^*}$, while spectral equivalence occurs for $\mathbf{D}_a = \mathbf{D}_b$, which is much weaker.

3.2. *Finite Element Approximations*

Now consider finite element approximations of the solution of the boundary-value problem $Au = f$, where A is a uniformly elliptic operator of the form (3.1a), (3.1b). Let $\mathbf{W}_h$ be a finite dimensional subspace of $\mathbf{D}_A$ with basis $\{\phi_1, \ldots, \phi_n\}$. Let $\mathbf{V}_h \subseteq \mathbf{L}_2(\Omega)$ have the basis $\{\psi_1, \ldots, \psi_n\}$. Let P_h, Q_h be the $\mathbf{L}_2(\Omega)$ orthogonal projections onto $\mathbf{W}_h$ and $\mathbf{V}_h$, respectively. The finite element operator is then

$$A_h = Q_h A P_h. \tag{3.28}$$

If A_h is one-to-one on $\mathbf{W}_h$, we can denote its inverse from $\mathbf{V}_h$ to $\mathbf{W}_h$ by A_h^{-1}. The finite element approximation to u is given by

$$u_h = A_h^+ f = A_h^{-1} Q_h f. \tag{3.29}$$

Now let $\mathbf{M}_h$ be the "mass matrix"

$$(\mathbf{M}_h)_{ij} = \langle \psi_i, \psi_j \rangle_{\mathbf{L}_2} \tag{3.30}$$

and A_h be the "stiffness matrix"

$$(\mathbf{A}_h)_{ij} = \langle \psi_i, A\phi_j \rangle_{\mathbf{L}_2}. \tag{3.31}$$

The matrix associated with the operator A_h in the bases $\{\phi_i\}$ and $\{\psi_i\}$ is given by

$$\mathbf{M}_h^{-1} \mathbf{A}_h. \tag{3.32}$$

We will state some results on the uniform equivalence of the finite element operators and then relate these to bounds for their respective stiffness matrices. We assume the spaces $\mathbf{W}_h$ and $\mathbf{V}_h$ are chosen so that the finite element operators satisfy bounds similar to (3.4).

THEOREM 3.8. *Let A, B, and Ω satisfy the hypothesis of Theorem* 3.1. *Let $\mathbf{W}_h \subseteq \mathbf{D} = \mathbf{D}_A \cap \mathbf{D}_B$ and $\mathbf{V}_h \subseteq \mathbf{L}_2(\Omega)$ be chosen such that there are constants $\hat{K}_1(A)$, $\hat{K}_1(B)$ independent of h and*

$$\|u\|_{\mathbf{H}_{2m}} \le \hat{K}_1(A) \|A_h u\|_{\mathbf{L}_2}, \tag{3.33a}$$

$$\|u\|_{\mathbf{H}_{2m}} \le \hat{K}_1(B) \|B_h u\|_{\mathbf{L}_2} \tag{3.33b}$$

for $u \in \mathbf{W}_h$. Then

$$\{A_h\} \sim_{\mathbf{L}_2} \{B_h\} \qquad \text{on } \mathbf{D},$$

$$\{A_h^+\} \sim_{\mathbf{H}_{2m}} \{B_h^+\} \qquad \text{on } \mathbf{L}_2(\Omega).$$

Proof. The proof follows from the bounds above, the bound $\|A_h u\|_{\mathbf{L}_2} \le \|Au\|_{\mathbf{L}_2}$ for $u \in W_h$, and bounds (3.4) and (3.5). □

The bounds (3.4) and (3.5) make the hypotheses (3.33a), (3.33b) equivalent to the hypotheses (2.29a), (2.29b) of Theorem 2.15, which could then be used to prove the first result. We also obtain a result analogous to that for Corollary 3.2.

COROLLARY 3.9. *Let A, B, $\mathbf{W}_h$, $\mathbf{V}_h$, and Ω satisfy the hypothesis of Theorem 3.8. In addition, let $\mathbf{V}_h \subseteq \mathbf{D}_* = \mathbf{D}_{A^*} \cap \mathbf{D}_{B^*}$, let $\hat{A}_h = P_h A^* Q_h$, $\hat{B}_h = P_h B^* Q_h$, and assume there are constants $\hat{K}_1(A^*)$, $\hat{K}_2(B^*)$ independent of h such that*

$$\|v\|_{\mathbf{H}_{2m}} \le \hat{K}_1(A^*)\|\hat{A}_h v\|_{\mathbf{L}_2},$$

$$\|v\|_{\mathbf{H}_{2m}} \le \hat{K}_1(B^*)\|\hat{B}_h v\|_{\mathbf{L}_2}$$

for $v \in \mathbf{V}_h$. Then

$$\{A_h^+\} \sim_{\mathbf{L}_2} \{B_h^+\} \qquad \text{on } \mathbf{L}_2(\Omega).$$

Proof. Since $\hat{A}_h$ and $\hat{B}_h$ satisfy all the hypotheses of Theorem 3.8, we have $\{\hat{A}_h\} \sim_{\mathbf{L}_2} \{\hat{B}_h\}$ on $\mathbf{D}$. Theorem 2.5 yields $(\hat{A}_h^{-1})^* \sim_{\mathbf{L}_2} (\hat{B}_h^{-1})^*$ on $\mathbf{V}_h$. (Notice that the adjoint here is with respect to the *restricted* spaces $\mathbf{V}_h$ and $\mathbf{W}_h$; $\hat{A}_h^*$ is a map from $\mathbf{W}_h \to \mathbf{V}_h$.) Moreover, a closer look at the proof of Theorem 2.5 reveals that equivalence holds with the same bounds. Thus, $\{(\hat{A}_h^{-1})^*\} \sim_{\mathbf{L}_2} \{(\hat{B}_h^{-1})^*\}$ on $\mathbf{D}_*$. Since $\mathbf{W}_h \subseteq \mathbf{D}$ we have

$$\hat{A}_h^* = Q_h A P_h$$

and

$$(\hat{A}_h^{-1})^* = (Q_h A P_h)^{-1} = A_h^{-1}.$$

Since $A_h^+ = A_h^{-1} Q_h$, the result follows. □

The condition that $\mathbf{V}_h \subseteq \mathbf{D}_{A^*} \cap \mathbf{D}_{B^*}$ is not unreasonable in that both $\mathbf{D}_{A^*}$ and $\mathbf{D}_{B^*}$ are dense in $\mathbf{L}_2(\Omega)$. However, it is not a necessary condition. In Theorem 3.12 a different line of proof is used where the hypotheses require optimal convergence and inverse bounds to prove uniform $\mathbf{L}_2$ norm equivalence. Of course, unless $\mathbf{V}_h \subseteq \mathbf{D}_{A^*} \cap \mathbf{D}_{B^*}$ these hypotheses most likely will not be satisfied. We remark that Theorem 2.12 implies that if A_h^+ and B_h^+ converge pointwise in $\mathbf{L}_2$ norm to A^{-1} and B^{-1}, then $\{A_h^+\} \sim_{\mathbf{L}_2} \{B_h^{+1}\}$ on $\mathbf{L}_2(\Omega)$ only if $A^{-1} \sim_{\mathbf{L}_2} B^{-1}$ on $\mathbf{L}_2(\Omega)$; that is, only if $\mathbf{D}_{A^*} = \mathbf{D}_{B^*}$. Finally, we may achieve results on the matrices associated with these operators.

THEOREM 3.10. *Let $\mathbb{A}_h$, $\mathbb{B}_h$ be the stiffness matrices associated with the finite element operators A_h and B_h. Let the mass matrices $\mathbb{M}_h$ satisfy*

$C_{l_2}(\mathbf{M}_h) \le c_0$ *independent of* h. *If* $\{A_h\} \dot{\sim}_{\mathbf{L}_2} \{B_h\}$, *then*

$$\{\mathbf{A}_h\} \sim_{l_2} \{\mathbf{B}_h\}.$$

If $\{A_h^{-1}\} \dot{\sim}_{\mathbf{L}_2} \{B_h^{-1}\}$, *then*

$$\{\mathbf{A}_h^{-1}\} \sim_{l_2} \{\mathbf{B}_h^{-1}\}.$$

Proof. Suppose $u_h = \sum_i \alpha_i \phi_i \in \mathbf{W}_h$. Let $\boldsymbol{\alpha}^{\mathrm{T}} = (\alpha_1, \ldots, \alpha_{N_h})$, then

$$\frac{\|A_h u_h\|_{\mathbf{L}_2}^2}{\|B_h u_h\|_{\mathbf{L}_2}^2} = \frac{\langle \mathbf{M}_h(\mathbf{M}_h^{-1}\mathbf{A}_h)\boldsymbol{\alpha}, (\mathbf{M}_h^{-1}\mathbf{A}_h)\boldsymbol{\alpha}\rangle_{l_2}}{\langle \mathbf{M}_h(\mathbf{M}_h^{-1}\mathbf{B}_h)\boldsymbol{\alpha}, (\mathbf{M}_h^{-1}\mathbf{B}_h)\boldsymbol{\alpha}\rangle_{l_2}} \le C(\mathbf{M}_h)\frac{\|\mathbf{A}_h\boldsymbol{\alpha}\|_{l_2}^2}{\|\mathbf{B}_h\boldsymbol{\alpha}\|_{l_2}^2}.$$

A similar bound on the reciprocal of the left-hand side yields the first result. The second result follows in a similar fashion. □

Let us now consider the weak form of the problem. As we saw from Theorem 3.5, under the proper hypotheses the continuous operators are equivalent in the $\mathbf{H}_m$ and the appropriate $\mathbf{H}_m^*$ norms. The same holds true for the finite element operators. Let $\mathbf{W}_n \subseteq \mathbf{D}_a \subseteq \mathbf{H}_m$, $\mathbf{V}_h \subseteq \mathbf{T}_a \subseteq \mathbf{H}_m$. The finite element approximation to the weak form of the problem (3.22) is to find $u_h \in \mathbf{W}_h$ such that

$$a(u_h, v_h) - \gamma_a(u_h, v_h) = \langle f, v_h \rangle_{\mathbf{L}_2} \tag{3.34}$$

for every $v_h \in \mathbf{V}_h$. If, as before, we let $\{\phi_i\}$ and $\{\psi_i\}$ be bases for $\mathbf{W}_h$ and $\mathbf{V}_h$, respectively, then the stiffness matrix $\mathbf{A}_h$ is given by

$$(\mathbf{A}_h)_{ij} = a(\phi_j, \psi_i) - \gamma_a(\phi_j, \psi_i), \tag{3.35}$$

and the mass matrix is as in (3.30). If we let a_h be the operator from $\mathbf{W}_h \to \mathbf{V}_h^*$, then the natural basis for $\mathbf{V}_h^*$ is $\{\langle \psi_i, \cdot \rangle_{\mathbf{L}_2}\}$. The matrix associated with a_h in these bases is again $\mathbf{M}_h^{-1}\mathbf{A}_h$. The finite element approximation is then

$$u_h = a_h^+ f = a_h^{-1} E Q_h f, \tag{3.36}$$

where Q_h is the $\mathbf{L}_2$ orthogonal projection onto $\mathbf{V}_h$ and E is as defined in (3.20a), (3.20b). Suppose a satisfies the bound (3.13), then a_h also satisfies (3.13). Suppose that a also satisfies (3.14) and (3.15) and that $\mathbf{W}_h$ and $\mathbf{V}_h$ are chosen so that there exists $\hat{M}_1(a)$ independent of h and

$$\sup_{v_h \in \mathbf{V}_h} \frac{|a(u_h, v_h) - \gamma_a(u_h, v_h)|}{\|v_h\|_{\mathbf{H}_m}} \ge \frac{1}{\hat{M}_1(a)}\|u_h\|_{\mathbf{H}_m} \tag{3.37}$$

for $u_h \in \mathbf{W}_h$ and

$$\sup_{u_h \in \mathbf{W}_h} |a(u_h, v_h) - \gamma_a(u_h, v_h)| > 0 \tag{3.38}$$

for $v_h \in \mathbf{V}_h$. Then,

$$\|a_h(u_h)\|_{\mathbf{V}_h^*} \leq M_2(a)\|u_h\|_{\mathbf{H}_m} \tag{3.39a}$$

for $u_h \in \mathbf{W}_h$, and

$$\|a_h^{-1}E(v_h)\|_{\mathbf{H}_m} \leq \hat{M}_1(a)\|E(v_h)\|_{\mathbf{V}_h^*} \tag{3.39b}$$

for $v_h \in \mathbf{V}_h$. If (3.39a), (3.39b) hold for two families of operators a_h and b_h, then they are uniformly equivalent families.

THEOREM 3.11. *Let a and b satisfy bounds* (3.13), (3.14), *and* (3.15). *Let* $\mathbf{W}_h \subseteq \mathbf{D}_a \cap \mathbf{D}_b$ *and* $\mathbf{V}_h \subseteq \mathbf{T}_a \cap \mathbf{T}_b$ *be chosen so that* (3.37) *and* (3.38) *hold for both a and b. Then,*

$$\{a_h^+\} \sim_{\mathbf{H}_m} \{b_h^+\} \qquad \text{on } \mathbf{L}_2(\Omega).$$

Further,

$$a_h \sim_{\mathbf{V}_h^*} b_h \qquad \text{on } \mathbf{W}_h$$

with bounds independent of h.

Proof. The results follow directly from the application of (3.39a), (3.39b) for both a and b. □

Remark. We cannot say that $\{a_h\}$ is uniformly $\mathbf{V}_h^*$ norm equivalent to $\{b_h\}$ because the $\mathbf{V}_h^*$ norm depends upon h.

Remark. Since $\mathbf{D}_a$ and $\mathbf{D}_b$ are closed in $\mathbf{H}_m$, $\mathbf{D}_a \cap \mathbf{D}_b$ is closed. Thus, a_h^+ and b_h^+ will converge pointwise to a^{-1} and b^{-1} in the $\mathbf{H}_m$ norm only if $\mathbf{D}_a = \mathbf{D}_b$. As we have remarked before, it is frequently the case that $\mathbf{D}_a = \mathbf{T}_a$ and $\mathbf{D}_b = \mathbf{T}_b$, which would then imply that $\mathbf{W}_h, \mathbf{V}_h \subseteq \mathbf{D}_a = \mathbf{D}_b$.

Unfortunately, equivalence in the $\mathbf{H}_m$ norm does not imply the l_2 equivalence of the matrices $\mathbf{A}_h$. This is because if $v = \Sigma\alpha_j\psi_j \in \mathbf{V}_h$, then

$$\|a_h^{-1}v\|_{\mathbf{H}_m}^2 = \langle \hat{\mathbf{M}}_h(\mathbf{A}_h^{-1}\mathbf{M}_h)\boldsymbol{\alpha}, (\mathbf{A}_h^{-1}\mathbf{M}_h)\boldsymbol{\alpha}\rangle_{l_2}, \tag{3.40}$$

where $\hat{\mathbf{M}}_h$ is the mass matrix in the $\mathbf{H}_m$ inner product

$$(\hat{\mathbf{M}}_h)_{ij} = \langle \psi_i, \psi_j \rangle_{\mathbf{H}_m}. \tag{3.41}$$

In practical applications, $C_{l_2}(\hat{\mathbf{M}}_h)$ is not bounded independent of h.

This apparent difficulty can be avoided by assuming optimal convergence and an inverse bound on the finite element operators. A version of the following theorem can be found in Bramble and Pasciak [6] for $m = 1$. We remark that $m = 1$ is the case in which the hypotheses are most likely to be satisfied.

THEOREM 3.12. *Suppose A and B are uniformly elliptic operators on $\mathbf{H}_{2m}(\Omega)$ such that* (3.4) *and* (3.5) *hold for A^* and B^* and $\mathbf{D}_{A^*} = \mathbf{D}_{B^*}$. Let A and B give rise to a and b such that* (3.13), (3.14), *and* (3.15) *hold. Let $\mathbf{W}_h$ and $\mathbf{V}_h$ be chosen so that there are constants $K_3(a)$, $K_3(b)$, $K_4(a)$, and $K_4(b)$ independent of h such that*

$$\|a_h^+ f - a^{-1} f\|_{\mathbf{L}_2} \leq h^{2m} K_3(a) \|f\|_{\mathbf{L}_2}, \tag{3.42a}$$

$$\|b_h^+ f - b^{-1} f\|_{\mathbf{L}_2} \leq h^{2m} K_3(b) \|f\|_{\mathbf{L}_2} \tag{3.42b}$$

for $f \in \mathbf{L}_2(\Omega)$, and

$$\|a_h u\|_{\mathbf{L}_2} \leq h^{-2m} K_4(a) \|u\|_{\mathbf{L}_2}, \tag{3.43a}$$

$$\|b_h u\|_{\mathbf{L}_2} \leq h^{-2m} K_4(b) \|u\|_{\mathbf{L}_2} \tag{3.43b}$$

for $u \in \mathbf{W}_h$. Then

$$\{a_h^+\} \sim_{\mathbf{L}_2} \{b_h^+\} \qquad \text{on } \mathbf{L}_2(\Omega).$$

Proof. By Theorem 3.3, $A^{-1} \sim_{\mathbf{L}_2} B^{-1}$ on $\mathbf{L}_2(\Omega)$. The bounds (3.13), (3.14), and (3.15) imply that $A^{-1} = a^{-1}E$ and $B^{-1} = b^{-1}E$ on $\mathbf{L}_2(\Omega)$. Using (3.42a), (3.4) for A^*, (3.5) for B^*, (3.42b), and (3.43b), we have

$$\begin{aligned}
\|a_h^+ f\|_{\mathbf{L}_2} &\leq \|a^{-1} f\|_{\mathbf{L}_2} + h^{2m} K_3(a) \|f\|_{\mathbf{L}_2} \\
&\leq K_2(A^*) K_1(B^*) \|b^{-1} f\|_{\mathbf{L}_2} + h^{2m} K_3(a) \|f\|_{\mathbf{L}_2} \\
&\leq K_2(A^*) K_1(B^*) \|b_h^+ f\|_{\mathbf{L}_2} \\
&\quad + h^{2m}\big(K_3(a) + K_2(A^*) K_1(B^*) K_3(b)\big) \|f\|_{\mathbf{L}_2} \\
&\leq \big[K_2(A^*) K_1(B^*) \\
&\qquad + K_4(b)\big(K_3(a) + K_2(A^*) K_1(B^*) K_3(b)\big)\big] \|b_h^+ f\|_{\mathbf{L}_2}.
\end{aligned}$$

The reverse bound is found in a similar manner. □

COROLLARY 3.13. *Suppose the mass matrices $\mathbf{M}_h$ satisfy $C_{l_2}(\mathbf{M}_h) \leq c_0$. Under the hypothesis of Theorem* 3.12,

$$\{\mathbf{A}_h^{-1}\} \sim_{l_2} \{\mathbf{B}_h^{-1}\}.$$

Proof. The proof is as in Theorem 3.10. □

3.3. *Finite Difference Approximations*

In general, it is not so easy to obtain similar results for finite difference discretizations. The results of D'Yakanov [13] can be used to show uniform equivalence in spectrum of the discretizations using centered differences of positive, self-adjoint, uniformly elliptic operators on an n-dimensional cube with Dirichlet boundary conditions. Drya [11] has proved the necessary estimates to show uniform equivalence in norm for centered difference approximations of uniformly elliptic operators with positive definite symmetric part without mixed derivatives and Dirichlet boundary conditions for a convex polygonal grid region whose sides match the grid exactly. (In general, Ω_h is not convex even when Ω is convex.) He also has treated the case with mixed derivatives for a uniform grid. We will show uniform equivalence in norm of several variants of the centered difference approximation to invertible uniformly elliptic operators on two-dimensional rectangles with either Dirichlet or Neumann boundary conditions on each edge. Our approach could also be used to extend the results of [11] to the general, invertible, uniformly elliptic operator with Dirichlet boundary conditions on a convex polygonal grid domain.

The rectangle is perhaps the most important case because in this case if the operator B is separable and B_h is its finite difference discretization, then there are fast algorithms for obtaining $B_h^{-1}g$ [30, 31]. The papers [7, 14, 34] are significant precisely because they exploit this fact. In this case, there is a very useful result by Nitsche and Nitsche [28]. While the paper [28] deals only with the case of Dirichlet boundary conditions, it is easy to extend the results to the more general case where along an entire side one requires either $u = 0$ or $\partial u/\partial n = 0$. We sketch this more general result in Lemma A.1 of the Appendix.

Before discussing the discrete case, it is illustrative to examine more closely the arguments used in the continuous case. For simplicity, we take Ω to be the unit square. The arguments apply to any rectangle whose sides are parallel to the axes. Let

$$\Omega = \{(x, y): 0 < x < 1, 0 < y < 1\}. \tag{3.44}$$

Let A be an invertible, uniformly elliptic partial differential operator of second order defined on Ω. That is, A is of the form

$$Au = -\left[a(x, y)\frac{\partial^2 u}{\partial x^2} + 2b(x, y)\frac{\partial^2 u}{\partial x\,\partial y} + c(x, y)\frac{\partial^2 u}{\partial y^2}\right] + d(x, y)\frac{\partial u}{\partial x} + e(x, y)\frac{\partial u}{\partial y} + \sigma(x, y)u, \tag{3.45a}$$

with boundary conditions of form

$$u \text{ or } \frac{\partial u}{\partial n} = 0 \tag{3.45b}$$

on each edge. For example, we could have the boundary conditions

$$u(x,0) = \frac{\partial u}{\partial y}(x,1) = 0, \qquad 0 \le x \le 1, \tag{3.46a}$$

$$u(0, y) = \frac{\partial u}{\partial y}(1, y) = 0, \qquad 0 \le y \le 1. \tag{3.46b}$$

The coefficients a, b, c, d, e, σ are assumed to be $\mathbf{C}^1(\overline{\Omega})$. Since A is uniformly elliptic, there are positive constants

$$q = q(A), \qquad Q = Q(A), \tag{3.47a}$$

with

$$0 < q \le Q, \tag{3.47b}$$

and for all $(x, y) \in \overline{\Omega}$,

$$q(\xi^2 + \eta^2) \le a(x, y)\xi^2 + 2b(x, y)\xi\eta + c(x, y)\eta^2 \le Q(\xi^2 + \eta^2). \tag{3.48}$$

Since A is invertible, there is a constant $K_0 = K_0(A)$ such that for every $u(x, y) \in \mathbf{H}_2(\overline{\Omega})$, which satisfies the boundary conditions (in the appropriate weak form), we have

$$\|u\|_{\mathbf{L}_2} \le K_0 \|Au\|_{\mathbf{L}_2}. \tag{3.49}$$

Next, we shall outline the proof that the estimate (3.49) can be improved to an estimate of the form

$$\|u\|_{\mathbf{H}_2} \le K_1 \|Au\|_{\mathbf{L}_2}, \tag{3.50}$$

where the $\mathbf{H}_2$ norm is given by

$$\|u\|^2_{\mathbf{H}_2} = \|u\|^2_{\mathbf{L}_2} + \left\|\frac{\partial u}{\partial x}\right\|^2_{\mathbf{L}_2} + \left\|\frac{\partial u}{\partial y}\right\|^2_{\mathbf{L}_2} + \left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{\mathbf{L}_2} + \left\|\frac{\partial^2 u}{\partial x\,\partial y}\right\|^2_{\mathbf{L}_2} + \left\|\frac{\partial^2 u}{\partial y^2}\right\|^2_{\mathbf{L}_2}. \tag{3.51}$$

The ideas for this estimate apply, with only slight modification, to prove analogous estimates for the associated finite difference equations. The technical details required for the discrete case are given later.

First, consider the special case where $A = A_0$ with

$$A_0 u = -\left[a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2}\right]. \tag{3.52}$$

Let $u(x, y) \in \mathbf{C}^3(\bar{\Omega})$ and satisfy the boundary conditions of A. The argument developed in Nitsche and Nitsche [28] shows that

$$\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{\mathbf{L}_2}^2 + 2\left\|\frac{\partial^2 u}{\partial x\,\partial y}\right\|_{\mathbf{L}_2}^2 + \left\|\frac{\partial^2 u}{\partial y^2}\right\|_{\mathbf{L}_2}^2 \le \frac{2Q^2}{q^4}\|A_0 u\|_{\mathbf{L}_2}^2. \tag{3.53}$$

Remark. The finite-difference analog of this estimate was proven in [28] for Dirichlet boundary conditions. The more general case described above is discussed in Lemma A.1 of the Appendix.

The next result proves (3.50) for more general operators.

LEMMA 3.14. *Suppose A is given by* (3.45a), (3.45b), *and has a bounded inverse*; *that is*, (3.49) *holds. Then, there is a constant $K_1 = K_1(A)$ such that* (3.50) *holds. The constant K_1 depends on the coefficients a, b, c, d, e, σ, and their derivatives.*

Proof. The operator A can be rewritten as

$$Au = -\left[\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(b\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right]$$
$$+\left(d + \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right)\frac{\partial u}{\partial x} + \left(e + \frac{\partial c}{\partial y} + \frac{\partial b}{\partial x}\right)\frac{\partial u}{\partial y} + \sigma u. \tag{3.54}$$

Remark. In the discrete case, it is necessary to show that a finite difference operator, which is written as the analog of (3.45a) (see (3.65)), may also be written as an analog of (3.54); that is, as (3.67) and a controllable perturbation. This result is contained in Lemma A.3 of the Appendix—specifically (A.11) and (A.12).

Using this representation, multiplying by u, and integrating by parts, we find

$$\iint_\Omega\left[a\left(\frac{\partial u}{\partial x}\right)^2 + 2b\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + c\left(\frac{\partial u}{\partial y}\right)^2\right]dx\,dy \le T(u, Au), \tag{3.55a}$$

where

$$T(u, Au) = \|Au\|_{\mathbf{L}_2}\cdot\|u\|_{\mathbf{L}_2} + \left[d_0\left\|\frac{\partial u}{\partial x}\right\|_{\mathbf{L}_2} + e_0\left\|\frac{\partial u}{\partial y}\right\|_{\mathbf{L}_2}\right]\cdot\|u\|_{\mathbf{L}_2} + \sigma_0\|u\|_{\mathbf{L}_2}^2 \tag{3.55b}$$

with

$$d_0 = \left\| d + \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} \right\|_\infty, \qquad \left\| e + \frac{\partial c}{\partial y} + \frac{\partial b}{\partial x} \right\|_\infty, \qquad \sigma_0 = \|\sigma\|_\infty. \quad (3.55c)$$

Using (3.48) and combining (3.55a)–(3.55c), we have

$$q\left(\left\| \frac{\partial u}{\partial x} \right\|_{\mathbf{L}_2}^2 + \left\| \frac{\partial u}{\partial y} \right\|_{\mathbf{L}_2}^2 \right) \leq \|Au\|_{\mathbf{L}_2} \|u\|_{\mathbf{L}_2}$$
$$+ \left(d_0 \left\| \frac{\partial u}{\partial x} \right\|_{\mathbf{L}_2} + e_0 \left\| \frac{\partial u}{\partial y} \right\|_{\mathbf{L}_2} \right) \|u\|_{\mathbf{L}_2} + \sigma_0 \|u\|_{\mathbf{L}_2}^2. \quad (3.56)$$

Remark. The finite-difference analog of this last argument follows from Lemma A.2 of the Appendix.

Substituting (3.49) into the right-hand side of (3.56), we are left with a quadratic inequality in $\|\partial u/\partial x\|_{\mathbf{L}_2}$ and $\|\partial u/\partial y\|_{\mathbf{L}_2}$. Solving this yields

$$\left\| \frac{\partial u}{\partial x} \right\|_{\mathbf{L}_2} + \left\| \frac{\partial u}{\partial y} \right\|_{\mathbf{L}_2} \leq k_0 \|Au\|_{\mathbf{L}_2} \quad (3.57)$$

for some constant k_0. Observe that

$$A_0 u = Au - \left[d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + \sigma u \right].$$

If we take the norm of both sides and substitute (3.57) and (3.49) into the right-hand side, we see that there is a constant k_1 such that

$$\|A_0 u\|_{\mathbf{L}_2} \leq k_1 \|Au\|_{\mathbf{L}_2}. \quad (3.58)$$

Using (3.49), (3.53), (3.57), and (3.58), we obtain the estimate (3.50). □

We now turn our attention to finite difference discretizations of the operator A. Let p and l be integers and set

$$\Delta x = \frac{1}{p+1}, \qquad \Delta y = \frac{1}{l+1}, \qquad h = \max(\Delta x, \Delta y), \quad (3.59)$$

$$\Omega_h = \{(x_k, y_j) \in \Omega;\ x_k = k\,\Delta x,\ y_j = j\Delta y\}, \quad (3.60a)$$

$$\partial\Omega_h = \{(x_k, y_j) \in \partial\Omega;\ x_k = k\,\Delta x,\ y = j\Delta y\}, \quad (3.60b)$$

$$\bar{\Omega} = \Omega_h \cup \partial\Omega_h. \quad (3.60c)$$

Note. If $(x_k, y_j) \in \partial\Omega_h$, then either $k = 0$ or $p + 1$ or $j = 0$ or $l + 1$.

Let $\mathbf{S}_h$ denote the set of grid vectors $V = \{V_{k,j}\}$ defined on $\overline{\Omega}_h$ that satisfies the appropriate discrete boundary conditions. Thus, if the boundary conditions associated with A require

(i) $u(0, y) = 0$, then $V_{0,j} = 0$, $j = 0, 1, \ldots, l + 1$,

(ii) $u_y(x, 1) = 0$, then $V_{k,l+1} = V_{k,l}$, $k = 1, 2, \ldots, p$,

and so on.

Remark. In the case of the boundary condition (ii) above, one would probably choose Δy somewhat differently so that

$$y_l = 1 - \tfrac{1}{2}\Delta y, \qquad y_{l+1} = 1 + \tfrac{1}{2}\Delta y.$$

However, such a modification has no effect on our analysis. Hence, for purposes of this discussion, we formulate the discrete spaces as above.

Let $G(x, y)$ be a function defined on $\overline{\Omega}$. We write

$$G_{k,j} = G(x_k, y_j), \qquad G_{k+1/2,j} = G\left(x_k + \tfrac{1}{2}\Delta x, y_j\right), \tag{3.61}$$

etc. Let $V \in \mathbf{S}_h$; we denote the usual forward, backward, and centered difference quotients by subscripts as

$$[V_x]_{k,j} = \frac{1}{\Delta x}\left[V_{k+1,j} - V_{k,j}\right], \tag{3.62a}$$

$$[V_{\bar{x}}]_{k,j} = \frac{1}{\Delta x}\left[V_{k,j} - V_{k-1,j}\right], \tag{3.62b}$$

$$[V_{\hat{x}}]_{k,j} = \frac{1}{2\,\Delta x}\left[V_{k+1,j} - V_{k-1,j}\right], \tag{3.62c}$$

with similar notation for difference quotients in the y-directions. Let T_x, T_y denote the shift operators

$$[T_x V]_{k,j} = V_{k+1,j}, \qquad [T_y V]_{k,j} = V_{k,j+1}, \tag{3.63}$$

$$[T_x^{-1} V]_{k,j} = V_{k-1,j}, \qquad [T_y^{-1} V]_{k,j} = V_{k,j-1}. \tag{3.64}$$

With this notation, we are able to describe finite difference operators that correspond to different representations of the operator A.

Case 1. Consider the representation (3.45a). Let $V \in \mathbf{S}_h$. We define

$$A_h V = -\left[aV_{x\bar{x}} + 2bV_{\hat{x}\hat{y}} + cV_{y\bar{y}}\right] + dV_{\hat{x}} + cV_{\hat{y}} + \sigma V. \tag{3.65}$$

Case 2. Consider the representation (3.54). Let $V \in \mathbf{S}_h$. Let

$$\tilde{a}(x, y) = a\left(x + \tfrac{1}{2}\Delta x, y\right), \tag{3.66a}$$

$$\tilde{c}(x, y) = c\left(x, y + \tfrac{1}{2}\Delta y\right). \tag{3.66b}$$

Define

$$\begin{aligned} \tilde{A}_h V = &-\left[(\tilde{a}V_x)_{\bar{x}} + (bV_{\hat{x}})_{\hat{y}} + \left(bV_{\hat{y}}\right)_{\hat{x}} + \left(\tilde{c}V_y\right)_{\bar{y}}\right] \\ &+\left(d + \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right)V_{\hat{x}} + \left(e + \frac{\partial c}{\partial y} + \frac{\partial b}{\partial x}\right)V_{\hat{y}} + \sigma V. \end{aligned} \tag{3.67}$$

Case 3. The differential operator A may also be represented as

$$\begin{aligned} Au = &-\left[\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(b\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial y}{\partial x}\right) + \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)\right] \\ &+\frac{1}{2}\left[\hat{d}\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(\hat{d}u)\right] + \frac{1}{2}\left[\hat{e}\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}(\hat{e}u)\right] + \hat{\sigma}u, \end{aligned} \tag{3.68a}$$

where

$$\hat{d} = \left(d + \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right), \qquad \hat{e} = \left(e\frac{\partial c}{\partial y} + \frac{\partial b}{\partial x}\right), \tag{3.68b}$$

$$\hat{\sigma} = \left[\sigma - \left(\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + 2\frac{\partial^2 b}{\partial x\,\partial y} + \frac{\partial d}{\partial x} + \frac{\partial e}{\partial y}\right)\right]. \tag{3.68c}$$

Let $V \in \mathbf{S}_h$ and define

$$\begin{aligned} \hat{A}_h V = &-\left[(\tilde{a}V_x)_{\bar{x}} + \left(bV_{\hat{y}}\right)_{\hat{x}} + (bV_{\hat{x}})_{\hat{y}} + \left(\tilde{c}V_y\right)_{\bar{y}}\right] \\ &+\tfrac{1}{2}\left[\hat{d}V_{\hat{x}} + (\hat{d}V)_{\hat{x}}\right] + \tfrac{1}{2}\left[\hat{e}V_{\hat{y}} + (\tilde{e}V)_{\hat{y}}\right] + \hat{\sigma}V. \end{aligned} \tag{3.69}$$

Remark. Each of these representations has its advantages. We have already seen the value of the first two in the proof of Lemma 3.14. The representation (3.69) has other advantages. For example, this representation is used in [14] together with the condition $\hat{\sigma} \geq 0$. In that special case

$$\operatorname{Re}\left(\hat{A}_h V, V\right) \geq 0.$$

Since our work only requires the analog of (3.49), we will not make special use of this discretization. However, our analysis treats this representation as well.

Before we discuss the discrete analog of Lemma 3.14, we must first introduce some norms and seminorms on $\mathbf{S}_h$. For every $V \in \mathbf{S}_h$, we set

$$\|V\|_g = \left(\Delta x\, \Delta y \sum_{k=1}^{p} \sum_{j=1}^{l} |V_{kj}|^2\right)^{1/2}, \tag{3.70a}$$

$$|V|_{g,1} = \left[\Delta x\, \Delta y \sum_{j=1}^{l} \sum_{k=0}^{p} (V_x)_{kj}^2 + \Delta x\, \Delta y \sum_{j=0}^{l} \sum_{k=1}^{p} (V_y)_{kj}^2\right]^{1/2}, \tag{3.70b}$$

$$|V|_{g,2} = \left\{\|V_{x\bar{x}}\|_g^2 + 2\|V_{\hat{x}\hat{y}}\|_g^2 + \|V_{y\bar{y}}\|_g^2\right\}^{1/2}, \tag{3.70c}$$

$$\|V\|_{g,1} = \left\{\|V\|_g^2 + |V|_{g,1}^2\right\}^{1/2}, \tag{3.70d}$$

$$\|V\|_{g,2} = \left\{\|V\|_{g,1}^2 + |V|_{g,2}^2\right\}^{1/2}. \tag{3.70e}$$

LEMMA 3.15. *Let $A(h)$ be one of the finite difference operators A_h, $\tilde{A}_h$, or $\hat{A}_h$. Assume there are constants $\tilde{K}_0 = \tilde{K}_0(A) > 0$, $h_0 > 0$ such that for all h, $0 < h \le h_0$, we have*

$$\|V\|_g \le \tilde{K}_0 \|A(h)V\|_g, \tag{3.71}$$

for all $V \in \mathbf{S}_h$. Then there is a constant $\tilde{K}_1 = \tilde{K}_1(A)$ such that

$$\|V\|_{g,2} \le \tilde{K}_1 \|A(h)V\|_g, \qquad 0 < h \le h_0. \tag{3.72}$$

Proof. The proof follows from Lemmas A.1, A.2, and A.3 of the Appendix, which provide the finite-difference analogs of the arguments given in the proof of Lemma 3.14. The complete proof is given in the Appendix. □

Remark. The restriction $0 < h \le h_0$ in the conclusion of (3.72) is not a serious restriction since we generally require that $h \to 0$ in any sequence of successively finer meshes. Thus, there are only a finite number of cases for which $h > h_0$. And, for that finite set of finite-dimensional spaces, the norms $\|\ \|_g$ and $\|\ \|_{g,2}$ are equivalent.

LEMMA 3.16. *Let $A(h)$ be one of the finite difference operators A_h, $\tilde{A}_h$, $\hat{A}_h$. There is a constant $\tilde{K}_2 = \tilde{K}_2(A)$ such that*

$$\|A(h)V\|_g \le \tilde{K}_2 \|V\|_{g,2}. \tag{3.73}$$

Proof. When $A(h) = A_h$ this estimate is immediate because

$$\|V_{\hat{x}}\|_g^2 + \|V_{\hat{y}}\|_g^2 \le |V|_{g,1}^2.$$

For the other operators, this estimate follows from the estimate for A_h and Lemma A.3 (see (A.16) and (A.12)). □

LEMMA 3.17. *Let the boundary conditions for A be Dirichlet boundary conditions, that is, $u = 0$ on the entire boundary, $\partial\Omega$. Let $A(h)$ be one of the finite difference operators A_h, $\tilde{A}_h$, $\hat{A}_h$. Then $A(h)^*$ is a difference approximation to A^* with properties similar to those of A. In particular, if there is a constant $\tilde{K}_0(A)$ such that* (3.71) *holds, then there is a constant $\tilde{K}_0^* = \tilde{K}_0(A^*)$ and*

$$\|V\|_g \leq \tilde{K}_0(A^*)\|A(h)^*V\|_g. \tag{3.74}$$

Further, there are constants $\tilde{K}_1^* = \tilde{K}_1(A^*)$ and $\tilde{K}_2^* = \tilde{K}_2(A^*)$ such that

$$\|V\|_{g,2} \leq \tilde{K}_1^*\|A(h)^*V\|_g, \tag{3.75a}$$

$$\|A(h)^*V\|_g \leq \tilde{K}_2^*\|V\|_{g,2}. \tag{3.75b}$$

Proof. The estimate (3.74) follows immediately from the elementary facts

$$\|A(h)\|_g = \|A(h)^*\|_g,$$
$$\left[A(h)^{-1}\right]^* = \left[A(h)^*\right]^{-1}.$$

Hence, it is only necessary to verify the statement that $A(h)^*$ is a difference approximation to A^* with properties similar to those of $A(h)$. This fact follows from Lemma A.3 (see (A.11) and (A.12)) of the Appendix and the simple observation that (see (3.64))

$$\begin{aligned}\hat{A}(h)^* = &-\left[(\hat{a}V_x)_{\bar{x}} + (bV_{\hat{y}})_{\hat{x}} + (bV_{\hat{x}})_{\hat{y}} + (\hat{C}V_y)_{\bar{y}}\right]\\ &-\tfrac{1}{2}\left[\hat{d}V_{\hat{x}} + (\hat{d}V)_{\hat{x}} + \hat{e}V_{\hat{y}} + (\hat{e}V)_{\hat{y}} + \hat{\sigma}V\right]. \ \square\end{aligned}$$

THEOREM 3.18. *Let A and B be two elliptic operators of the form* (3.45a), (3.45b) (*defined on the unit square* Ω) *with the same boundary conditions. Let $A(h)$ be one of A_h, $\tilde{A}_h$, $\hat{A}_h$, and let $B(h)$ be one of B_h, $\tilde{B}_h$, $\hat{B}_h$. Assume that the hypothesis of Lemma* 3.15 *applies to both $A(h)$ and $B(h)$. Let $\tilde{K}_0(A)$ and $\tilde{K}_0(B)$ be the constants in the assumed estimates* (3.71). *Let $\tilde{K}_1(A)$, $\tilde{K}_1(B)$, $\tilde{K}_2(A)$, and $\tilde{K}_2(B)$ be the constants given by Lemmas* 3.15 *and* 3.16. *Then*

$$\begin{aligned}C\left(A(h)B(h)^{-1}\right) &= \|A(h)B(h)^{-1}\|_g\|B(h)A(h)^{-1}\|_g\\ &\leq \left[\tilde{K}_1(A)\tilde{K}_0(B)\right]\left[\tilde{K}_1(B)\tilde{K}_0(A)\right].\end{aligned} \tag{3.76a}$$

In the special case where A and B satisfy Dirichlet boundary conditions, let $\tilde{K}_1(A^*)$, $\tilde{K}_1(B^*)$, $\tilde{K}_2(A^*)$, and $\tilde{K}_2(B^*)$ be given by Lemmas 3.15, 3.16, *and*

3.17. *Then*

$$C\big(B(h)^{-1}A(h)\big) = \|B(h)^{-1}A(h)\|_g \|A(h)^{-1}B(h)\|_g$$
$$\leq \big[\tilde{K}_1(A^*)\tilde{K}_0(B^*)\big]\big[\tilde{K}_1(B^*)\tilde{K}_0(A^*)\big]. \quad (3.76b)$$

Proof. The estimate (3.76a) follows immediately from Lemmas 3.15 and 3.16. The estimate (3.76b) follows from Lemma 3.17 and the observation

$$\big\{A(h)^*\big[B(h)^{-1}\big]^*\big\}^* = B(h)^{-1}A(h). \ \square$$

4. Concluding Remarks

This work is motivated by the desire to construct a preconditioning strategy that yields bounds like (1.6), (1.8), or (1.10) independent of the mesh parameter h. Theorem 2.12 shows that such a strategy must be based upon an operator that is equivalent, or an operator whose inverse is equivalent in the appropriate norm. Thus, equivalence is a necessary condition.

The discussion of Section 3 shows that while not all uniformly elliptic operators are equivalent, the equivalence classes are quite large. Since the bounds (1.6), (1.8), and (1.10) depend upon certain condition numbers, we use this to establish a measure of the distance between two equivalent operators. Suppose A, B: $\mathbf{W} \to \mathbf{V}$ are one-to-one. If either A and B or A^{-1} and B^{-1} are bounded, then Theorem 2.2 states that $A \doteq_{\mathbf{V}} B$ only if $\mathbf{D}_A = \mathbf{D}_B$. In this case AB^{-1}: $\mathbf{R}_B \to \mathbf{R}_A$ is one-to-one and onto and

$$C_{\mathbf{V}}(AB^{-1}) = \|AB^{-1}\|_{\mathbf{V}} \|BA^{-1}\|_{\mathbf{V}} < \infty. \quad (4.1)$$

Let either A or A^{-1} be bounded and let

$$\mathbf{E}(A) = \{\text{equivalence class of } A \text{ under } \doteq_{\mathbf{V}}\}. \quad (4.2)$$

Within $\mathbf{E}(A)$ we may define the following semi-measure

$$d(A, B) = \log\big(C_{\mathbf{V}}(AB^{-1})\big), \quad (4.3)$$

where

$$C_{\mathbf{V}}(AB^{-1}) = \|AB^{-1}\|_{\mathbf{V}} \|BA^{-1}\|_{\mathbf{V}}.$$

For every A, B, $D \in \mathbf{E}(A)$, we have

(i) $d(A, A) = 0$,

(ii) $d(A, B) = d(B, A)$,

(iii) $d(A, D) \leq d(A, B) + d(B, D)$.

This is a semi-measure because $d(A, B) = 0$ does not imply $A = B$. It does imply, however, that $AB^{-1} = U$, where U: $\mathbf{R}_B \to \mathbf{R}_A$ is a unitary operator.

In this measure $\mathbf{E}(A)$ is unbounded. It is possible to choose B so that $d(A, B)$ is arbitrarily large. This leads to the conclusion that while equivalence may be necessary to yield bounds independent of h, it is by no means sufficient to produce a good preconditioning strategy. One must choose a B close to A in this measure.

Consider a class of operators $\mathbf{G}$. Suppose a subset $\mathbf{F} \subseteq \mathbf{G}$ can be identified such that for $B \in \mathbf{F}$ equations of the type $Bu = f$ are easily solved. Given $A \in \mathbf{G}$, we seek $B \in \mathbf{F}$ that satisfies

$$\min_{B \in \mathbf{F}} d(A, B) \tag{4.4}$$

if bounds of type (1.10) are sought. (Alternatively, we may seek $B \in \mathbf{F}$ that satisfies

$$\min_{B \in \mathbf{F}} d(A^{-1}, B^{-1}) \tag{4.5}$$

if bounds of type (1.8) are sought.) If $\mathbf{E}(A) \cap \mathbf{F}$ is not empty, then bounds independent of h can be established. The overall effectiveness of this strategy for the class $\mathbf{G}$ can be measured by the maximum distance from $\mathbf{F}$ to $\mathbf{G}$; that is,

$$d(\mathbf{F}, \mathbf{G}) = \max_{A \in G} \min_{B \in F} d(A, B). \tag{4.6}$$

We remark that this strategy was used in Bank [4] to motivate an algorithm for preconditioning the linear systems arising from a finite difference approximation to a second-order, nonseparable, self-adjoint, elliptic operator on a rectangle. In this work, the preconditioning consists of a diagonal scaling followed by the inverse of a separable operator. The scaling and separable operator are chosen to minimize bounds on the condition of the preconditioned system.

As a final remark, we note that most incomplete LU factorization techniques yield preconditionings $\mathbf{B}_h$ that are known to *not* be uniformly equivalent in norm to the associated discretizations $\mathbf{A}_h$ because the condition of $\mathbf{A}_h \mathbf{B}_h^{-1}$ is not uniformly bounded (cf. van der Vorst [32]). We conjecture that the inverses $\mathbf{B}_h^{-1}$ do not converge to the inverse of any elliptic second-order differential operator. Efforts to find sparse LU factorizations that yield uniformly equivalent families for the Laplace operator lead to the discrete analogue of the Cauchy–Riemann equations and difficulties with stability (cf. Hyman and Manteuffel [22], Liniger [24]).

APPENDIX

In this Appendix, we develop the technical tools to prove Lemmas 3.14 and 3.15. The basic idea is clear enough. Suppose $A(h)$ is the "principal part" of A_h as given by (3.65); that is, $A(h) = A_{0,h}$, where

$$A_{0,h}V = -\left[aV_{x\bar{x}} + 2bV_{\hat{x}\hat{y}} + cV_{y\bar{y}}\right]. \tag{A.1}$$

Then, as in the analytic case, we obtain an estimate on $|V|_{g,2}$ similar to (3.51). On the other hand, suppose $A(h)$ is the "principal part" of $\tilde{A}_h$ as given by (3.41); that is, $A(h) = A_{1,h}$, where

$$A_{1,h}V = -\left[(\tilde{a}V_x)_{\bar{x}} + (bV_{\hat{x}})_{\hat{y}} + (bV_{\hat{y}})_{\hat{x}} + (\tilde{c}V_y)_{\bar{y}}\right]. \tag{A.2}$$

Then, as in the analytic case, summation by parts (instead of integration by parts) yields an estimate on $|V|_{g,1}$. Finally, lower order terms are estimated by making use of the basic inequality

$$\alpha\beta \le \frac{1}{2}\left[\frac{\alpha^2}{\varepsilon^2} + \varepsilon^2\beta^2\right] \tag{A.3}$$

for any $\varepsilon \neq 0$. Thus, the essential point is to be able to represent one "principal part" in terms of the other and lower order terms. However, first we state the precise results for the two principal parts.

LEMMA A.1. *Let $V \in \mathbf{S}_h$. Let $A_{0,h}$ be given by* (A.1), *then*

$$|V|_{g,2}^2 \le \frac{2Q^2}{q^4}\|A_{0,h}V\|_g^2. \tag{A.4}$$

Proof. This result is proved in Nitsche and Nitsche [28] for the case where $\mathbf{S}_h$ is described by homogeneous Dirichlet boundary conditions,

$$V = 0 \qquad \text{on } \partial\Omega_h.$$

In the general case, we follow the argument of [28] and prove that, under the more general boundary conditions, summation by parts yields

$$\sum_{k=1}^{p}\sum_{j=1}^{l}(V_{\bar{x}\bar{y}})_{k,j}^2 = \sum_{k=1}^{p}\sum_{j=1}^{l}(V_{x\bar{x}})_{k,j}(V_{y\bar{y}})_{k,j}.$$

Then following the argument of [28] we find that

$$\Delta x\,\Delta y\sum_{k=1}^{p}\sum_{j=1}^{l}\left[(V_{x\bar{x}})(V_{y\bar{y}}) - (V_{\hat{x}\hat{y}})^2\right]_{k,j} \ge 0. \tag{A.5}$$

The estimate (A.4) now follows easily. For example, let us bound $\|V_{x\bar{x}}\|_g$ and $\|V_{\hat{x}\hat{y}}\|_g$. Multiply (A.1) by $(1/c)V_{x\bar{x}}$ to obtain

$$\left[(V_{x\bar{x}})(V_{y\bar{y}}) - (V_{\hat{x}\hat{y}})^2\right] + \frac{1}{c}\left[a(V_{x\bar{x}})^2 + 2bV_{x\bar{x}}V_{\hat{x}\hat{y}} + c(V_{\hat{x}\hat{y}})^2\right]$$
$$= -\frac{1}{c}[A_{0,h}V][V_{x\bar{x}}].$$

Using (3.48), we have

$$\left[(V_{x\bar{x}})(V_{y\bar{y}}) - (V_{\hat{x}\hat{y}})^2\right] + \frac{q}{Q}\left[(V_{x\bar{x}})^2 + (V_{\hat{x}\hat{y}})^2\right] \le -\frac{1}{c}[A_{0,h}V][V_{x\bar{x}}].$$

We multiply by $\Delta x\,\Delta y$ and sum over the mesh. Using (A.5) and (A.3), we obtain

$$\|V_{x\bar{x}}\|_q^2 + \|V_{\hat{x}\hat{y}}\|_g^2 \le \frac{Q}{2q^2}\left[\frac{1}{\varepsilon^2}\|A_{0,h}V\|_g^2 + \varepsilon^2\|V_{x\bar{x}}\|_g^2\right].$$

Setting $\varepsilon = q^2/Q$ yields

$$\|V_{x\bar{x}}\|_g^2 + 2\|V_{\hat{x}\hat{y}}\|_g \le \frac{Q^2}{q^4}\|A_{0,h}V\|_g^2.$$

Repeating the argument above but now multiplying through by $(1/a)V_{y\bar{y}}$ yields

$$\|V_{y\bar{y}}\|_g^2 + 2\|V_{\hat{x}\hat{y}}\|_g^2 \le \frac{Q^2}{q^4}\|A_{0,h}V\|_g^2.$$

Adding these last two inequalities yields the result. □

LEMMA A.2. *Let $V \in \mathrm{S}_h$. Let $A_{1,h}$ be given by* (A.2). *Then, there is a constant $h_0 > 0$ such that, for $0 < h \le h_0$, we have*

$$|V|_{g,1}^2 \le \frac{2}{q}\left[\Delta x\,\Delta y \sum_{j,k}(A_{1,h}V)_{k,j}V_{k,j}\right]. \tag{A.6}$$

Proof. This result is contained in Lemma 3.4 of [5].

LEMMA A.3. *Let $\tilde{a}(x, y)$ and $\tilde{c}(x, y)$ be given by* (3.76a), (3.76b). *Let*

$$\bar{a}_{k,j} = \frac{1}{\Delta x}[a_{k,j} - a_{k-(1/2),j}], \qquad \hat{a}_{kj} = \frac{1}{\Delta x}[a_{k+(1/2),j} - a_{k,j}], \tag{A.7a}$$

$$\bar{c}_{k,j} = \frac{1}{\Delta y}[c_{k,j} - c_{k,j-1/2}], \qquad \hat{c}_{kj} = \frac{1}{\Delta y}[c_{k,j+1/2} - c_{k,j}]. \tag{A.7b}$$

Then

$$(\tilde{a}V_x)_{\bar{x}} = aV_{x\bar{x}} + \bar{a}V_{\bar{x}} + \hat{a}V_x, \tag{A.8a}$$

$$(\tilde{c}V_y)_{\bar{y}} = cV_{y\bar{y}} + \bar{c}V_{\bar{y}} + \hat{c}V_y, \tag{A.8b}$$

and

$$(bV_{\hat{y}})_{\hat{x}} = bV_{\hat{y}\hat{x}} + \tfrac{1}{2}b_xT_xV_{\hat{y}} + \tfrac{1}{2}b_xT_x^{-1}V_{\hat{y}}, \tag{A.9a}$$

$$(bV_{\hat{x}})_{\hat{y}} = bV_{\hat{x}\hat{y}} + \tfrac{1}{2}b_yT_yV_{\hat{x}} + \tfrac{1}{2}b_yT_y^{-1}V_{\hat{x}}. \tag{A.9b}$$

Thus,

$$A_{1,h}V = A_{0,h}V - \left[\hat{a}V_x + \bar{a}V_{\bar{x}} + \tfrac{1}{2}b_yT_yV_{\hat{x}} + \tfrac{1}{2}b_yT_y^{-1}V_{\hat{x}}\right] - \left[\hat{c}V_y + \bar{c}V_{\bar{y}} + \tfrac{1}{2}b_xT_xV_{\hat{y}} + \tfrac{1}{2}b_xT_x^{-1}V_{\hat{y}}\right], \tag{A.10}$$

and there are linear operators $\tilde{E}_h$ and $\hat{E}_h$ defined on S_h and constants $\tilde{E}$ and $\hat{E}$ such that

$$\tilde{A}_hV = A_hV + \tilde{E}_hV, \tag{A.11a}$$

$$\hat{A}_hV = A_hV + \hat{E}_hV, \tag{A.11b}$$

$$\|\tilde{E}_hV\|_g \le \tilde{E}|V|_{g,1}, \tag{A.12a}$$

$$\|\hat{E}_hV\|_g \le \hat{E}\left[|V|_{g,1} + \|V\|_g\right]. \tag{A.12b}$$

The constant $\tilde{E}$ depends only on the coefficients a, b, c, d, e and their first derivatives, while the constant $\hat{E}$ depends only on the coefficients a, b, c, d, e and their first and second derivatives.

Proof. The identities (A.8) and (A.9) follow from a simple computation based on the definitions (3.62a)–(3.62c). The identities (A.10) and (A.11) follow immediately from the definitions. Finally, the bounds (A.12) follow immediately from the formulae for $\tilde{a}$, $\hat{a}$, $\bar{a}$, etc. □

We are now in a position to prove Lemma 3.15.

Proof of Lemma 3.15. Let $A(h) = \tilde{A}_h$. Then

$$A_{1,h}V = \tilde{A}_hV - \hat{d}V_{\hat{x}} - \hat{e}V_{\hat{y}} - \sigma V. \tag{A.13}$$

Multiply by V and sum over the grid. Using (A.6) of Lemma A.2, we have

$$|V|_{g,1}^2 \le \frac{2}{q}\left[\|\tilde{A}_hV\|_g \cdot \|V\|_g + \|\hat{d}\|_\infty \|V_{\hat{x}}\|_g \|V\|_g + \|\hat{e}\|_\infty \|V_{\hat{y}}\|_g \|V\|_g + \|\sigma\|_\infty \|V\|_g^2\right]. \tag{A.14}$$

Using (A.3) and (3.71), we obtain

$$|V|_{g,1}^2 \le K_3 \|\tilde{A}_h V\|^2, \tag{A.15}$$

where the constant K_3 depends on $\tilde{K}_0(A)$ and the coefficients a, b, c, d, e, σ and their first derivatives. Using (A.11a), we have

$$A_h V = \tilde{A}_h V - \tilde{E}_h V.$$

Hence

$$A_{0,h} V = \tilde{A}_h V - \tilde{E}_h V - dV_{\hat{x}} - eV_{\hat{y}} - \sigma V. \tag{A.16}$$

If we take the norm of both sides of (A.16), we have

$$\|A_{0,h}V\|_g \le \|\tilde{A}_h V\|_g + \|\tilde{E}_h V\|_g + \|d\|_\infty \|V_{\hat{x}}\|_g + \|e\|_\infty \|V_{\hat{y}}\|_g + \|\sigma\|_\infty \|V\|_g. \tag{A.17}$$

Using (A.12a), (A.15), and (3.71), we see that

$$\|A_{0,h}V\|_g \le K_4 \|\tilde{A}_h V\|_g$$

for some constant K_4 that depends only on the coefficients of A and their derivatives. Applying Lemma A.1, we have

$$|V|_{g,2}^2 \le \frac{2Q^2}{q^4} K_4^2 \|\tilde{A}_h V\|_g^2. \tag{A.18}$$

Together with (A.15) and (3.71), we have established (3.72) for $\tilde{A}_h$. The other cases follow in a similar way. □

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