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Oscillatory Effects of Retarded Actions

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1. INTRODUCTION

An interesting question in the study of retarded differential equations (R.D.E. for short), whose answer is of great practical importance, is the following: When does the presence of a delay make the solutions of an R.D.E. qualitatively different from those of the corresponding equation without delay?

Here we are concerned with the following more specific question: When does the introduction of a delay change the oscillatory character of an ordinary differential equation? This question is extremely difficult. Since the delays are very crucial in this case, one has to invent techniques sensitive to the presence of delays. For some contributions in this direction the reader is referred to Kamenskii [3], Ladas and Lakshmikantham [4], Wong [9], Gustafson [2], Ladas, Ladde, and Papadakis [5], Papadakis [7], Ladas, Lakshmikantham, and Papadakis [6], and Sficas and Staikos [8].

In this paper we obtain sufficient conditions under which the R.D.E.

$$y^{(2n)}(t) - p(t)\,y(t-\tau) = 0, \qquad n \geqslant 1, \tag{1}$$

where $p \in C[[0, \infty), R_+]$ and $\tau > 0$, has oscillatory solutions. Our results generalize the results in [7] and are in the spirit of the results in [6]. Although the hypotheses under which we prove oscillations here are, in general, incompatible with those in [6] when $p(t)$ is constant, our hypotheses are weaker than those in [6].

For simplicity, the delay τ is assumed to be constant. As is customary, a solution is said to be oscillatory if it has arbitrarily large zeros.

2.

For the sake of clarity we first develop a series of lemmas about solutions of the R.D.E.

$$y^{(n)}(t) + (-1)^{n+1}\,p(t)\,y(t-\tau) = 0, \qquad n \geqslant 1, \tag{2}$$

where $p(t) \in C[[0, \infty), R_+]$, $p(t) \not\equiv 0$, and τ is a positive constant.

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LEMMA 1. *If* $y(t)$ *is a solution of* (2), *then for all* $t, s \in [0, \infty)$

$$y(s) = \sum_{k=0}^{n-1} \frac{y^{(k)}(t)}{k!} (s - t)^k + \frac{1}{(n-1)!} \int_s^t (u - s)^{n-1} p(u)\, y(u - \tau)\, du. \tag{3}$$

Proof. Applying Taylor's formula with remainder (see [1, p. 157]) to the function $y(s)$, and using (2), we get

$$\begin{aligned} y(s) &= \sum_{k=0}^{n-1} \frac{y^{(k)}(t)}{k!} (s - t)^k + \frac{1}{(n-1)!} \int_t^s (s - u)^{n-1} y^{(n)}(u)\, du \\ &= \sum_{k=0}^{n-1} \frac{y^{(k)}(t)}{k!} (s - t)^k + \frac{1}{(n-1)!} \int_t^s (s - u)^{n-1} (-1)^n p(u)\, y(u - \tau)\, du \\ &= \sum_{k=0}^{n-1} \frac{y^{(k)}(t)}{k!} (s - t)^k + \frac{1}{(n-1)!} \int_s^t (u - s)^{n-1} p(u)\, y(u - \tau)\, du. \end{aligned}$$

LEMMA 2. *If* $y(t)$ *is a nonnegative and nonincreasing solution of* (2), *then for* $t - \tau \leqslant s \leqslant t$ *the following inequality holds:*

$$y(s) \left[1 - \frac{1}{(n-1)!} \int_s^t (u - s)^{n-1} p(u)\, du\right] \geqslant \sum_{k=0}^{n-1} \frac{y^{(k)}(t)}{k!} (s - t)^k. \tag{4}$$

Proof. If $s \leqslant u \leqslant t$, then $s - \tau \leqslant u - \tau \leqslant t - \tau \leqslant s$ and therefore $y(u - \tau) \geqslant y(s)$. Using this inequality in (3), inequality (4) follows immediately.

LEMMA 3. *If* $y(t)$ *is a bounded and nonoscillatory solution of* (2), *then for sufficiently large* t

$$y^{(k)}(t)\, y^{(k+1)}(t) \leqslant 0, \qquad 0 \leqslant k \leqslant n - 1. \tag{5}$$

Proof. Let $y(t)$ be a bounded and nonoscillatory solution of (2). Without loss of generality assume that $y(t) > 0$ for $t \geqslant t_0$. Then, from (2), it follows that

$$(-1)^n y^{(n)}(t) \geqslant 0, \qquad t \geqslant t_0 + \tau. \tag{6}$$

Using (6) and the fact that if $f^{(k)}(t) f^{(k+1)}(t) \geqslant 0$ (but f is not identically constant) for $k \geqslant 1$ then $\lim_{t\to\infty} |f(t)| = +\infty$, we conclude, in view of the boundedness of $y(t)$, that there exists a $t_1 \geqslant t_0$ such that

$$(-1)^k y^{(k)}(t) \geqslant 0, \qquad t \geqslant t_1 \qquad \text{for } k = 0, 1, \dots, n.$$

Therefore, for $t \geqslant t_1$ and $0 \leqslant k \leqslant n - 1$,

$$y^{(k)}(t)\, y^{(k+1)}(t) = -(-1)^k y^{(k)}(t) \cdot (-1)^{k+1} (t) \leqslant 0$$

and the proof is complete.

The following oscillatory result about Eq. (2) is an immediate consequence of Lemma 2 (set $s = t - \tau$ in (4)) and Lemma 3.

COROLLARY 1. *Assume that*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} (u - t + \tau)^{n-1} p(u)\, du > (n-1)!. \tag{7}$$

Then every bounded solution of Eq. (2) *is oscillatory.*

LEMMA 4. *If $y(t)$ is a nonnegative and nonincreasing solution of* (2), *then for $t - \tau \leqslant s \leqslant t$ the following inequality holds:*

$$\begin{aligned}
(-1)^n y^{(n-1)}(t+\tau) &\geqslant \frac{(-1)^{n-1} y^{(n-1)}(t)}{(n-1)!} \\
&\times \left[\int_{t-\tau}^{t} [(t-s)^{n-1} p(s+\tau) - (-1)^n (t-s-\tau)^k p(s)]\, ds - (n-1)!\right] \\
&+ \sum_{k=0}^{n-2} \frac{(-1)^k y^{(k)}(t)}{k!} \int_{t-\tau}^{t} [(t-s)^k p(s+\tau) - (-1)^n (t-s-\tau)^k p(s)]\, ds \\
&+ (-1)^n \int_{t-\tau}^{t} y(s+\tau)\, p(s)\, ds.
\end{aligned} \tag{8}$$

Proof. Multiplying both side of (4) by $p(s+\tau)$ and using (2) we obtain

$$\begin{aligned}
(-1)^n y^{(n)}(s+\tau) &\left[1 - \frac{1}{(n-1)!}\int_{s}^{t} (u-s)^{n-1} p(u)\, du\right] \\
&\geqslant \sum_{k=0}^{n-1} \frac{y^{(k)}(t)}{k!} (s-t)^k p(s+\tau).
\end{aligned} \tag{9}$$

Set

$$F(s) = 1 - \frac{1}{(n-1)!}\int_{s}^{t} (u-s)^{n-1} p(u)\, du.$$

Then,

$$F^{(k-1)}(s) = \frac{(-1)^n}{(n-k)!}\int_{s}^{t} (s-u)^{n-k} p(u)\, du, \qquad k = 2, 3, \ldots, n \tag{10}$$

and

$$F^{(n)}(s) = (-1)^{n-1} p(s). \tag{11}$$

Integration by parts n times yields

$$\int_{t-\tau}^{t} y^{(n)}(s-\tau)\,F(s)\,ds$$
$$= \sum_{k=1}^{n} (-1)^k\, y^{(n-k)}(s+\tau)\, F^{(k-1)}(s)\,\Big|_{t-\tau}^{t} + (-1)^n \int_{t-\tau}^{t} y(s+\tau)\,F^{(n)}(s)\,ds. \tag{12}$$

Now inequality (8) follows by integrating both sides of (9) with respect to s from $t-\tau$ to t and by using (12), (10), and (11).

3. Oscillations of Eq. (1)

THEOREM 1. *Assume that for every $t_0 > 0$ there exists $t > t_0$ such that the following two hypotheses are satisfied.*

$$\int_{t-\tau}^{t} [(t-s)^{2n-1}\,p(s+\tau) - (t-s-\tau)^{2n-1}\,p(s)]\,ds \geqslant (2n-1)!. \tag{H_1}$$

$$\int_{t-\tau}^{t} [(t-s)^{k}\,p(s+\tau) - (t-s-\tau)^{k}\,p(s)]\,ds \geqslant 0 \qquad \text{for } k = 0, 1, 2,\ldots, 2n-2. \tag{H_2}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (1). Without loss of generality we assume that $y(t) > 0$ for $t \geqslant t_0$. Then by Lemma 3 there is a $t_1 \geqslant t_0$ such that

$$(-1)^k\,y^{(k)}(t) \geqslant 0 \qquad \text{for } t \geqslant t_1 \quad \text{and} \quad k = 1, 2,\ldots, 2n.$$

Now replacing n by $2n$ in Lemma 4, we obtain

$$y^{(2n-1)}(t+\tau)$$
$$\geqslant -\frac{y^{(2n-1)}(t)}{(2n-1)!}\left[\int_{t-\tau}^{t} [(t-s)^{2n-1}\,p(s+\tau) - (t-s-\tau)^{2n-1}\,p(s)]\,ds - (2n-1)!\right]$$
$$+ \sum_{k=0}^{2n-2} \frac{(-1)^k\,y^{(k)}(t)}{k!} \int_{t-\tau}^{t} [(t-s)^{k}\,p(s+\tau) - (t-s-\tau)^{k}\,p(s)]\,ds$$
$$+ \int_{t-\tau}^{t} y(s+\tau)\,p(s)\,ds. \tag{13}$$

In view of hypotheses (H_1) and (H_2) there exists a $t > t_1$ such that the right-hand side of (13) is positive while its left-hand side is nonpositive. This contradiction completes the proof.

Remark 1. When Eq. (1) is of second order and $p(t)$ is a nonnegative τ-periodic function of t (in particular, a positive constant) or a positive and increasing function of t, then hypothesis (H_2) of Theorem 1 is automatically satisfied and hypothesis (H_1) is weaker than [5, (3.7)].

Remark 2. If $p(t)$ is a positive constant p, then hypothesis (H_2) of Theorem 1 is automatically satisfied and (H_1) becomes

$$2p\tau^{2n} \geqslant (2n)!$$

which is weaker than $p\tau^{2n} \geqslant (2n)!$, which is the corresponding hypothesis of [6, Theorem 2.1].

So far we have proved that under hypotheses (H_1) and (H_2) all bounded solutions of Eq. (1) are oscillatory. This result enables us to show now that Eq. (1) does indeed have oscillatory solutions. We need some notation first. It is well known that the R.D.E. (1) together with the initial conditions

$$y(t) = \phi(t), \quad 0 \leqslant t \leqslant \tau; \qquad y^{(i)}(\tau) = y_i\,, \quad i = 1, 2,..., 2n - 2 \tag{14}$$

and

$$y^{(2n-1)}(\tau) = A \tag{15}$$

where $\phi \in C[[0, \tau], R]$, and y_i, $A \in R$, $i = 1, 2,..., 2n - 2$, has a unique solution on $[\tau, \infty)$. The initial function ϕ and the $2n - 2$ constants $y_1, y_2,..., y_{2n-2}$ are assumed to be given but fixed for the remainder of this section. However, we allow A to vary in R. For each $A \in R$, the unique solution of (1), (14), and (15) is denoted by $y(t, A)$.

Let us introduce the following subsets of the real line R:

$$K^{-\infty} = \{A \in R: \lim_{t\to\infty} y(t, A) = -\infty\}$$

$$K^{+\infty} = \{A \in R: \lim_{t\to\infty} y(t, A) = +\infty\}$$

$$K^{0} = \{A \in R: \lim_{t\to\infty} y(t, A) = 0\}$$

$$K^{\sim} = \{A \in R: y(t, A) \text{ oscillates}\}.$$

Then, under the additional hypothesis

$$\int_0^\infty p(t)\,dt = +\infty, \tag{16}$$

we have (see [6])

$$R = K^{-\infty} \cup K^{+\infty} \cup K^0 \cup K^{\sim}$$

and there exist real numbers a and b with $a \leqslant b$ such that

$$K^{-\infty} = (-\infty, a), \qquad K^{+\infty} = (b, +\infty).$$

Thus, clearly,

$$K^0 \cup K^{\sim} \neq \varnothing .$$

But under the hypotheses of Theorem 1, $K^0 \subset K^{\sim}$, thus $K^{\sim} \neq \varnothing$ and in fact for every $A \in [a, b]$ the solution $y(t, A)$ of (1), (14), and (15) is oscillatory. In summary, we have proved the following result:

THEOREM 2. *Assume that* (H_1), (H_2), *and* (16) *are satisfied. Then for every given initial function* $\phi \in C[[0, \tau], R]$ *and any given set of* $n - 2$ *real constants* $y_1, y_2, \ldots, y_{2n-2}$ *there exists* $A \in R$ *such that* $y(t, A)$ *is oscillatory.*

We close this section by stating two interesting open problems.

OPEN PROBLEM 1. Under what conditions on $p(p(t) > 0)$, τ, ϕ, and $y_1, y_2, \ldots, y_{2n-2}$ is the set K^0 nonempty?

If such conditions could be found and if (16) holds then the system (1) and (14) would have absolutely no oscillatory solution for any choice of $A \in R$, i.e., the delay τ in this case would be literally "harmless."

OPEN PROBLEM 2. Under what conditions on p $(p(t) > 0)$, τ, ϕ, and $y_1, y_2, \ldots, y_{2n-2}$ does the set $K^0 \cup K^{\sim}$ have more than one element?

If such conditions could be found and if (16) holds then the system (1) and (14) would have an oscillatory solution for every $A \in [a, b]$ where $a < b$, i.e., the delay in this case would be literally "harmful."

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