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Retract rational fields ☆

Ming-chang Kang

Department of Mathematics and Taida Institute of Mathematical Sciences, National Taiwan University, Taipei, Taiwan

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ABSTRACT

Let k be an infinite field. The notion of retract k-rationality was introduced by Saltman in the study of Noether's problem and other rationality problems. We will investigate the retract rationality of a field in this paper. Theorem 1: Let $k \subset K \subset L$ be fields. If K is retract k-rational and L is retract K-rational, then L is retract k-rational. Theorem 2: For any finite group G containing an abelian normal subgroup H such that G/H is a cyclic group, for any complex representation $G \to GL(V)$, the fixed field $\mathbb{C}(V)^G$ is retract $\mathbb{C}$-rational. Theorem 3: If G is a finite group, then all the Sylow subgroups of G are cyclic if and only if $\mathbb{C}_\alpha(M)^G$ is retract $\mathbb{C}$-rational for all G-lattices M, for all short exact sequences $\alpha : 0 \to \mathbb{C}^\times \to M_\alpha \to M \to 0$. Because the unramified Brauer group of a retract $\mathbb{C}$-rational field is trivial, Theorems 2 and 3 generalize previous results of Bogomolov and Barge respectively (see Theorems 5.9 and 6.1).

1. Introduction

Let k be a field, and L be a finitely generated field extension of k. L is called k-rational (or rational over k) if L is purely transcendental over k, i.e. L is isomorphic to some rational function field over k. L is called stably k-rational if $L(y_1, \dots, y_m)$ is k-rational for some $y_1, \dots, y_m$ which are algebraically independent over L. L is called k-unirational if L is k-isomorphic to a subfield of some k-rational field extension of k. It is easy to see that "k-rational" $\Rightarrow$ "stably k-rational" $\Rightarrow$ "k-unirational".

Let G be a finite group acting on the rational function field $k(x_g\colon\ g \in G)$ by k-automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Denote by $k(G)$ the fixed subfield, i.e. $k(G) = k(x_g\colon\ g \in G)^G$. Noether's problem asks, under what situation, the field $k(G)$ is k-rational.

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E-mail address: kang@math.ntu.edu.tw.

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Note that, if k is an infinite field and $k(G)$ is k-rational (resp. stably k-rational), then there exists a generic G-Galois extension over k [Sa2, Theorem 5.1]. On the other hand, when Hilbert's irreducibility theorem is valid for k (e.g. if k is any algebraic number field), it is not difficult to see that the existence of a generic G-Galois extension over k implies that there is a Galois field extension K over k such that $\mathrm{Gal}(K/k) \simeq G$, i.e. the inverse Galois problem for the pair (k, G) is solvable (see, for example, [Sw1, Theorem 3.3]). In the study of generic Galois extensions and generic division algebras, Saltman was led to the notion of retract k-rationality [Sa1,Sa4], which is the main subject of this paper.

Definition 1.1. (See [Sa1, p. 130], [Sa4, Definition 3.1].) Let k be an infinite field and L be a field containing k. L is called retract k-rational, if there are some affine domain A over k and k-algebra morphisms $\varphi : A \to k[X_1, \dots, X_n][1/f]$, $\psi : k[X_1, \dots, X_n][1/f]$ where $k[X_1, \dots, X_n]$ is a polynomial ring over k, $f \in k[X_1, \dots, X_n]\backslash\{0\}$, satisfying that

(i) L is the quotient field of A, and
(ii) $\psi \circ \varphi = 1_A$, the identity map on A.

In the above definition of retract k-rationality, it is required that k is an infinite field because this assumption guarantees the existence of sufficiently many k-specializations when we apply the notion of retract rationality to other concepts or problems. Here is a geometric picture of retract rationality. Suppose that L is retract k-rational over k. Then there are quasi-projective varieties V and W defined over k, a dominating k-morphism $p : V \to W$ satisfying that $k(W) = L$, $k(V)$ is k-rational and p has a section, i.e. a k-morphism $s : W \to V$ with $ps = 1_W$.

Another related notion is discussed by Colliot-Thélène and Sansuc [CTS2]. A field L over k is called a direct factor of a k-rational field if there is a field L' over k such that the quotient field of $L \otimes_k L'$ is k-rational (in particular, the k-algebra $L \otimes_k L'$ is an integral domain). It is known that, if L is the function field of some algebraic torus T over k, then L is retract k-rational if and only if it is a direct factor of some k-rational field [CTS2, Proposition 7.4].

Return to Noether's problem.

Theorem 1.2. *(See [Sa2,Sa4,De].) Let k be an infinite field and G be a finite group. The following statements are equivalent*:

(i) *$k(G)$ is retract k-rational.*
(ii) *There is a generic G-Galois extension over k.*
(iii) *There exists a generic G-polynomial over k.*

Proof. (i) $\Leftrightarrow$ (ii) by [Sa2, Theorem 5.3], [Sa4, Theorem 3.12]. The equivalence of (i), (ii), (iii) was proved in [De,DM]. $\square$

It is not difficult to verify that, if k is an infinite field, then "k-rational" $\Rightarrow$ "stably k-rational" $\Rightarrow$ "retract k-rational" $\Rightarrow$ "k-unirational". Thus, if $k(G)$ is not retract k-rational, then $k(G)$ is not stably k-rational (and is not k-rational, in particular). This is the strategy for showing that $\mathbb{C}(G)$ is not $\mathbb{C}$-rational for some group G of order p^9 by Saltman in [Sa3] (where p is any prime number). On the other hand, if $k(G)$ is k-rational, then $k(G)$ is retract k-rational.

We remark that the direction of the implication "rational" $\Rightarrow$ "stably rational" $\Rightarrow$ "retract rational" $\Rightarrow$ "k-unirational" cannot be reversed. There is a field extension L of $\mathbb{C}$ such that L is stably $\mathbb{C}$-rational, but not $\mathbb{C}$-rational [BCTSSD]. If C_p denotes the cyclic group of order p, then $\mathbb{Q}(C_p)$ is retract $\mathbb{Q}$-rational, but not stably $\mathbb{Q}$-rational when $p = 47$, 113 or 233, etc. (see Theorem 3.7 and the remark after its proof). $\mathbb{Q}(C_8)$ is $\mathbb{Q}$-unirational, but not retract $\mathbb{Q}$-rational (see Theorem 2.9); for finitely generated field extensions over $\mathbb{C}$ which are $\mathbb{C}$-unirational, but not retract $\mathbb{C}$-rational, see [Sa3,Bo,CHKK]. On the other hand, we don't know whether there is a field extension L of $\mathbb{C}$ such that L is retract $\mathbb{C}$-rational, but is not stably $\mathbb{C}$-rational. The reader is referred to the papers [MT,CTS3] for surveys of the rationality problems, and to Swan's paper [Sw1] for Noether's problem.

In this paper, we will prove a transitivity theorem for retract rationality in Theorem 4.2. Then we will show that $\mathbb{C}(V)^G$ is retract $\mathbb{C}$-rational where $G \to GL(V)$ is any complex representation and G is a finite group containing an abelian normal subgroup H such that G/H is a cyclic group (see Theorem 5.10). Because of Theorem 3.2, Theorem 5.10 may be regarded as a generalization of a result of Bogomolov (see Theorem 5.9). Finally we will show that, if G is a finite group, then all the Sylow subgroups of G are cyclic if and only if $\mathbb{C}_\alpha(M)^G$ is retract $\mathbb{C}$-rational for all G-lattices M, for all short exact sequences $\alpha: 0 \to \mathbb{C}^\times \to M_\alpha \to M \to 0$. This result generalizes a theorem of Barge (see Theorem 6.1).

An application of the transitivity theorem is Theorem 5.4, which asserts that $k(G)$ is retract k-rational is equivalent to the retract k-rationality of $k(M)^G$ where M is any faithful G-lattice with $[M]^{fl}$ invertible. We remark that Theorems 2.9 and 3.7, due to Voskresenskii and Saltman respectively, are of interest themselves. The proofs of these two theorems are included for the convenience of the reader.

We remark that there is a notion, called the property $\mathrm{Rat}(G/k)$ by Serre [GMS, p. 86], which is slightly stronger than the existence of a generic G-Galois extension over k. We define it as follows.

Definition 1.3. (See [GMS, pp. 11, 86].) Let k be an infinite field and G be a finite group. We say that the property $\mathrm{Rat}(G/k)$ holds for the pair (G,k), if there exists a versal G-torsor over L where L is some k-rational field extension.

In order to explain this property, we define first the notion of a G-Galois covering.

Definition 1.4. (See [Mi1, p. 43], [Mi2, p. 41], [Sw1, Proposition 2.1].) Let G be a finite group. Let $R \subset S$ be commutative rings such that the group G acts on S by R-automorphisms of S with $R = S^G$ where S^G is the ring of invariants of S under the action of G. We say that S is a Galois covering of R with group G (for short, S is a G-Galois covering of R), if the morphism $h: S \otimes_R S \to \prod_{\sigma \in G} S$ defined below is an isomorphism where we define $h(s_1 \otimes s_2) = (\dots, h_\sigma(s_1 \otimes s_2), \dots)_\sigma \in \prod_{\sigma \in G} S$ with $h_\sigma(s_1 \otimes s_2) = s_1 \cdot \sigma(s_2)$ (i.e. the σ-th coordinate of $h(s_1 \otimes s_2)$ is $s_1 \cdot \sigma(s_2)$). We also say that $Spec(S) \to Spec(R)$ is a G-Galois covering if S is a G-Galois covering of R.

The above definition can be globalized. Namely, when V, W are schemes or algebraic varieties defined over a field k and $V \to W$ is a faithfully flat morphism, we can define by the similar way the notion that $V \to W$ is a G-Galois covering.

A G-Galois covering $V \to W$ is nothing but a G-torsor of W, i.e. a principal homogeneous space over W under G ([Mi2, Example 11.3, p. 76], [Mi1, pp. 120 and 43–44]). If $R \subset S$ are commutative rings, the notion that S is a G-Galois covering of R defined here is equivalent to the notion that S is a Galois extension of R with group G in the sense of Galois extensions of commutative rings [Sw1, Proposition 2.1]. Since G is a finite group, the assumption of faithful flatness in [Mi1, p. 43], [Mi2, p. 43] guarantees that the morphism is affine and finite (by the faithfully flat descent [Mi1, p. 20]); when both V and W are affine schemes, the assumption of faithful flatness for $V \to W$ is redundant by [Sw1, Corollary 2.2].

Now we may rephrase Serre's property $\mathrm{Rat}(G/k)$ as follows.

Definition 1.5. Let k be an infinite field and G be a finite group. We say that the property $\mathrm{Rat}(G/k)$ holds for the pair (G,k), if there exists a G-Galois covering $V \to W$ where W is a smooth k-rational variety defined over k satisfying the following condition: For any field k' containing k, any G-Galois covering A of k', any nonempty open subset $U \subset W$, there exists a point $x \in U(k') \subset W$ such that $Spec(A) \simeq V \times_W Spec(k')$ where the fibre product $V \times_W Spec(k')$ is defined via the morphism $Spec(k') \to \{x\} \subset W$.

Here is an affine version. The property $\mathrm{Rat}(G/k)$ holds, if there is a G-Galois covering S of R satisfying that (i) R and S are affine k-algebra, (ii) R is a localized polynomial ring, i.e. $R = k[X_1, \dots, X_n][1/f]$ for some non-zero polynomial f, (iii) for any field k' containing k, any G-Galois covering A of k', any $r \in R \setminus \{0\}$, there is a k-morphism $\phi: R \to k'$ such that $\phi(r) \neq 0$ and $A \simeq S \otimes_\phi k'$.

We claim that, if k is an infinite field and G is a finite group, then "$k(G)$ is stably k-rational" $\Rightarrow$ "the property $\mathrm{Rat}(G/k)$ holds" $\Rightarrow$ "there is a generic G-Galois extension over k".

For the implication "$k(G)$ is stably k-rational" $\Rightarrow$ "the property $\mathrm{Rat}(G/k)$ holds", the same proof of [Sw1, Theorem 4.2] works as well in this situation; in particular, we rely on Kuyk's Lemma, i.e. [Sw1, Lemma 4.5].

As to the implication "the property $\mathrm{Rat}(G/k)$ holds" $\Rightarrow$ "there is a generic G-Galois extension over k", suppose that $V \to W$ is the G-Galois covering given in Definition 1.5. Choose an affine open subset W_0 of W such that $W_0 \simeq \mathit{Spec}(R)$ for some localized polynomial ring R (use Lemma 4.1, if necessary). Consider the G-Galois covering $V \times_W W_0 \to W_0$. The fibre product $V \times_W W_0$ is an affine variety because $V \to W$ is a G-torsor and G is a finite constant group scheme. Write $V \times_W W_0 = \mathit{Spec}(S)$. Then the pair (R, S) satisfies the conditions for a generic G-Galois extension over k (see [Sa2, Definition 1.1] for its definition).

By Theorem 1.2, we find that, if k is an infinite field and G is a finite group, then "$k(G)$ is stably k-rational" $\Rightarrow$ "the property $\mathrm{Rat}(G/k)$ holds" $\Rightarrow$ "$k(G)$ is retract k-rational". We don't know whether the two notions "$\mathrm{Rat}(G/k)$ holds" and "$k(G)$ is retract k-rational" are equivalent or not.

In [Ku] Kunyavskii studies the birational classification of 3-dimensional algebraic tori over a field k. He gives a list of all those tori which are k-rational; the remaining ones are not stably k-rational. In a private communication during 2009 Kunyavskii informed me that, from the proof in [Ku], it is not difficult to deduce that a 3-dimensional algebraic torus over k is not retract k-rational if and only if it is not stably k-rational.

We organize this paper as follows. We review basic notions of multiplicative group actions in Section 2. In Section 3 Saltman's work on retract rationality is reviewed. The transitivity theorem of retract rationality is proved in Section 4. Applications are given in Section 5 where Theorem 5.10 is the main result. In Section 6, we study the fixed subfields of monomial actions; Theorem 6.6 is the generalization of Barge's Theorem.

Standing notations. In discussing retract rationality, we always assume that the ground field is infinite (see Definition 1.1). Thus, throughout this paper, we will assume that k is an infinite field, unless otherwise specified. A finitely generated field extension L of k is called a k-field for short. $k(x_1, \dots, x_n)$ or $k(X_1, \dots, X_n)$ denotes the rational function field of n variables over k. For emphasis, recall $k(G) = k(x_g\colon\ g \in G)^G$.

We denote by ζ_n a primitive n-th root of unity in some extension field of k. When we write $\zeta_n \in k$, it is understood that $\operatorname{char} k = 0$ or $\operatorname{char} k = p > 0$ with $p \nmid n$. Similarly, when we write $\operatorname{char} k \nmid n$, it is understood that $\operatorname{char} k = 0$ or $\operatorname{char} k = p > 0$ with $p \nmid n$.

For brevity, we will call $k[X_1, \dots, X_n][1/f]$ a localized polynomial ring when $k[X_1, \dots, X_n]$ is a polynomial ring and $f \in k[X_1, \dots, X_n]\backslash\{0\}$ (see the definition of retract rationality in Definition 1.1). An affine domain over k or an affine k-domain (or simply an affine domain) is an integral domain of the form $k[\alpha_1, \dots, \alpha_m]$ for finitely many elements $\alpha_1, \dots, \alpha_m$.

All the groups in this article are finite groups. C_n denotes the cyclic group of order n. $\mathbb{Z}[\pi]$ is the group ring of the finite group π over $\mathbb{Z}$. The exponent of a group G is the least common multiple of the orders of elements in G.

2. Multiplicative group actions

Let π be a finite group. A π-lattice M is a finitely generated $\mathbb{Z}[\pi]$-module such that M is a free abelian group when it is regarded as an abelian group.

For a π-lattice M, $k[M]$ denotes the Laurent polynomial ring and $k(M)$ is the quotient field of $k[M]$. Explicitly, if $M = \bigoplus_{1 \leqslant i \leqslant m} \mathbb{Z} \cdot x_i$ as a free abelian group, then $k[M] = k[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$ and $k(M) = k(x_1, \dots, x_m)$. Since π acts on M, it will act on $k[M]$ and $k(M)$ by k-automorphisms, i.e. if $\sigma \in \pi$ and $\sigma \cdot x_j = \sum_{1 \leqslant i \leqslant m} a_{ij} x_i \in M$, then we define the multiplicative action of σ on $k[M]$ and $k(M)$ by $\sigma \cdot x_j = \prod_{1 \leqslant i \leqslant m} x_i^{a_{ij}}$.

The multiplicative action of π on $k(M)$ is called a purely monomial action in [HK1]. If π is a group acting on the rational function field $k(x_1, \dots, x_m)$ by k-automorphism such that $\sigma \cdot x_j = c_j(\sigma) \cdot$

$\prod_{1 \leqslant i \leqslant m} x_i^{a_{ij}}$ where $\sigma \in \pi$, $a_{ij} \in \mathbb{Z}$ and $c_j(\sigma) \in k \backslash \{0\}$, such a multiplicative group action is called a monomial action.

Definition 2.1. Let $M = \bigoplus_{1 \leqslant j \leqslant m} \mathbb{Z} \cdot x_j$ be a π-lattice and π act on $k(M) = k(x_1, \dots, x_m)$ by purely monomial k-automorphisms. The fixed field, denoted by $k(M)^\pi$, is defined as $k(M)^\pi = \{f \in k(x_1, \dots, x_m)\colon\ \sigma \cdot f = f$ for any $\sigma \in \pi\}$. This field $k(M)^\pi$ was designated as $k(M, \pi)$ by Saltman in [Sa5].

On the other hand, the fixed field for a monomial action is denoted by $k_\alpha(M)^\pi$ (here α designates the extension of $\mathbb{Z}[\pi]$-modules associated to the monomial action, which will be defined below). Precisely, if π acts on $k(M) = k(x_1, \dots, x_m)$ by monomial k-automorphisms, define M_α to be the (multiplicatively written) $\mathbb{Z}[\pi]$-module generated by $x_1, \dots, x_m$ and $k^\times$ $(:= k \backslash \{0\})$ in $k(x_1, \dots, x_m) \backslash \{0\}$. Thus we obtain a short exact sequence of $\mathbb{Z}[\pi]$-modules $0 \to k^\times \to M_\alpha \to M \to 0$; label this short exact sequence (or the module extension) as α. Define $k_\alpha(M)^\pi = \{f \in k(x_1, \dots, x_m)\colon\ \sigma \cdot f = f$ for any $\sigma \in \pi\}$.

Note that $k_\alpha(M)^\pi$ of this article agrees with the notation of Saltman in [Sa6, p. 538]; our notation $k_\alpha(M)^\pi$ also agrees with Saltman's notation in [Sa7, p. 535], except that, M_α in [Sa7] is the multiplicative subgroup generated by $x_1, \dots, x_m$ and μ where k is assumed to be algebraically closed and μ denotes the group of all roots of unity in $k^\times$.

Definition 2.2. Let K be a k-field, π be a finite group, and $M = \bigoplus_{1 \leqslant j \leqslant m} \mathbb{Z} \cdot x_j$ be a π-lattice. Suppose that π acts on K by k-automorphisms of K and π acts on $K(M)$ by monomial k-automorphism, i.e. $\sigma \cdot x_j = c_j(\sigma) \cdot \prod_{1 \leqslant i \leqslant m} x_i^{a_{ij}}$ where $\sigma \in \pi$, $c_j(\sigma) \in K \backslash \{0\}$, $a_{ij} \in \mathbb{Z}$. We will denote the fixed field by $K_\alpha(M)^\pi$ where $\alpha : 0 \to K^\times \to M_\alpha \to M \to 0$ is the associated extension of this monomial action of π. If π acts on $K(M)$ by purely monomial automorphisms, we will write $K(M)^\pi$ for $K_\alpha(M)^\pi$.

Note that it is not necessary to assume that the action of π on the k-field K is faithful. In case π acts faithfully on K and acts on $K(M)$ by purely monomial k-automorphisms, then $K(M)^\pi$ is just the function field of some algebraic torus defined over K^π split by K and with character group M (see [Vo2]).

We recall some basic facts of the theory of flabby (flasque) π-lattices developed by Voskresenskii, Endo and Miyata, Colliot-Thélène and Sansuc, etc. [Vo2,CTS1]. We refer the reader to [Sw1,Sw2,Lo] for a quick review of the theory.

Definition 2.3. A π-lattice M is called a permutation lattice if M has a $\mathbb{Z}$-basis permuted by π. M is called an invertible (or permutation projective) lattice, if it is a direct summand of some permutation lattice. A π-lattice M is called a flabby (or flasque) lattice if $H^{-1}(\pi', M) = 0$ for any subgroup π' of π (note that all the cohomology groups in this paper, in particular $H^{-1}(\pi', M)$, are the Tate cohomology groups). Similarly, M is called coflabby if $H^1(\pi', M) = 0$ for any subgroup π' of π. More generally, if N is a $\mathbb{Z}[\pi]$-module, we will say that N is H^1 trivial if $H^1(\pi', N) = 0$ for any subgroup π' of π.

Let $\mathcal{L}_\pi$ be the set of all π-lattices. We define a similarity relation on $\mathcal{L}_\pi$: If $M_1, M_2 \in \mathcal{L}_\pi$, then $M_1 \sim M_2$ if and only if $M_1 \oplus Q_1 \simeq M_2 \oplus Q_2$ for some permutation lattices Q_1 and Q_2. The set of all similarity classes is denoted by $\mathcal{L}_\pi/\sim$; $[M]$ denotes the similarity class containing M in $\mathcal{L}_\pi/\sim$. Note that the operation of the direct sum in $\mathcal{L}_\pi$ induces a commutative monoid structure on $\mathcal{L}_\pi/\sim$.

Lemma 2.4. *(See [Sw1, Lemma 8.4], [Le, Proposition 1.2].)*

(1) *If E is an invertible π-lattice, then E is flabby and coflabby.*
(2) *If E is an invertible π-lattice and C is a coflabby π-lattice, then any short exact sequence $0 \to C \to N \to E \to 0$ splits.*

Theorem 2.5 *(Endo and Miyata). (See [Sw2, Theorem 3.4], [Lo, 2.10.1].) Let π be a finite group. Then all the flabby π-lattices are invertible if and only if all the Sylow subgroups of π are cyclic.*

Theorem 2.6 *(Colliot-Thélène and Sansuc). (See [Sw1, Lemma 8.5], [Lo, Lemma 2.6.1].) For any π-lattice M, there is a short exact sequence of π-lattices $0 \to M \to P \to F \to 0$ where P is a permutation lattice and F is a flabby lattice.*

Definition 2.7. The exact sequence $0 \to M \to P \to F \to 0$ in the above theorem is called a flabby resolution of the π-lattice M. The flabby class of M, denoted by $[M]^{fl}$, is defined as $[M]^{fl} = [F] \in \mathcal{L}_\pi/\sim$. Note that $[M]^{fl}$ is well defined: If $[M] = [M']$, $[M]^{fl} = [F]$ and $[M']^{fl} = [F']$, then $F \oplus Q \simeq F' \oplus Q'$ for some permutation lattices Q and Q', and therefore $[F] = [F']$ (see, for example, [Sw1, Lemma 8.7]).

When we say that $[M]^{fl}$ is invertible, we mean that $[M]^{fl} = [E]$ for some invertible lattice E.

Theorem 2.8. *(See Saltman [Sa4, Theorem 3.14].) Let K be a finite Galois field extension of k with $\pi = \mathrm{Gal}(K/k)$. For any π-lattice M, $K(M)^\pi$ is retract k-rational if and only if $[M]^{fl}$ is invertible.*

As an application of Theorem 2.8, we prove the following theorem.

Theorem 2.9. *(See Voskresenskii [Vo1].) Let k be an infinite field with $\operatorname{char} k \neq 2$. If $k(\zeta_{2^n})$ is not a cyclic extension of k, then $k(C_{2^n})$ is not retract k-rational. Thus $k(C_{2^n})$ is not rational over k.*

Proof. Write $C_{2^n} = \langle \sigma \rangle$ and $V = \bigoplus_{0 \leqslant i \leqslant 2^n-1} k \cdot x(\sigma^i)$ be the regular representation space of C_{2^n}. Then $k(C_{2^n}) = k(x(\sigma^i)\colon\ 0 \leqslant i \leqslant 2^n-1)^{\langle\sigma\rangle}$.

Let $\zeta = \zeta_{2^n}$ and $\pi = \mathrm{Gal}(k(\zeta)/k)$. Extend the actions of σ and π to $k(\zeta)(x(\sigma^i)\colon\ 0 \leqslant i \leqslant 2^n-1)$ so that π acts trivially on $x(\sigma^i)$ and σ acts trivially on $k(\zeta)$. For $0 \leqslant i \leqslant 2^n-1$, define

$$y_i = \sum_{0 \leqslant j \leqslant 2^n-1} \zeta^{-ij} \cdot x(\sigma^j) \in \bigoplus_{0 \leqslant j \leqslant 2^n-1} k(\zeta) \cdot x(\sigma^j).$$

It follows that $k(C_{2^n}) = k(x(\sigma^i)\colon\ 0 \leqslant i \leqslant 2^n-1)^{\langle\sigma\rangle} = \{k(\zeta)(x(\sigma^i)\colon\ 0 \leqslant i \leqslant 2^n-1)^\pi\}^{\langle\sigma\rangle} = k(\zeta)(y_i\colon\ 0 \leqslant i \leqslant 2^n-1)^{\langle\sigma,\pi\rangle}$.

It is easy to see that $\sigma \cdot y_i = \zeta^i y_i$ for $0 \leqslant i \leqslant 2^n-1$. Moreover, if $\tau_t \in \pi$ is defined by $\tau_t(\zeta) = \zeta^t$, then $\tau_t(y_i) = y_{ti}$ for $0 \leqslant i \leqslant 2^n-1$ (note that the subscript ti of y_{ti} is taken modulo 2^n). It follows that $k(\zeta)(y_i\colon\ 0 \leqslant i \leqslant 2^n-1)^{\langle\sigma,\pi\rangle} = k(\zeta)(y_i\colon\ 1 \leqslant i \leqslant 2^n-1)^{\langle\sigma,\pi\rangle}(y_0)$.

Let N be the multiplicative subgroup of $k(\zeta)(y_i\colon\ 1 \leqslant i \leqslant 2^n-1)\backslash\{0\}$ generated by $y_1, y_2, \ldots, y_{2^n-1}$. Since π acts on $N = \langle y_i\colon\ 1 \leqslant i \leqslant 2^n-1\rangle$, N is a π-lattice. Similarly, π acts on $\langle\zeta\rangle \simeq \mathbb{Z}/2^n\mathbb{Z}$; thus we may regard $\mathbb{Z}/2^n\mathbb{Z}$ as a finite $\mathbb{Z}[\pi]$-module (note that $\tau_t \cdot \bar{i} = \overline{ti}$ for any $\bar{i} \in \mathbb{Z}/2^n\mathbb{Z}$). Define a π-morphism Φ by

$$\Phi : N \to \mathbb{Z}/2^n\mathbb{Z}$$

where, for any monomial $y = \prod_{1 \leqslant j \leqslant 2^n-1} y_j^{\lambda_j}$ with $\lambda_j \in \mathbb{Z}$, define $\Phi(y) = \sigma(y)/y$ (note that $\sigma(y)/y \in \langle\zeta\rangle$, and thus can be regarded as an element of $\mathbb{Z}/2^n\mathbb{Z}$).

Define $M = \mathrm{Ker}(\Phi)$, which is a π-lattice. It follows that $k(\zeta)(y_i\colon\ 1 \leqslant i \leqslant 2^n-1)^{\langle\sigma,\pi\rangle} = \{k(\zeta) \times (y_i\colon\ 1 \leqslant i \leqslant 2^n-1)^{\langle\sigma\rangle}\}^\pi = k(\zeta)(M)^\pi$.

We compare the above construction with that in [Le, p. 310]. It is clear that $N \simeq \mathbb{Z}^{C(q)}$ where $q = 2^n$ and $\mathbb{Z}^{C(q)}$ is Lenstra's notation. Thus $M \simeq I_q$ in Lenstra's notation. By [Le, Propositions 3.1 and 3.2], $H^1(\pi', I_q) = 0$ for any subgroup π' of π and $H^{-1}(\pi_0, I_q) \simeq \mathbb{Z}/2\mathbb{Z}$ if π_0 is the unique subgroup of π isomorphic to $C_2 \times C_2$ (also see [Vo2, p. 79]). Thus $I_q (\simeq M)$ is coflabby, but not flabby.

Let $0 \to M \to P \to F \to 0$ by any flabby resolution of M so that P is permutation and F is flabby. Suppose that F is invertible. By Lemma 2.4, this exact sequence splits. Thus $P \simeq M \oplus F$. In particular, M is invertible. By Lemma 2.4 again, M is flabby. This leads to a contradiction to the previous assertion that M is not flabby.

Apply Theorem 2.8. We find that $k(\zeta)(M)^\pi$ $(= k(C_{2^n}))$ is not retract k-rational. $\square$

Corollary 2.10. *Let k be a field such that $k(\zeta_{2^n})$ is not cyclic over k. Let $G = H \rtimes C_{2^n}$ where H is a normal subgroup of the finite group G with C_{2^n} acting on it. Then $k(G)$ is not retract k-rational.*

Proof. Suppose $k(G)$ is retract k-rational. By Theorem 3.5, $k(C_{2^n})$ is retract k-rational, which contradicts to Theorem 2.9. $\square$

Remark. Let M be the π-lattice defined in the proof of Theorem 2.9. Voskresenskii showed that $[M]^{fl}$, the flabby class of M, is not invertible ([Vo1, pp. 97–99], [Vo2, p. 79]); the same result was obtained by Lenstra [Le, pp. 310–311]. Saltman showed that $\mathbb{Q}(G)$ is not retract $\mathbb{Q}$-rational if G is a finite abelian group containing an element of order 2^n with $n \geqslant 3$ by using Wang's counter-example to Grunwald Theorem [Sa2, Theorem 5.11]. Sonn generalized Saltman's Theorem and proved that $\mathbb{Q}(G)$ is not retract $\mathbb{Q}$-rational if G is any finite group containing a normal subgroup H such that $G/H \simeq C_{2^n}$ with $n \geqslant 3$ [So].

3. Criteria of retract rationality

In this section we recall several results about retract rationality, which will be used subsequently. First we define the unramified Brauer group of a k-field L.

Definition 3.1. (See [Sa3], [Sa5, p. 226].) Let L be a k-field. Define the unramified Brauer group of L over k, denoted by $\mathrm{Br}_{v,k}(L)$, as $\mathrm{Br}_{v,k}(L) = \bigcap_R \mathrm{Br}(R) \subset \mathrm{Br}(L)$ where R runs over all discrete k-valuation rings whose quotient fields are equal to L, and $\mathrm{Br}(R)$ denotes the Brauer group of R. See [Bo, Section 3], [Sa7, Theorem 12] for more results about unramified Brauer groups.

Theorem 3.2. *(See Saltman [Sa5, Section 2].)*

(i) *Let L be a k-field. If L is retract k-rational, then $\mathrm{Br}_{v,k}(L) \simeq \mathrm{Br}(k)$. In particular, when k is algebraically closed and L is retract k-rational, then $\mathrm{Br}_{v,k}(L) = 0$.*
(ii) *If $K \subset L$ are k-fields and L is retract K-rational, then $\mathrm{Br}_{v,k}(K) \simeq \mathrm{Br}_{v,k}(L)$.*

Note that $\mathrm{Br}_{v,\mathbb{C}}(L) = 0$ is just a necessary condition for a $\mathbb{C}$-field L to be retract $\mathbb{C}$-rational. It is not a sufficient condition. In fact, Peyre shows that, there is a group G of order p^{12} such that $\mathbb{C}(G)$ is not retract $\mathbb{C}$-rational but $\mathrm{Br}_{v,\mathbb{C}}(\mathbb{C}(G)) = 0$ [Pe].

Definition 3.3. (See [Sa5].) Let $K \subset L$ be k-field. K is called a dense retraction of L if there is a regular affine K-algebra R such that (i) the quotient field of R is L, and (ii) for any $r \in R\backslash\{0\}$, there is a K-algebra morphism $\varphi : R[1/r] \to K$.

We will prove in Lemma 5.2 that, if L is retract k-rational, then k is a dense retraction of L.

Now consider retract rationality. We reformulate Saltman's results of [Sa2] in terms of retract rationality by applying Theorem 1.2.

Lemma 3.4.

(i) *(See [Sa4, Proposition 3.6].) Let L be a k-field, $L(x_1, \dots, x_n)$ be the rational function field over L. Then L is retract k-rational if and only if so is $L(x_1, \dots, x_n)$.*
(ii) *(See [Sa2, Theorems 1.5 and 3.1].) Let $G = G_1 \times G_2$. Then $k(G)$ is retract k-rational if and only if so are $k(G_1)$ and $k(G_2)$.*
(iii) *(See [Sa5, Lemma 1.1].) Let $K \subset L$ be k-fields. If L is retract k-rational and K is a dense retraction of L, then K is retract k-rational.*
(iv) *(See [Sa5, Theorem 1.3].) Let K be a finite Galois field extension of k with $\pi = \mathrm{Gal}(K/k)$, and M be any π-lattice. Then k is a dense retraction of $K(M)^{\pi}$.*

Theorem 3.5. *(See [Sa2, Theorems 3.1 and 3.5].) Let $G = N \rtimes G_0$ where N is a normal subgroup of G with G_0 acting on N.*

(1) *If $k(G)$ is retract k-rational, so is $k(G_0)$.*
(2) *Assume furthermore that N is abelian and $\gcd\{|N|, |G_0|\} = 1$. If both $k(N)$ and $k(G_0)$ are retract k-rational, so is $k(G)$.*

Remark. For more results about sufficient conditions to ensure that $k(N \rtimes G_0)$ is retract k-rational, see [Ka2, Theorems 1.11, 1.12, 1.13 and 4.3].

We recall a reduction theorem for Noether's problem.

Theorem 3.6. *(See [KP, Theorem 1.1].) Let k be a field with $\operatorname{char} k = p > 0$ and $\tilde{G}$ be a group extension defined by $1 \to C_p \to \tilde{G} \to G \to 1$. Then $k(\tilde{G})$ is rational over $k(G)$.*

Theorem 3.7. *(See Saltman [Sa4, Theorem 4.12].) Let k be an infinite field and G be a finite abelian group of exponent $e = 2^r m$ with $2 \nmid m$. Then $k(G)$ is retract k-rational if and only if either $\operatorname{char} k = 2$, or $k(\zeta_{2^r})$ is a cyclic extension over k.*

Proof. If $\operatorname{char} k = p > 0$ and $p \mid e$, choose an element $g \in G$ of order p. Consider $1 \to \langle g \rangle \to G \to G/\langle g \rangle \to 1$ and apply Theorem 3.6. Since $k(G)$ is rational over $k(G/\langle g \rangle)$, it follows that $k(G)$ is retract k-rational if and only if so is $k(G/\langle g \rangle)$ by Lemma 3.4. Thus we may assume that $\gcd\{\operatorname{char} k, |G|\} = 1$ without loss of generality.

Write $G \simeq \prod_q C_q$ where these q's are some prime powers with $\gcd\{\operatorname{char} k, q\} = 1$. By Lemma 3.4, it suffices to check whether each $k(C_q)$ is or is not retract k-rational.

If $\operatorname{char} k = 2$, then q is an odd integer by the above assumption. Thus $k(C_q)$ is retract k-rational by [Sa2, Theorem 2.1]. From now on, we assume that $\operatorname{char} k \neq 2$.

By [Sa2, Theorem 2.1], $k(C_q)$ is retract k-rational if q is odd or q is even with $k(\zeta_q)$ being cyclic over k. When $k(\zeta_q)$ is not cyclic over k, then $k(C_q)$ is not retract k-rational by Theorem 2.9. □

Remark. Voskresenskii shows that, if $G = C_{2^r}$, then $k(G)$ is k-rational $\Leftrightarrow$ $k(G)$ is retract k-rational $\Leftrightarrow$ either $\operatorname{char} k = 2$ or $k(\zeta_{2^r})$ is cyclic over k [Vo2, p. 79]. For any odd prime number p, $\mathbb{Q}(C_p)$ is always retract $\mathbb{Q}$-rational by the above theorem, while $\mathbb{Q}(C_{47})$ is not $\mathbb{Q}$-rational by Swan (see, for example, [Le, p. 299]), and thus not stably $\mathbb{Q}$-rational by [Le, Proposition 5.6].

Here is another criterion for retract rationality.

Example 3.8. (See [Ka2, p. 2763].) Let k be any infinite field, G be a non-abelian p-group of exponent p and of order p^3 or p^4. Then $k(G)$ is retract k-rational.

4. A transitivity theorem

Before proving the transitivity theorem, we recall a lemma due to Swan.

Lemma 4.1. *(See [Sw1, Lemma 4.3].) Let L be a k-field, R_1 and R_2 be affine k-domains contained in L such that the quotient fields of R_1 and R_2 are equal to L. Then there are $r_1 \in R_1 \backslash \{0\}$, $r_2 \in R_2 \backslash \{0\}$ such that $R_1[1/r_1] = R_2[1/r_2]$.*

Theorem 4.2. *Let $K \subset L$ be k-fields. If K is retract k-rational and L is retract K-rational, then L is retract k-rational.*

Proof. Geometrically this result looks clear. Here is a rigorous proof.

Step 1. By assumptions, there exist an affine K-domain B, an affine k-domain S, localized polynomial ring $K[X_1,\dots,X_n][1/f]$, $k[Y_1,\dots,Y_m][1/g]$, and K-algebra morphisms $\varphi: B \to K[X_1,\dots,X_n][1/f]$, $\psi: K[X_1,\dots,X_n][1/f] \to B$, k-algebra morphisms $\varphi_1: S \to k[Y_1,\dots,Y_m][1/g]$, $\psi_1: k[Y_1,\dots,Y_m][1/g] \to S$ satisfying that

(i) the quotient fields of B and S are L and K respectively, and
(ii) $\psi \circ \varphi = 1_B$, $\psi_1 \circ \varphi_1 = 1_S$.

We will find a subring A of L and an affine k-domain R of K such that

(i) $A = R[\alpha_1,\dots,\alpha_t]$ for some $\alpha_1,\dots,\alpha_t \in L$,
(ii) the quotient fields of A and R are L and K respectively, and
(iii) the above morphisms φ, ψ, φ_1, ψ_1 are "well defined" for A and R, i.e. the "natural extensions" of these morphisms (still denoted by φ, ψ, φ_1, ψ_1, by abusing the notations) $\varphi: A \to R[X_1,\dots,X_n][1/f]$, $\psi: R[X_1,\dots,X_n][1/f] \to A$, $\varphi_1: R \to k[Y_1,\dots,Y_m][1/g_0]$, $\psi_1: k[Y_1,\dots,Y_m][1/g_0] \to R$ are well defined (where $g_0 = gg_1$ for some non-zero polynomial g_1) and satisfy $\psi \circ \varphi = 1_A$, $\psi_1 \circ \varphi_1 = 1_R$.

The above assertion seems obvious in some sense, although a formal proof is tedious. We provide the proof in the following.

Note that in choosing the localized polynomials $K[X_1,\dots,X_n][1/f]$ and $k[Y_1,\dots,Y_m][1/g]$, we may assume that $X_1,\dots,X_n,Y_1,\dots,Y_m$ are algebraically independent over K. In fact, these subrings may be chosen from the rational function field $K(X_1,\dots,X_n,Y_1,\dots,Y_m)$.

Write $B = K[\alpha_1,\alpha_2,\dots,\alpha_t]$ for some $\alpha_1,\dots,\alpha_t \in L$. Let R_1 be an affine k-domain whose quotient field is K. Thus the quotient field of $R_1[\alpha_1,\dots,\alpha_t]$ is L.

We will enlarge R_1 by adjoining additional elements of K to R_1. First $f \in K[X_1,\dots,X_n]$. Adjoin all the coefficients of f into R_1. Then consider $\varphi(\alpha_j)$ for $1 \leqslant j \leqslant t$. Since $\varphi(\alpha_j) = f_j/f^l$ for some $f_j \in K[X_1,\dots,X_n]$. Adjoin all the coefficients of all these f_j to R_1 also. Call this new affine k-domain R_2. It follows that $f \in R_2[X_1,\dots,X_n]$ and $\varphi: R_2[\alpha_1,\dots,\alpha_t] \to R_2[X_1,\dots,X_n][1/f]$ is well defined.

Now consider $\psi(X_1),\dots,\psi(X_n)$ and $\psi(1/f)$. They lie in $B = K[\alpha_1,\dots,\alpha_t]$. Thus they belong to the subring $R_2[\alpha_1,\dots,\alpha_t][1/\beta]$ for a fixed element $\beta \in K\backslash\{0\}$. Adjoin $1/\beta$ to R_2. Call this affine k-domain R_3. We conclude that the R_3-algebra morphisms $\varphi: R_3[\alpha_1,\dots,\alpha_t] \to R_3[X_1,\dots,X_n][1/f]$, $\psi: R_3[X_1,\dots,X_n][1/f] \to R_3[\alpha_1,\dots,\alpha_t]$ are well defined and satisfy $\psi \circ \varphi = 1$.

Consider the affine k-domain S. Apply Lemma 4.1. We find $r \in R_3\backslash\{0\}$ and $r_1 \in S\backslash\{0\}$ so that $R_3[1/r] = S[1/r_1]$. Define $R = R_3[1/r]$ and $A = R[\alpha_1,\dots,\alpha_t]$.

Note that $\varphi_1(r_1) = g_1/g^{l'}$ for some non-zero polynomial $g_1 \in k[Y_1,\dots,Y_m]$. Define $g_0 = g \cdot g_1$. Then $\varphi_1: S[1/r_1] \to k[Y_1,\dots,Y_m][1/g_0]$ is well defined. It is not difficult to check that, in the morphism $\psi_1: k[Y_1,\dots,Y_m][1/g] \to S$, the element $\psi_1(g)$ is a unit in S. Thus $\psi_2(g_1) = r_1 u$ for some unit $u \in S$. It follows that $\psi_1: k[Y_1,\dots,Y_m][1/g_0] \to S[1/r_1]$ is also well defined. Thus, the k-algebra morphisms $\varphi_1: R \to k[Y_1,\dots,Y_m][1/g_0]$ and $\psi_2: k[Y_1,\dots,Y_m][1/g_0] \to R$ satisfying $\psi_1 \circ \varphi_1 = 1_R$. So are the R-algebra morphisms $\varphi: A \to R[X_1,\dots,X_n][1/f]$ and $\psi: R[X_1,\dots,X_n][1/f] \to A$. Done.

Step 2. Let $C_0 := R[X_1,\dots,X_n][1/f]$. Then we have R-algebra morphisms $\varphi: A \to C_0$ and $\psi: C_0 \to A$ with $\psi \circ \varphi = 1_A$. Note that $A = R[\alpha_1,\dots,\alpha_t]$ is an affine k-domain whose quotient field is L. We will define a localized polynomial C related to A and C_0.

Since $f \in R[X_1,\dots,X_n]$, write $f = \sum_\lambda a_\lambda \cdot X^\lambda$ where $X^\lambda = X_1^{\lambda_1} X_2^{\lambda_2} \cdots X_n^{\lambda_n}$ and $a_\lambda \in R$. Write $\varphi_1(a_\lambda) = b_\lambda / g_0^N$ for all λ where $b_\lambda \in k[Y_1,\dots,Y_m]$. Define f_0 and h by

$$f_0 = \left(\sum_\lambda b_\lambda X^\lambda\right) / g_0^N,$$

$$h = \sum_\lambda b_\lambda X^\lambda \in k[X_1,\dots,X_n,Y_1,\dots,Y_m]\backslash\{0\}.$$

Define $C = k[X_1, \dots, X_n, Y_1, \dots, Y_m][1/(g_0h)]$.

From the k-algebra morphism $R \xrightarrow{\varphi_1} k[Y_1, \dots, Y_m][1/g_0] \xrightarrow{\psi_1} R$, extend the base to $k[X_1, \dots, X_n]$, i.e. define k-algebra morphisms $\varphi_2 : R[X_1, \dots, X_n] \to k[X_1, \dots, X_n, Y_1, \dots, Y_m][1/g_0]$ and $\psi_2 : k[X_1, \dots, X_n, Y_1, \dots, Y_m][1/g_0] \to R[X_1, \dots, X_n]$ by requiring that both φ_2 and ψ_2 are morphisms over $k[X_1, \dots, X_n]$ and define $\varphi_2(r) = \varphi_1(r)$ for any $r \in R$, $\psi_2(G) = \psi_1(G)$ for any $G \in k[Y_1, \dots, Y_m][1/g_0]$.

Note that $f \in R[X_1, \dots, X_n]$ and $\varphi_2(f) = f_0 = h/g_0^N$ by the above definition. Hence $\varphi_2 : C_0 = R[X_1, \dots, X_n][1/f] \to C = k[X_1, \dots, X_n, Y_1, \dots, Y_m][1/(g_0h)]$ is well defined.

Moreover, from the relation $b_\lambda = g_0^N \cdot \varphi_1(a_\lambda)$, we get $\psi_1(b_\lambda) = \psi_1(g_0)^N \cdot (\psi_1 \circ \varphi_1)(a_\lambda) = a_\lambda \cdot \psi_1(g_0)^N$. Note that $\psi_1(g_0)$ is a unit in R. It follows that $\psi_2(h) = \psi_2(\sum_\lambda b_\lambda X^\lambda) = \sum_\lambda \psi_2(b_\lambda)X^\lambda = \sum_\lambda \psi_1(b_\lambda)X^\lambda = \psi_1(g_0)^N \cdot \sum_\lambda a_\lambda X^\lambda = \psi_1(g_0)^N \cdot f$ is also a unit in C_0 since $1/f \in C_0$. Thus $\psi_2 : C = k_0[X_1, \dots, X_n, Y_1, \dots, Y_m][1/(g_0h)] \to C_0 = R[X_1, \dots, X_n][1/f]$ is also well defined. Clearly $\psi_2 \circ \varphi_2 = 1_{C_0}$.

Step 3. Note that we have the following diagram

$$\begin{tikzcd}
A \arrow[r, "\varphi"] & C_0 \arrow[r, "\psi"] \arrow[d, "\varphi_2"'] & A \\
 & C \arrow[d, "\psi_2"'] & \\
A \arrow[r, "\varphi"] & C_0 \arrow[r, "\psi"] & A
\end{tikzcd}$$

define $\tilde{\varphi} = \varphi_2 \circ \varphi$ and $\tilde{\psi} = \psi \circ \psi_2$. It follows that $\tilde{\psi} \cdot \tilde{\varphi} = 1_A$. Thus L is retract k-rational. $\square$

5. Applications

We recall a known result which will be used subsequently.

Theorem 5.1. *(See [HK3, Theorem 1].) Let L be any field and G be a finite group acting on $L(x_1, \dots, x_m)$, the rational function field of m variables over a field L. Suppose that*

(i) *for any $\sigma \in G$, $\sigma(L) \subset L$;*
(ii) *the restriction of the action of G to L is faithful;*
(iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL_m(L)$ and $B(\sigma)$ is an $m \times 1$ matrix over L.

Then there exist $z_1, \dots, z_m \in L(x_1, \dots, x_m)$ such that $L(x_1, \dots, x_m) = L(z_1, \dots, z_m)$ and $\sigma(z_j) = z_j$ for any $\sigma \in G$, any $1 \leqslant j \leqslant m$.

Lemma 5.2. *Let $K \subset L$ be k-field. If L is retract K-rational, then K is a dense retraction of L.*

Proof. Let A be an affine K-domain whose quotient field is L arising from the definition of retract K-rationality. Let $K[X_1, \dots, X_n][1/f]$ be the localized polynomial ring and $\varphi : A \to K[X_1, \dots, X_n][1/f]$, $\psi : K[X_1, \dots, X_n][1/f]$ be the K-morphisms satisfying $\psi \circ \varphi = 1_A$.

Since the singular locus of A defines a non-zero ideal I in A, we may choose any non-zero element $\alpha \in I$; then replace A by $A[\alpha]$ and replace $K[X_1, \dots, X_n][1/f]$ by $K[X_1, \dots, X_n][1/(f\phi(r))]$. Thus we may assume that A is a regular domain from the beginning. For any $r \in A\backslash\{0\}$, let $g = \varphi(r)$. Find a K-morphism $\Phi : K[X_1, \dots, X_n][1/f] \to K$ such that $\Phi(fg) \neq 0$. Then $\Phi \circ \phi : A \to K$ is the required map. $\square$

We consider an application of Theorem 4.2.

Recall Theorem 2.8 provides a criterion of retract rationality for $K(M)^G$ when G is faithful on K and M is a G-lattice (it is unnecessary to assume that M is a faithful G-lattice). Now we consider the retract rationality for $k(M)^G$ where G acts trivially on the field k.

Theorem 5.3. *Let G be a finite group acting trivially on the field k, and M be a faithful G-lattice.*

(i) *(See Saltman [Sa5, Corollary 1.6].) If $k(M)^G$ is retract k-rational, then $k(G)$ is also retract k-rational.*
(ii) *(See Saltman [Sa5, Proposition 1.7].) If $0 \to M \to N \to E \to 0$ is an exact sequence of G-lattices where E is invertible, then $k(N)^G$ is retract $k(M)^G$-rational.*

We may wonder whether some criterion of retract rationality for $k(M)^G$ is available. Although we cannot find a complete solution, we are able to answer this question when $[M]^{fl}$ is invertible.

Theorem 5.4. *Let G be a finite group. For any G-lattice M in the following statements, it is assumed that G acts on $k(M)$ by purely monomial k-automorphisms. The following statements are equivalent*:

(i) *$k(G)$ is retract k-rational*;
(ii) *$k(M)^G$ is retract k-rational for some faithful permutation G-lattice M*;
(iii) *$k(M)^G$ is retract k-rational for some faithful G-lattice M such that $[M]^{fl}$ is invertible*;
(iv) *$k(M)^G$ is retract k-rational for all faithful permutation G-lattices M*;
(v) *$k(M)^G$ is retract k-rational for all faithful G-lattices M satisfying that $[M]^{fl}$ are invertible*;
(vi) *$k(M)^G$ is retract k-rational for some faithful G-lattice M.*

Proof. (i) $\Rightarrow$ (vi) by taking $M = \mathbb{Z}[G]$.

(vi) $\Rightarrow$ (i) by Theorem 5.3.

The implications "(v) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)" are easy. It remains to show that "(i) $\Rightarrow$ (v)".

For any faithful G-lattice M with $[M]^{fl}$ invertible, we will show that $k(M)^G$ is retract k-rational.

Define $N := \mathbb{Z}[G]$ and consider $k(M \oplus N)^G$.

By Theorem 5.1, $k(M \oplus N)^G = \{k(M)(z_1, \dots, z_l)\}^G$ where $\sigma \cdot z_j = z_j$ for any $\sigma \in G$, any $1 \leqslant j \leqslant l = |G|$. Thus $k(M \oplus N)^G = k(M)^G(z_1, \dots, z_l)$. It follows that $k(M)^G$ is retract k-rational if and only if so is $k(M \oplus N)^G$ by Lemma 3.4.

Now $k(M \oplus N)^G \simeq k(N \oplus M)^G = \{k(N)(M)\}^G$ is retract rational over $k(N)^G$ by Theorem 2.8. Since $k(N)^G = k(G)$ is retract k-rational. Apply Theorem 4.2. We find that $k(M \oplus N)^G$ is retract k-rational.

Here is another proof of "(i) $\Rightarrow$ (v)".

Suppose that $k(G)$ is retract k-rational and M is a faithful G-lattice with $[M]^{fl}$ invertible.

Let $0 \to M \to P \to E \to 0$ be the flabby resolution of M where P is a permutation lattice and E is an invertible lattice because $[M]^{fl}$ is invertible. By Theorem 5.3, we find that $k(P)^G$ is retract rational over $k(M)^G$. Thus $k(M)^G$ is a dense retraction of $k(P)^G$ by Lemma 5.2. We may apply Lemma 3.4 to show that $k(M)^G$ is retract k-rational, if it is known that $k(P)^G$ is retract k-rational. Since $k(G) = k(\mathbb{Z}[G]^G)$, we may apply Theorem 5.1 twice to $k(P \oplus \mathbb{Z}[G])^G$ as the preceding proof. Thus we find that $k(G)$ is retract k-rational if and only if so is $k(P)^G$. Done. $\square$

Remarks. (i) It is known that $k(M)^G$ is k-rational for any G-lattice M with $\operatorname{rank}_{\mathbb{Z}}(M) \leqslant 3$. See [HK1, HK2,HR].

(ii) Note that [Ka2, Theorem 4.3] may be regarded as a hybrid of Theorem 2.8 and the above Theorem 5.4 (with the help of the following Theorem 5.5).

Theorem 5.5. *Let $G = \langle\sigma\rangle$ be a cyclic group of order $n = 2^r m$ where $2 \nmid m$. The following statements are equivalent*:

(i) *$k(G)$ is retract k-rational.*
(ii) *$k(M)^G$ is retract k-rational for any G-lattice M.*
(iii) *Either* $\operatorname{char} k = 2$ *or* $\operatorname{char} k \neq 2$ *such that $k(\zeta_{2^r})$ is a cyclic extension of k.*

Proof. The equivalence of (i) and (iii) follows from Theorem 3.7.

(ii) $\Rightarrow$ (i) by Theorem 5.4.

(i) $\Rightarrow$ (ii) If M is a faithful G-lattice, then $[M]^{fl}$ is invertible by Theorem 2.5. Hence $k(M)^G$ is retract k-rational by Theorem 5.4.

If M is not faithful, find a normal subgroup H of G so that M is a faithful lattice over G/H. Let $n' = 2^s m'$ be the order of G/H with $2 \nmid m'$. Since $n' \mid n$ and $k(\zeta_{2^r})$ is cyclic over k, it follows that $k(\zeta_{2^s})$ is also cyclic over k. Thus $k(G/H)$ is retract k-rational by Theorem 3.7. Now we may apply the same arguments in the preceding paragraph to the group G/H. $\square$

We recall a theorem in group theory.

Theorem 5.6. *(See [Za, Theorem 11, p. 175].) Let G be a finite group. Then the following two statements are equivalent*:

(i) *All the Sylow subgroups of G are cyclic.*
(ii) *G is of the form $G = \langle\sigma, \tau\rangle$ with relations $\sigma^m = \tau^n = 1$, $\tau\sigma\tau^{-1} = \sigma^r$ where m, n, r are positive integers satisfying*

$$\gcd\{(r-1)n, m\} = 1 \quad \textit{and} \quad r^n \equiv 1 \pmod{m}.$$

Note that, in the condition (ii) of the above theorem, if $r = 1$, it is understood as "$\gcd\{n, m\} = 1$".

The following result is an extension of Theorem 5.5. We choose to formulate only one direction among the various directions of implication.

Theorem 5.7. *Let G be a finite group satisfying the property in Theorem* 5.6. *If $k(G)$ is retract k-rational, then $k(M)^G$ is retract k-rational for any G-lattice M.*

Proof. By the same method as in the proof of Theorem 5.5, we may assume that M is faithful. Then apply Theorem 2.5 for such a group G, and use Theorem 5.4. $\square$

Now we consider an application of Theorem 5.5.

Recall two previous results about the rationality problem and unramified Brauer groups.

Theorem 5.8. *(See Kang [Ka1, Theorem 1.4].) Let k be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H so that G/H is cyclic of order n,* (ii) *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain, and* (iii) *$\zeta_e \in k$ where e is the exponent of G. If $G \to GL(V)$ is any finite-dimensional linear representation of G over k, then $k(V)^G$ is rational over k. In particular, $k(G)$ is k-rational.*

Theorem 5.9. *(See Bogomolov [Bo, Lemma 4.9].) Let G be a finite group containing an abelian normal subgroup H such that G/H is cyclic. Then* $\mathrm{Br}_{v,\mathbb{C}}(\mathbb{C}(G)) = 0$.

What we will prove next is that, with the same assumptions as in Theorem 5.9, $\mathbb{C}(G)$ is retract $\mathbb{C}$-rational. Hence it is not surprising that $\mathrm{Br}_{v,\mathbb{C}}(\mathbb{C}(G)) = 0$ in this situation.

Theorem 5.10. *Let k be an infinite field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H so that G/H is cyclic, and* (ii) *$\zeta_{e'} \in k$ with $e' = lcm\{exp(H), ord(\tau)\}$ where τ is some element in G and the image of τ in G/H generates the cyclic group G/H. If $G \to GL(V)$ is any linear representation of G on the k-vector space V, then $k(V)^G$ is retract k-rational. In particular, $k(G)$ is retract k-rational.*

Proof. Step 1. We will go over the proof of Theorem 5.8 in the paper [Ka1]. By [Ka1, Corollary 3.2], the proof of Theorem 5.8 is valid under the weaker assumption on $\zeta_{e'}$. We will show that $k(V)^G = k(M)^\pi(Y_1, \dots, Y_r)$ where $\pi = G/H = \langle \bar{\tau} \rangle$ and M is a π-lattice.

Note that the assumption that $\mathbb{Z}[\zeta_n]$ is a unique factorization domain is used in the proof of [Ka1, Theorem 2.2]. This theorem asserts that $k(M)$ is π-isomorphic to $k(L)$, a fact which appears only in Step 5 of the proof of [Ka1, Theorem 1.4, line 7 from the bottom on p. 1218].

On the other hand, in Step 4 of the proof of [Ka1, Theorem 1.4], it is known that $k(V)^G = k(y(i,j){:}\ 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant d_i - 1)^G(Y_1, \dots, Y_r) = k(z(i,j){:}\ 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant d_i - 1)^G(Y_1, \dots, Y_r)$ where $G = \langle H, \tau \rangle$ acts on these $z(i,j)$ by

$$\begin{aligned} &\tau : z(i,1) \mapsto z(i,2) \mapsto \cdots \mapsto z(i, d_i - 1) \mapsto \Bigl(\prod_{1 \leqslant j \leqslant d_i - 1} z(i,j) \Bigr)^{-1}, \\ &\sigma : z(i,j) \mapsto \Psi_i\bigl(\tau^{-(j-1)} \sigma \tau^{j-1}\bigr) z(i,j) \end{aligned} \tag{1}$$

where $\sigma \in H$ and $1 \leqslant j \leqslant d_i - 1$.

The first two paragraphs of Step 5 of the proof of [Ka1, Theorem 1.4] shows that $k(z(i,j){:}\ 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant d_i - 1)^H = k(M)$. Hence $k(V)^G = k(M)^{\langle \tau \rangle}(Y_1, \dots, Y_r)$. From formula (1), it is clear that τ acts on $k(M)$ by purely monomial k-automorphisms.

Step 2. By Fischer's Theorem ([Sw1, Theorem 6.1], [KP, Corollary 1.5]), $k(G/H)$ is k-rational; thus it is retract k-rational. Applying Theorem 5.5, we find that $k(M)^{\langle \tau \rangle}$ is retract k-rational. By Lemma 3.4, $k(V)^G$ is retract k-rational.

In particular, take a k-vector space V whose dual space is equal to $\bigoplus_{g \in G} k \cdot x(g)$, the regular representation of G. We find that $k(G) = k(V)^G$ is retract k-rational. □

Remark. Compare Theorem 5.10 with Proposition 5.2 in [Ka2] (and also Theorems 1.11, 1.12 and Corollary 5.1 there). There the assumption $\zeta_{e'} \in k$ is waived, while other assumptions, e.g. the group extension $1 \to H \to G \to C_n \to 1$ splits and the structures of some Galois extensions over k, are required.

6. Monomial actions

Recall the definition of the fixed field $k_\alpha(M)^G$ of a monomial action of G (see Definition 2.2). Throughout this section, G acts trivially on k. We will generalize the following theorem of Barge.

Theorem 6.1. *(See Barge [Ba, Theorem IV-1].) Let G be a finite group. The following two statements are equivalent:*

(i) *All the Sylow subgroups of G are cyclic.*
(ii) $\mathrm{Br}_{v,\mathbb{C}}(\mathbb{C}_\alpha(M)^G) = 0$ *for all G-lattices M, for all short exact sequences of $\mathbb{Z}[G]$-modules $\alpha : 0 \to \mathbb{C}^\times \to M_\alpha \to M \to 0$.*

First we recall an H^1 trivial embedding theorem due to Saltman.

Theorem 6.2. *(See Saltman [Sa7, Proposition 2].) Let G be a finite group, M be a G-lattice. If $\alpha : 0 \to k^\times \to M_\alpha \to M \to 0$ is an exact sequence of $\mathbb{Z}[G]$-modules, then there is an exact sequence $\beta : 0 \to k^\times \to N_\beta \to N \to 0$ satisfying that* (i) *N is a G-lattice,* (ii) *$M_\alpha \subset N_\beta$,* (iii) *N_β is H^1 trivial, i.e. $H^1(G', N_\beta) = 0$ for any subgroup $G' \subset G$, and* (iv) *N_β/M_α is a permutation G-lattice.*

Theorem 6.3. *Let G be a finite group. Then $k(G)$ is retract k-rational if and only if $k_\alpha(M)^G$ is retract k-rational for any invertible G-lattice M, for any short exact sequence of $\mathbb{Z}[G]$-modules $\alpha : 0 \to k^\times \to M_\alpha \to M \to 0$ with G acting faithfully on M_α.*

Proof. It suffices to show that "only if" part.

Suppose that $k(G)$ is retract k-rational and $\alpha : 0 \to k^\times \to M_\alpha \to M \to 0$ is the given extension.

Choose a G-lattice N such that $M \oplus N$ is a permutation G-lattice. Denote $P = M \oplus N$.

We extend the action of G from $k_\alpha(M)$ to $k_\alpha(M \oplus N)$ by requiring G acts on $k(N)$ by purely monomial k-automorphisms. Then G acts faithfully on $k_\alpha(M \oplus N)$.

Since $M \oplus N = P$, it follows that G acts on $k_\alpha(P) = k_\alpha(M \oplus N)$ by monomial k-automorphisms. Moreover, if $P = \bigoplus_{1 \leqslant i \leqslant n} \mathbb{Z} \cdot x_i$ and G permutes $\{x_i \colon 1 \leqslant i \leqslant n\}$, then G acts on $k_\alpha(P) = k(x_1, \ldots, x_n)$ by linear k-automorphisms, i.e. for any $\sigma \in G$, any $1 \leqslant i \leqslant n$, $\sigma \cdot x_i = a_i(\sigma) x_j$ where j depends on i and $a_i(\sigma)$ is some non-zero element in k depending on σ and i.

Consider $k_\alpha(P \oplus Q)^G$ where $Q = \mathbb{Z}[G]$. By Theorem 5.1, $k_\alpha(P \oplus Q)^G$ is rational over $k_\alpha(P)^G$; apply the same theorem again, $k_\alpha(P \oplus Q)^G \simeq k_\alpha(Q \oplus P)^G$ is rational over $k_\alpha(Q)^G = k(G)$. Since $k(G)$ is retract k-rational, so is $k_\alpha(P)^G$ by Lemma 3.4.

On the other hand, consider $k_\alpha(M \oplus N)^G$ ($\simeq k_\alpha(P)^G$). By Lemma 3.4, $k_\alpha(M)^G$ is a dense retraction of $k_\alpha(M \oplus N)^G$ (note that G acts faithfully on $k_\alpha(M)$). Since $k_\alpha(P)^G$ is retract k-rational, so is $k_\alpha(M)^G$ again by Lemma 3.4. $\square$

Lemma 6.4. *Let G be a finite group. Assume that $k(\bar{G})$ is retract k-rational for all quotient groups $\bar{G}$ of the group G. Let M be a G-lattice and $\alpha : 0 \to k^\times \to M_\alpha \to M \to 0$ be a short exact sequence of $\mathbb{Z}[G]$-modules satisfying that*

(i) *denoting $H = \{\sigma \in G \colon \sigma$ acts trivially on $M_\alpha\}$, then there is a short exact sequence of $\mathbb{Z}[G/H]$-modules $\beta : 0 \to k^\times \to N_\beta \to N \to 0$ where, regarding N_β and N as $\mathbb{Z}[G]$-modules, N is a G-lattice, N_β is H^1 trivial, $M_\alpha \subset N_\beta$, and N_β / M_α is a permutation G-lattice*;

(ii) *N is H^1 trivial*; *and*

(iii) *$[M]^{fl}$ is an invertible G-lattice.*

Then $k_\alpha(M)^G$ is retract k-rational.

Remark. The assumption (i) can be achieved by Theorem 6.2. On the other hand, the assumption (ii) is essential. In fact, Saltman proves that, if k is an infinite filed with $\operatorname{char} k \neq 2$ and $\sigma : k(x, y, z) \to k(x, y, z)$ is a k-automorphism defined by $\sigma(x) = a/x$, $\sigma(y) = b/y$, $\sigma(z) = c/z$ where $a, b, c \in k \backslash \{0\}$ satisfying $[k(\sqrt{a}, \sqrt{b}, \sqrt{c}) : k] = 8$, then $k(x, y, z)^{\langle \sigma \rangle}$ is not retract k-rational [Sa8]. The above theorem is not applicable to Saltman's example, because N is not H^1 trivial for any embedding of M_α into an H^1 trivial module N_β.

Proof. Replace the group G by G/H where H is the subgroup in the assumption (i). We may assume that the G-module M_α is faithful.

Let N_β be any H^1 trivial embedding of M_α satisfying the assumptions (i), (ii) and (iii).

Since N_β / M_α is a permutation G-lattice, $k_\beta(N_\beta) = k_\alpha(M_\alpha)(x_1, \ldots, x_n)$ for some $x_1, \ldots, x_n$ satisfying that, for any $\sigma \in G$, $\sigma(x_i) = a_i(\sigma) \cdot x_j$ for some x_j and some $a_i(\sigma) \in k_\alpha(M_\alpha) \backslash \{0\}$. By Theorem 5.1, $k_\beta(N_\beta)^G$ is rational over $k_\alpha(M_\alpha)^G$. By Lemma 3.4, $k_\beta(N_\beta)^G$ is retract k-rational if and only if so is $k_\alpha(M_\alpha)^G$.

From the snake lemma of the following diagram

$$\begin{tikzcd}
0 \arrow[r] & k^\times \arrow[r] \arrow[d, equal] & M_\alpha \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k^\times \arrow[r] & N_\beta \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

we find that $N_\beta/M_\alpha \simeq N/M$ is a permutation G-lattice. By [Sw2, Lemma 3.1], $[N]^{fl} = [M]^{fl}$ is an invertible lattice.

Let $0 \to N \to P \to N' \to 0$ be a flabby resolution of N, i.e. P is a permutation lattice and N' is a flabby lattice. By Lemma 2.4, this short exact sequence splits, i.e. $P \simeq N \oplus N'$. Hence N is an invertible G-lattice. Thus $k_\beta(N_\beta)^G$ is retract k-rational by Theorem 6.3. $\square$

Lemma 6.5. *Let G be a finite group satisfying the property in Theorem* 5.6 *and $k(\bar{G})$ is retract k-rational for all quotient groups $\bar{G}$ of the group G. Let $\alpha : 0 \to k^\times \to M_\alpha \to M \to 0$ be a short exact sequence of $\mathbb{Z}[G]$-modules where M is a G-lattice. Assume the exact sequence α satisfies assumptions* (i) *and* (ii) *in Lemma* 6.4. *Then $k_\alpha(M)^G$ is retract k-rational.*

Proof. By Lemma 6.4, it remains to show that the assumption (iii) is valid for α, i.e. $[M]^{fl}$ is invertible. But this follows from Theorems 2.5 and 5.6. $\square$

Remark. Lemma 6.5 was proved by Saltman when $G \simeq C_p$ where p is a prime number and $\zeta_p \in k$ [Sa7, Lemma 11].

The next result is a generalization of Theorem 6.1 and is valid for any field k which is algebraically closed and $\operatorname{char} k \nmid |G|$. But we choose to present our result when k is the field of complex numbers.

Theorem 6.6. *Let G be a finite group. The following three statements are equivalent*:

(i) *All the Sylow subgroups of G are cyclic.*
(ii) *$\mathbb{C}_\alpha(M)^G$ is retract $\mathbb{C}$-rational for all G-lattices M, for all short exact sequences of $\mathbb{Z}[G]$-modules $\alpha : 0 \to \mathbb{C}^\times \to M_\alpha \to M \to 0$.*
(iii) *$\mathrm{Br}_{v,\mathbb{C}}(\mathbb{C}_\alpha(M)^G) = 0$ for all G-lattices M, for all short exact sequences of $\mathbb{Z}[G]$-modules $\alpha : 0 \to \mathbb{C}^\times \to M_\alpha \to M \to 0$.*

Proof. (ii) $\Rightarrow$ (iii) by Theorem 3.2.

(iii) $\Rightarrow$ (i) by Theorem 6.1.

It remains to show that (i) $\Rightarrow$ (ii). We will apply Lemma 6.5.

Let G be a finite group satisfying the assumption (i) of this theorem. By Theorem 5.6, the group G and all of its quotient groups are metacyclic; thus Theorem 5.10 is applicable to these groups. It follows that $\mathbb{C}(\bar{G})$ is retract $\mathbb{C}$-rational for all quotient groups $\bar{G}$ of the group G.

For a short exact sequence of $\mathbb{Z}[G]$-modules $\alpha : 0 \to \mathbb{C}^\times \to M_\alpha \to M \to 0$, we will show that $\mathbb{C}_\alpha(M)^G$ is retract $\mathbb{C}$-rational. Replacing G by some quotient group G/H if necessary, we may assume that G acts faithfully on M_α.

In order to apply Lemma 6.5, we should check the validity of the assumptions (i) and (ii) of Lemma 6.5. The assumption (i) is valid by Theorem 6.2. As to the assumption (ii), we will show that $H^2(G', \mathbb{C}^\times) \to H^2(G', N_\beta)$ is injective for any subgroup $G' \subset G$, which is equivalent to the assumption (ii) of Lemma 6.5, because N_β is H^1 trivial.

Note that $H^2(G', \mathbb{C}^\times)$ is the trivial group, because we may consider $H^2(G'_p, \mathbb{C}^\times)$ where G'_p is a p-Sylow subgroup of G' and we find that $H^2(G'_p, \mathbb{C}^\times) \simeq H^0(G'_p, \mathbb{C}^\times) \simeq \mathbb{C}^\times/(\mathbb{C}^\times)^q = 0$ where q is the order of the cyclic group G'_p. Hence the result. $\square$

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