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Note

The intersection problem for star designs

Elizabeth J. Billington[a,*], D.G. Hoffman[b,1]

[a] Centre for Combinatorics, Department of Mathematics, The University of Queensland, Queensland 4072, Australia

[b] Department of Discrete and Statistical Sciences, 120 Math Annex, Auburn University, Auburn, AL 36849-5307, USA

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Abstract

We determine those triples (m,n,k) of integers for which there are two m-star designs on the same n-set having exactly k stars in common.

1. Introduction

If G is a graph without isolated vertices (all graphs here are simple; see [3] for our definitions and notation), we say B is a *G-design* if B is a finite set of graphs, each isomorphic to G, with the property that for every pair u,v of vertices in the union of their vertex set, there is at most one graph in B of which uv is an edge. If H is the union of the graphs in B, we say B is a *G-design on H*, so $\varepsilon(H)=|B|\varepsilon(G)$. If B is a G-design on the complete graph K_n, then B is called a *G-design of order n*. The *spectrum problem for G* is the problem of determining $\mathrm{Spec}(G)$; this is the set of all n for which there is a G-design of order n. The *intersection problem for G* is the problem of determining, for all $n\in\mathrm{Spec}(G)$, the set $I_G(n)$; this is the set of all k for which there are G-designs B_1 and B_2, on the same vertex set, with $|B_1\cap B_2|=k$.

We also define the set $J_G(n)$; this consists of all those non-negative integers k, with $k\varepsilon(G)\leqslant\binom{n}{2}$, *except* for $k=(\binom{n}{2}/\varepsilon(G))-1$. Obviously, $I_G(n)\subseteq J_G(n)$.

The first intersection problem solved was for $G=K_3$, in [6]. Subsequently, it was solved for several small graphs G, as well as for other combinatorial structures; see [1,2].

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Here we solve the intersection problem for an infinite family of graphs, the stars. For each positive integer m, the *m-star* S_m consists of a single vertex, *the center*, adjacent to each of m further vertices, and no further edges. (So S_m is the complete bipartite graph $K_{1,m}$.) The spectrum problem was solved by Huang [5] and Tarsi [8] independently:

Theorem 1. *For m,n positive integers, there is an m-star design of order n if and only if $2m|n(n-1)$, and either $n=1$ or $n \geqslant 2m$.*

Here is our result, in which we abbreviate $I_G(n)$ and $J_G(n)$ to $I_m(n)$ and $J_m(n)$, respectively, if G is the m-star:

Theorem 2. *Let m and n be positive integers, with $n \in \mathrm{Spec}(S_m)$. Then $I_m(n)=J_m(n)$, with exactly these exceptions:*
(1) $I_1(n)=\{\binom{n}{2}\}$.
(2) *If* $m \geqslant 3$, $I_m(2m)=J_m(2m)\backslash\{2m-3\}$.

We proceed to prove Theorem 2. We will assume $m \geqslant 2$ and $n \geqslant 2m$, as if either m or n is 1, the result is trivial.

2. Trades, weights, and tournaments

The G-designs T_1 and T_2 are defined to be *mates* if they are G-designs on the same graph H, and $T_1 \cap T_2 = \emptyset$. The G-design T is defined to be a *trade of volume t* if $|T|=t$, and T has at least one mate. Note that there are no trades of volume 1. The following lemma, the proof of which is left to the reader, reveals the significance of trades for our problem.

Lemma 1. *Let B_1 be a G-design of order n. Then there is a G-design B_2 of order n, on the same set of n vertices as B_1, with $|B_1 \cap B_2|=k$, if and only if B_1 contains a trade of volume $\binom{n}{2}/\varepsilon(G)-k$.*

An m-star design B on H defines a *weight function* ω from $V(H)$ to the non-negative integers as follows: for each vertex v of H, $\omega(v)$ is the number of stars of B for which v is the center. (So the weights sum to $\varepsilon(H)/m$.) We say B is *balanced* if for each pair v,w of vertices of H, $|\omega(v)-\omega(w)| \leqslant 1$. Implicit in Tarsi's proof of Theorem 1 from [8] is the following:

Theorem 3. *If there is an m-star design of order n, then there is a balanced one.*

An m-star design B on H also defines an orientation of H as follows: for every star in B, orient its edges away from its center. In the resulting digraph, each vertex v has out-degree $m\omega(v)$. (The converse is also true: if H has an orientation in which each vertex has outdegree divisible by m, then there is an m-star design on H. We exploit this fact in [4].) Now if B is an m-star design of order n, then H is K_n, and the resulting digraph is a *tournament*.

Moon, in [7], proved a nice result about tournaments; but first we need to recall a definition. A digraph D is said to be *strongly connected* if for every ordered pair (u, v) of vertices of D, there is a directed walk from u to v. The following lemma gives an equivalent definition; we again leave the proof to the reader.

Lemma 2. *D is strongly connected if and only if for every ordered pair (R, S) of non-empty sets with $R \cap S = \emptyset$ and $R \cup S = V(D)$, there is at least one arc of D whose tail is in S, and whose head is in R.*

Here is Moon's theorem:

Theorem 4. *Let D be a strongly connected tournament, let v be a vertex of D, and let $3 \leqslant t \leqslant v(D)$. Then D contains a directed cycle of length t containing v.*

We conclude this section with one more lemma. A *regular* tournament is one in which each vertex has the same indegree as outdegree. (Note that such a tournament must have an odd number of vertices.)

Lemma 3. *Every regular tournament is strongly connected.*

Proof. With (R, S) as in Lemma 2, note that the number of arcs directed from S to R is exactly $\frac{1}{2}|R||S|$, certainly a positive integer. □

3. The Proof of Theorem 2

Our strategy will be to show that certain star designs have trades of all possible sizes, and this will prove the theorem by Lemma 1. The next two lemmas describe the two types of trades we will use. We leave the proof of the first to the reader.

Lemma 4. *Let B be an m-star design on an sm-star S_{sm}. Then B contains a trade of volume t for all $0 \leqslant t \leqslant s$, except for $t = 1$.*

Lemma 5. *Let B be an m-star design whose associated digraph contains a directed cycle C of length t, at least s of whose vertices v have weight $\omega(v)$ greater than one. Then B contains a trade of volume $s + t$.*

Proof. Let the vertices and edges of the cycle C be indexed by the ring $\mathbb{Z}_t$ of integers $(\bmod\, t)$, so that for each $i \in \mathbb{Z}_t$, C contains the edge e_i oriented from v_i to

v_{i+1} (subscripts modulo t). Let $X \subseteq \mathbb{Z}_t$ with $|X| = s$, and $\omega(v_i) \geqslant 2$ for $i \in X$. For each $i \in \mathbb{Z}_t$, let G_i be the m-star in B containing e_i. For each $i \in X$, let H_i be any m-star of B centered at v_i other than G_i, and let f_i be an edge of H_i. We claim $T = \{G_i \mid i \in \mathbb{Z}_t\} \cup \{H_i \mid i \in X\}$ is a trade; in fact, we produce a mate. For each $i \in X$, let G_i' be the m-star obtained by replacing the edge e_i in G_i by f_i, and let H_i' be the m-star obtained by replacing the edge f_i in H_i by e_{i-1}. And for each $i \in \mathbb{Z}_t \backslash X$, let G_i' be the m-star obtained by replacing the edge e_i in G_i by e_{i-1}. Then $\{G_i' \mid i \in \mathbb{Z}_t\} \cup \{H_i' \mid i \in X\}$ is a mate to T. $\square$

Our proof of Theorem 2 divides into four cases.

Case 1. $n = 2m$: Let B be a balanced m-star design of order $2m$. (Actually, all m-star designs of order $2m$ are balanced!) This has one vertex, v_0, of weight 0, and the rest each have weight 1. If D is the resulting tournament, then $D \backslash v_0$ is a regular tournament. So by Lemma 3 and Theorem 4, $D \backslash v_0$, and hence D, contains a directed cycle of length t for each t in the range $3 \leqslant t \leqslant 2m-1$. So by Lemma 5, with $s = 0$, B contains a trade of volume t. Now Lemma 1 shows $I_m(2m)$ contains all non-negative integers up to $2m-4$. We already observed $2m-2 \notin I_m(2m)$, and trivially $2m-1 \in I_m(2m)$. So only one thing is left: is $2m-3 \in I_m(2m)$? It is easy to show that $1 \in I_2(4)$, but what about $m \geqslant 3$? This is taken care of by the next lemma, the proof of which we again leave to the reader.

Lemma 6. *Let T be an m-star trade of volume 2 on a graph H. If $m \geqslant 3$, then H is a $2m$-star.*

This proves $2m-3 \notin I_m(2m)$ for $m \geqslant 3$, because a $2m$-star has $2m+1$ vertices.

Case 2. $n = 2m+1$: Let $k \in I_m(2m)$, and let B_1 and B_2 be m-star designs of order $2m$ with $|B_1 \cap B_2| = k$. We convert these to designs of order $2m+1$ by adding two more m-stars centered at a new vertex v_0. Thus $k+2 \in I_m(2m+1)$. Also, these two new stars are a trade of volume 2 by Lemma 4, so also $k \in I_m(2m+1)$. From the case 1 results, we have proved $I_m(2m+1) = J_m(2m+1)$.

Case 3. $2m+2 \leqslant n \leqslant 4m+1$: Let B be a balanced m-star design of order n. Then each vertex has weight either 1 (a *small* vertex) or 2 (a *large* vertex). Note there must be at least one large vertex. Let D be the associated digraph, so small vertices have outdegree m, and large vertices have outdegree $2m$.

Lemma 7. *D is strongly connected.*

Proof. If not, there are sets R, S as in Lemma 2, with every arc between R and S directed from R to S. So B, restricted to S, must be a sub-m-star design of order $|S|$. But $|S| \neq 1$, (or else its only vertex would have outdegree 0) so $|S| \geqslant 2m$. But any vertex of R has outdegree at most $2m$, so $|S| \leqslant 2m$. Hence $|S| = 2m$, and every vertex of R has outdegree exactly $2m$. Now suppose u, v are two different vertices in R. The arc joining them must be oriented in some direction, say from u to v. But then u has outdegree at least $2m + 1$, a contradiction. Hence, $|R| = 1$, so $n = 2m + 1$, again a contradiction. $\square$

Now we use Theorem 4, and then Lemma 5, two times. First, with $3 \leqslant t \leqslant n - 1$, and $s = 0$; then with $t = n$ and $0 \leqslant s \leqslant n(n-1)/2m - n$. This shows B contains trades of all the required volumes except a volume of 2. But Lemma 4 takes care of this last case, since there is at least one large vertex.

Case 4. $4m + 2 \leqslant n$: Again, let B be a balanced m-star design of order n. This time, each vertex has weight at least 2, and there is at least one vertex of weight at least 3. This means that given $k \in J_m(n)$, we can find, for each vertex v, a non-negative integer $t_v \neq 1$, with $t_v \leqslant \omega(v)$, so that

$$\sum_v t_v = n(n-1)/2m - k.$$

By Lemma 4, for each vertex v, the stars of B centered at v contain a trade of volume t_v. The union of all these trades is a trade of volume $n(n-1)/2m - k$, so $k \in I_m(n)$ by Lemma 1.

This concludes the proof of Theorem 2.

We can get a little more out of the proof:

Theorem 5. *Let $k \in I_m(n)$. Then there exists a pair B_1, B_2 of balanced m-star designs of order n on the same n-set, with the same weight-function and $|B_1 \cap B_2| = k$, with exactly these exceptions*:
(1) $m = 2$, $n = 4$, $k = 1$;
(2) $m \geqslant 2$, $n = 2m + 1$, $k = 2m - 1$.

In (1), we can find appropriate balanced designs, but they cannot have the same weight function. In (2), we can find appropriate designs with the same weight function, but they cannot be balanced. But if $m = 2$ in (2), we can find a pair of designs, one of which is balanced.

Proof. The cases $m = 2$, $n = 4$ or 5 are easily verified. Our constructions in the proof of Theorem 2 do the rest, except in the case $n = 2m + 1$, where our construction uses designs that are not balanced. But the techniques of Case 3 can be used on a balanced design; we leave the details to the reader. (The techniques do not work if $k = 2m - 1$.) $\square$

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