Discrete Mathematics 66 (1987) 15–20
North-Holland

CRITICALLY (k, k)-CONNECTED GRAPHS

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Received 6 November 1985
Revised 17 November 1986

The vertex connectivity of a graph G is denoted by $\kappa(G)$ and the minimum degree of G is denoted by $\delta(G)$. A finite simple graph G is said to be critically (k, k)-connected if $\kappa(G) = \kappa(\bar{G}) = k$ and for each vertex v of G $\kappa(G - v) = k - 1$ or $\kappa(\bar{G} - v) = k - 1$, where $\bar{G}$ is the complement of G. The following result is proved: If G is a critically (k, k)-connected graph, $k \geq 2$, $\delta(G) \geq \frac{1}{2}(3k - 1)$ and $\delta(\bar{G}) \geq \frac{1}{2}(3k - 1)$, then $|V(G)| \leq 4k$. Furthermore, these bounds are sharp for $k \geq 3$.

1. Introduction

In this paper, we consider only finite simple graphs. The set of the vertices and the set of the edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The vertex connectivity of G is denoted by $\kappa(G)$ and the minimum degree of G is denoted by $\delta(G)$. A graph G is said to be critically k-connected if $\kappa(G) = k$ and $\kappa(G - v) = k - 1$ for each vertex v of G. Chartrand, Kaugars and Lick [1] have shown that if G is a critically k-connected graph, $k \geq 2$, then $\delta(G) < \frac{1}{2}(3k - 1)$ and that this bound is sharp.

Let $\bar{G}$ be the complement of G. A graph G is said to be critically (k, k)-connected if $\kappa(G) = \kappa(\bar{G}) = k$ and for each vertex v of G $\kappa(G - v) = k - 1$ or $\kappa(\bar{G} - v) = k - 1$. In this paper, we prove the following theorem.

Theorem A. *If G is a critically (k, k)-connected graph, $k \geq 2$, $\delta(G) \geq \frac{1}{2}(3k - 1)$ and $\delta(\bar{G}) \geq \frac{1}{2}(3k - 1)$, then $3k \leq |V(G)| \leq 4k$.*

We remark that this upper bound of $|V(G)|$ is sharp. Define $H_1 = K_k$, $H_2 = \bar{K}_k$, $H_3 = \bar{K}_k$ and $H_4 = K_k$. We define a graph G as follows:

$$V(G) = \bigcup_{i=1}^{4} V(H_i),$$

$$E(G) = \bigcup_{i=1}^{4} E(H_i) \cup \{(v, v') \mid v \in V(H_i),\ v' \in V(H_{i+1}),\ 1 \leq i \leq 3\}.$$

On the other hand for $k \geq 3$ the lower bound of minimum degree is sharp, since

0012-365X/87/$3.50

there are infinitely many critically (k, k)-connected graphs with $\delta(G) = m$ for each integer m satisfying $k \leqslant m < \frac{1}{2}(3k - 1)$. These examples will be given in Section 4.

2. Preliminaries

We will give a few definitions and some notation (cf. [2]). If H is a subset of $V(G)$, then we denote by $|H|$ the number of the vertices in H. A subset H of $V(G)$ may be identified with the induced subgraph $\langle H \rangle_G$ in G (or the induced subgraph $\langle H \rangle_{\bar{G}}$ in $\bar{G}$). A subset S of $V(G)$ is called a *k-cut* of G if the induced subgraph $G - S = \langle V(G) - S \rangle_G$ is disconnected and $|S| = k$. For a subset H of $V(G)$, set $N_G(H) = \bigcup_{x \in H} N_G(x) - H$, where $N_G(x)$ is the set of the vertices adjacent to x in G. Set $d_G(x) = |N_G(x)|$. A non-empty subset C of $V(G)$ is called a *fragment* of G if $N_G(C)$ is a $\kappa(G)$-cut of G and $V(G) - N_G(C) - C \neq \emptyset$. If C is a fragment of G and $N_G(C) = S$, then we say that C is a fragment with respect to S. A fragment C of G is called an *end* if $|N_G(C')| > \kappa(G)$ for any non-empty proper subset C' of C. Clearly an end C is a component of $G - N_G(C)$. An end of G with minimum cardinality is called an *atom* of G and its cardinality is denoted by a_G. We define $\bar{C} = G - (C \cup N_G(C))$ for a fragment C of G. We also define $\bar{C} = \bar{G} - (C \cup N_{\bar{G}}(C))$ for a fragment C of $\bar{G}$.

We remark that if $\delta(G) \geqslant \frac{1}{2}(3\kappa(G) - 1)$, then $a_G > \frac{1}{2}\kappa(G)$. In Section 3, we will prove the following Theorem B which is a generalization of Theorem A.

Theorem B. *If G is a critically (k, k)-connected graph, $k \geqslant 2$, $a_G > \frac{1}{2}k$ and $a_{\bar{G}} > \frac{1}{2}k$, then $|V(G)| \leqslant 4k$.*

The properties of ends in the following lemmas are essentially in our argument in Section 3. Lemma C follows from Lemma 3 in [4]. Lemma D follows from Satz 2 in [3]. (We remark that $|B| < |V(G)| - \frac{3}{2}\kappa(G)$ for any end B of G if $a_G > \frac{1}{2}\kappa(G)$.)

Lemma C (Mader). *If A is an end of G and $a_G > \frac{1}{2}\kappa(G)$, then $A \cap S = \emptyset$ for any $\kappa(G)$-cut S of G.*

Lemma D (Mader). *If $a_G > \frac{1}{2}\kappa(G)$ and B_1 and B_2 are two distinct ends of G, then $B_1 \cap B_2 = \emptyset$.*

3. A proof of Theorem B

Let $\mathscr{C}(G)$ (resp. $\mathscr{C}(\bar{G})$) be the set of the vertices contained in some $\kappa(G)$-cut of G (resp. $\kappa(\bar{G})$-cut of $\bar{G}$). Note that $V(G) = \mathscr{C}(G) \cup \mathscr{C}(\bar{G})$, if G is critically

(k, k)-connected. In this section we suppose G is critically (k, k)-connected, $k \geq 2$, $a_G > \frac{1}{2}k$ and $a_{\bar{G}} > \frac{1}{2}k$. Let $\{X_i\}$ $(i = 1, 2, \ldots)$ be all the ends of G and $\{Y_j\}$ $(j = 1, 2, \ldots)$ be all the ends of $\bar{G}$. Set $N_G(X_i) = S_i$, $N_{\bar{G}}(Y_j) = T_j$ for each i and j.

Lemma 1. *The following hold:*

(i) $X_i \cap \mathscr{C}(G) = \emptyset$ *and* $Y_j \cap \mathscr{C}(\bar{G}) = \emptyset$, *for all* i, j;

(ii) $X_i \cap X_{i'} = \emptyset$, $Y_j \cap Y_{j'} = \emptyset$ *for all* $i \neq i'$, $j \neq j'$;

(iii) $X_i \cap Y_j = \emptyset$ *for all* i, j.

Proof. Lemma C (resp. Lemma D) assures us of (i) (resp. (ii)). By Lemma C, $X_i \subseteq V(G) - \mathscr{C}(G) = \{\mathscr{C}(G) \cup \mathscr{C}(\bar{G})\} - \mathscr{C}(G) = \mathscr{C}(\bar{G}) - \mathscr{C}(G)$ and $Y_j \subseteq \mathscr{C}(G) - \mathscr{C}(\bar{G})$. Then $X_i \cap Y_j = \emptyset$. Hence (iii) holds. $\square$

From now on, we assume $|V(G)| > 3k$.

Lemma 2. *There exist only two ends* X_1, X_2 *of* G *(resp.* Y_1, Y_2 *of* $\bar{G}$*), and* $|\tilde{X}_i| \geq k+1$, $|\tilde{Y}_j| \geq k+1$, *for each* $1 \leq i, j \leq 2$. *Furthermore, we can assume* $T_1 \supseteq X_1$, $T_2 \not\supseteq X_1$, $T_1 \not\supseteq X_2$, $T_2 \supseteq X_2$, $S_1 \not\supseteq Y_1$, $S_2 \supseteq Y_1$, $S_1 \supseteq Y_2$ *and* $S_2 \not\supseteq Y_2$.

Proof. First we remark that in G, $Y_j \cup \tilde{Y}_j$ contains a complete bipartite graph with partite sets Y_j and $\tilde{Y}_j$ as its spanning subgraph. Our second remark is that if $Y_j \cup \tilde{Y}_j - S_i$ is connected in $G - S_i$ and $|X_i| < |V(G)| - 2k \leq |Y_j \cup \tilde{Y}_j - S_i|$, then $X_i \subseteq T_j$.

We claim that $|\tilde{X}_i|$, $|\tilde{Y}_j| > k$ for all i, j. Suppose $|\tilde{X}_i| \leq k$. Then $|X_i| = |V(G)| - |\tilde{X}_i| - |S_i| > k$. If $X_{i'}$ is an end in $\tilde{X}_i$, then

$$\sum_{\text{all } j} |Y_j| \leq |\tilde{X}_i - X_{i'}| + |S_i| < \tfrac{3}{2}k.$$

This implies $|Y_j| \leq k < |V(G)| - 2k$ for all j. If $T_j \not\supseteq \tilde{X}_i$, then $X_i \cup \tilde{X}_i - T_j$ is connected in $\bar{G} - T_j$ by the first remark. Hence, as a consequence of the second remark, $S_i \supseteq Y_j$. If $T_j \supseteq \tilde{X}_i$, then $S_i \supseteq Y_j$, since $X_i \cap Y_j = \emptyset$ by Lemma 1. In any case $S_i \supseteq Y_j$ for any j. Hence

$$k = |S_i| \geq \sum_{\text{all } j} |Y_j| > k,$$

a contradiction. In case $|\tilde{Y}_j| \leq k$ we can show a similar contradiction to the above case.

Now $|\tilde{X}_i|$, $|\tilde{Y}_j| > k$ for all i, j. Then $|X_i| < |V(G)| - 2k$ and $|Y_j| < |V(G)| - 2k$ for all i, j. We claim that if $T_j \not\supseteq X_i$, then $Y_j \subseteq S_i$. If $T_j \not\supseteq X_i$, then $(X_i \cup \tilde{X}_i) - T_j$ is connected in $\bar{G} - T_j$ by the first remark. Consequently $Y_j \subseteq S_i$ by the second remark. Now if $T_1 \not\supseteq X_1$, $T_2 \not\supseteq X_1$, then $Y_1 \cup Y_2 \subseteq S_1$, by the above claim. This is a contradiction. If $T_1 \supseteq X_1$ and $T_2 \supseteq X_1$, then $T_1 \not\supseteq X_2$ and $T_2 \not\supseteq X_2$, which is a contradiction similarly. Now we can assume $T_1 \supseteq X_1$ and $T_2 \not\supseteq X_1$ and there is no

T_3. Then there exist only two ends Y_1 and Y_2 in $\bar{G}$. Similarly we can show that there exist only two ends X_1 and X_2 in G. Hence the first statement holds.

Since $T_1 \supseteq X_1$ and $T_2 \not\supseteq X_1$, $T_1 \not\supseteq X_2$, $T_2 \supseteq X_2$ and by the above claim $Y_2 \subseteq S_1$, $Y_1 \subseteq S_2$. Hence the second statement holds. □

By Lemma 2 note that $G - S$ consists of two components for any k-cut S of G. One contains X_1 and the other contains X_2.

Lemma 3. *Suppose S is a k-cut of G. Then $S \not\supseteq Y_1$ or $S \not\supseteq Y_2$. If $S \not\supseteq Y_i$, then the component C of $G - S$ containing X_i is contained in T_i and the other component of $G - S$ has at least $k + 1$ vertices.*

Proof. The first claim holds since the fact that $S \supseteq Y_1$ implies that $S \not\supseteq Y_2$. Suppose $S \not\supseteq Y_i$. Since $S \not\supseteq Y_i$ and $\bar{Y}_i - S \neq \emptyset$, $(Y_i \cup \bar{Y}_i) - S$ is connected in $G - S$ and one component C of $G - S$ is contained in T_i. Since X_i is the only end of G contained in T_i, C must contain X_i. The last statement follows, since $|V(G)| > 3k$. □

Lemma 4. (i) *There is only one maximal element X_i^* in the set of all the fragments of G contained in T_i.*

(ii) *If S is a k-cut of G satisfying $S \not\supseteq Y_i$, then $\bar{X}_i^* \cap S = \emptyset$.*

(iii) $\bar{X}_1^* \cap \bar{X}_2^* \cap \mathscr{C}(G) = \emptyset$.

(iv) *$\bar{Y}_1^* \cap \bar{Y}_2^* \cap \mathscr{C}(\bar{G}) = \emptyset$ for the maximal element Y_i^* in the set of all the fragments of $\bar{G}$ contained in S_i $(i = 1, 2)$.*

(v) $\bar{X}_1^* \cap \bar{X}_2^* \cap \bar{Y}_1^* \cap \bar{Y}_2^* = \emptyset$.

Proof. Let C and C' be fragments of G contained in T_i. First we will show $C \cup C'$ is a fragment of G contained in T_i. Clearly $C \cup C' \subseteq T_i$. Note that

$$|\bar{C} \cap \bar{C}'| \geqslant |V(G)| - |C \cup C'| - |N_G(C)| - |N_G(C')|$$
$$\geqslant |V(G)| - |T_i| - 2k = |V(G)| - 3k > 0.$$

Since $C \cap C' \supseteq X_i \neq \emptyset$ and $\bar{C} \cap \bar{C}' \neq \emptyset$, $C \cup C'$ is also a fragment of G (cf. Lemma 2.1 in [2]). Hence (i) holds.

If C is a fragment of G with respect to S contained in T_i, then $C \subseteq X_i^*$. Hence $\bar{C} \supseteq \bar{X}_i^*$ and so $\bar{X}_i^* \cap S \subseteq \bar{C} \cap S = \emptyset$. Thus (ii) holds. If S is a k-cut of G, then $S \not\supseteq Y_1$ or $S \not\supseteq Y_2$. Hence from (ii), (iii) follows. By a similar argument for $\bar{G}$, (iv) holds. Since $V(G) = \mathscr{C}(G) \cup \mathscr{C}(\bar{G})$, (v) follows. □

With these lemmas, we are now prepared to prove Theorem B. Set $N_G(X_i^*) = S_i^*$ and $N_{\bar{G}}(Y_i^*) = T_i^*$.

Clearly $|\bar{X}_i^*| > k$ and $|\bar{Y}_i^*| > k$ for $i = 1, 2$. First we claim $V(G) - T_2^* - S_1^* \subseteq \bar{X}_1^*$. Note that $Y_1 - S_1^* \neq \emptyset$. (If $S_1^* \supseteq Y_1$, then $S_1^* \not\supseteq Y_2$ and the component of $G - S_1^*$

containing X_2 is contained in T_2 by Lemma 3, which contradicts $X_2 \subseteq \bar{X}_1^*$ and $|\bar{X}_1^*| > k$.) $S_2 \supseteq Y_2^* \supseteq Y_1$ by Lemma 2 and Lemma 4(iv). Since $Y_2^* - S_1^* \supseteq Y_1 - S_1^* \neq \emptyset$ and $\bar{Y}_2^* - S_1^* \neq \emptyset$, $V(G) - T_2^* - S_1^*$ is connected in $G - S_1^*$. Since $|V(G) - T_2^* - S_1^*| > k$, $V(G) - T_2^* - S_1^* \subseteq \bar{X}_1^*$ and the claim holds. Similarly $V(G) - T_1^* - S_2^* \subseteq \bar{X}_2^*$. Then

$$V(G) - (S_1^* \cup S_2^* \cup T_1^* \cup T_2^*) \subseteq \bar{X}_1^* \cap \bar{X}_2^*.$$

Similarly for $\bar{G}$

$$V(G) - (S_1^* \cup S_2^* \cup T_1^* \cup T_2^*) \subseteq \bar{Y}_1^* \cap \bar{Y}_2^*.$$

Then

$$V(G) - (T_1^* \cup T_2^* \cup S_1^* \cup S_2^*) \subseteq \bar{X}_1^* \cap \bar{X}_2^* \cap \bar{Y}_1^* \cap \bar{Y}_2^* = \emptyset,$$

by Lemma 4(v). Hence $|V(G)| \leq 4k$ and Theorem B holds.

4. Examples

For any given positive integer n and $k \geq 3$, we will give a critically (k, k)-connected graph $G(k, n)$ with $\delta(G(k, n)) = \lceil \frac{1}{2}(3k-3) \rceil$ and $\delta(\bar{G}(k, n)) = \lceil \frac{1}{2}(3k-1) \rceil$ and $|G(k, n)| = \lceil \frac{1}{2}(7k+1) \rceil + \lceil \frac{1}{2}k - 1 \rceil n$. It is convenient to set $k' = \lceil \frac{1}{2}k - 1 \rceil$ and $k'' = \lceil \frac{1}{2}(k+1) \rceil$. Define $H_1 = K_k$ and $H_2 = K_k$ and set $V(H_1) = \{v_1, v_2, \ldots\}$ and $V(H_2) = \{u_1, u_2, \ldots\}$. Also define $M_j = K_{k'}$ and set $V(M_j) = \{x_{j,1}, x_{j,2}, \ldots, x_{j,k'}\}$ for each j $(1 \leq j \leq n)$. Let $G_0(k, n)$ be a graph such that

$$V(G_0(k, n)) = V(H_1) \cup V(H_2) \cup \left(\bigcup_{j=1}^{n} V(M_j) \right)$$

and

$$\begin{aligned} E(G_0(k, n)) = {} & \left(\bigcup_{j=1}^{n} E(M_j) \right) \cup E(H_1) \cup E(H_2) \\ & \cup \{(v_i, x_{1,h}), (u_i, x_{n,h}) \mid 1 \leq i \leq k,\ 1 \leq h \leq k'\} \\ & \cup \left(\bigcup_{j=2}^{n} \{(x_{j-1,i}, x_{j,h}) \mid 1 \leq i,\ h \leq k'\} \right). \end{aligned}$$

Define $H_0 = \bar{K}_{k''}$, $L_1 = K_{k''-1}$ and $L_2 = K_{k'+1}$. We define $G(k, n)$ as follows:

$$\begin{aligned} V(G(k, n)) &= V(G_0(k, n) \cup V(L_1) \cup V(L_2) \cup V(H_0), \\ E(G(k, n)) &= E(G_0(k, n)) \cup E(L_1) \cup E(L_2) \\ & \quad \cup \{(v, v') \mid v \in V(L_i),\ v' \in V(H_i),\ 1 \leq i \leq 2\} \\ & \quad \cup \{(u, u') \mid u \in V(G_0(k, n)),\ u' \in V(H_0)\}. \end{aligned}$$

Now $d_{G(k,n)}(z) = \lceil \frac{1}{2}(3k-3) \rceil = \delta(G(k, n))$ for a vertex z of L_1 and $d_{\bar{G}(k,n)}(z') = \lceil \frac{1}{2}(3k-1) \rceil = \delta(\bar{G}(k, n))$ for a vertex z' of H_0. The graph given in Fig. 1 is $G(5, 4)$.

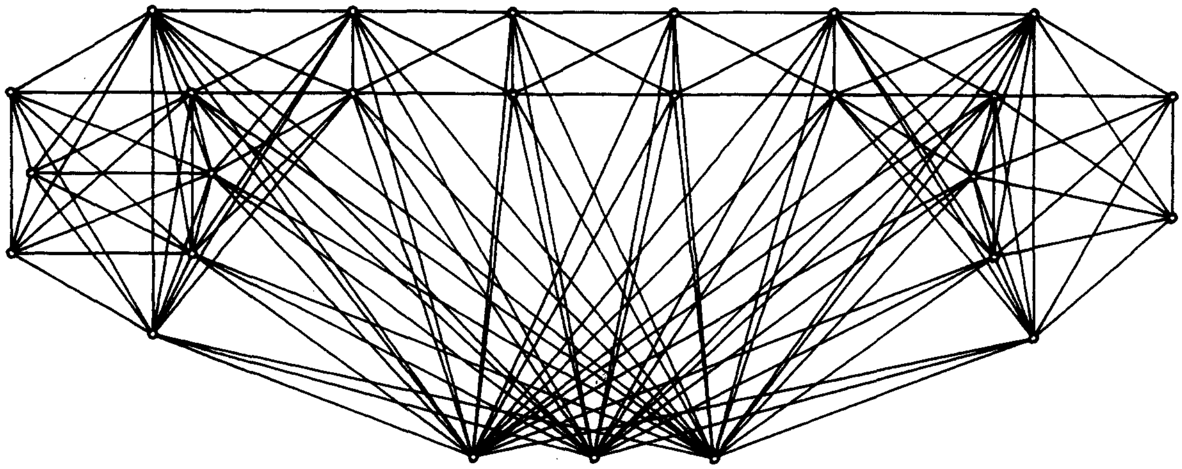

Fig. 1.

If we define $L_1 = K_{m-k+1}$ and $L_2 = K_{2k-m-1}$, for each fixed integer m satisfying $k \leq m < \frac{1}{2}(3k-1)$, then the graph $G'(k, n, m)$ with $n \geq 2$ constructed in the same way as $G(k, n)$ is a critically (k, k)-connected graph with $\delta(G'(k, n, m)) = m$ and $\delta(\bar{G}'(k, n, m)) = \lceil \frac{1}{2}(3k-1) \rceil$.

In [3] Hamidoune proved that a critically k-connected graph contains two vertices of degree not exceeding $\frac{3}{2}k - 1$ and there is a critically k-connected graph having exactly two vertices of degree not exceeding $\frac{3}{2}k - 1$ for each $k \geq 3$. On the other hand, there are infinitely many critically (k, k)-connected graphs having exactly one vertex of degree in G or in $\bar{G}$ not exceeding $\frac{3}{2}k - 1$ for every $k \geq 5$ and $k = 3$. For $k \geq 5$ and $k = 3$ the graphs $G'(k, n, k)$ with $n \geq 2$ are such examples.

By tedious calculation, we can show a critically $(2, 2)$-connected graph with $\delta(G) \geq 2$ and $\delta(\bar{G}) \geq 3$ satisfies $|V(G)| = 8$ or 9. There are infinitely many critically $(2, 2)$-connected graphs with $\delta(G) = \delta(\bar{G}) = 2$. If we define $H_1 = K_2$, $H_2 = K_2$, $M_j = K_1$ $(1 \leq j \leq n)$, $H_0 = K_1$, $L_1 = K_1$, and $L_2 = K_1$, then the graph $G''(2, n)$ constructed in the same way as $G(k, n)$ is one of such graphs for each positive integer n.

References

[1] G. Chartrand, A. Kaugars and D.R. Lick, Critically n-connected graphs, Proc. Amer. Math. Soc. 32 (1972) 63–68.

[2] Y. O. Hamidoune, On critically h-connected simple graphs, Discrete Math. 32 (1980) 257–262.

[3] W. Mader, Ein Eigenshaft der Atome endlicher Graphen, Arch. Math. 22 (1971) 333–336.

[4] W. Mader, Disjunkte Fragmente in kritisch n-fach zusammenhängenden Graphen, European J. Combin. 6 (1985) 353–359.