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Positive solutions for nonlinear discrete second-order boundary value problems with parameter dependence ☆

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ABSTRACT

In this paper, we consider the nonlinear discrete boundary value problem

$$\begin{cases} -\Delta\big[p(t-1)\Delta u(t-1)\big]+q(t)u(t)=\lambda f\big(t,u(t)\big), & t\in \mathbf{Z}[1,T],\\ u(0)=u(T), \qquad p(0)\Delta u(0)=p(T)\Delta u(T), \end{cases}$$

where λ is a positive parameter. By using the fixed point index theory, the criteria of the existence, multiplicity and nonexistence of positive solutions are established in terms of different values of λ.

1. Introduction

Let $\mathbf{R}, \mathbf{Z}, \mathbf{N}$ be the sets of real numbers, integers and natural numbers, respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}[a,b]=\{a, a+1, \ldots, b\}$ when $a \leqslant b$.

For some given positive integer T with $T>2$, we are concerned with the problem of existence, multiplicity, and nonexistence of positive solutions for the following boundary value problem (BVP for short)

$$\begin{cases} -\Delta\big[p(t-1)\Delta u(t-1)\big]+q(t)u(t)=\lambda f\big(t,u(t)\big), & t\in \mathbf{Z}[1,T],\\ u(0)=u(T), \qquad p(0)\Delta u(0)=p(T)\Delta u(T), \end{cases} \tag{1.1}$$

where $f:\mathbf{Z}[1,T]\times[0,+\infty)\to[0,+\infty)$ is continuous in the second variable; $p:\mathbf{Z}[0,T]\to(0,+\infty)$; $q:\mathbf{Z}[1,T]\to[0,+\infty)$ with $q(\cdot)\not\equiv 0$; λ is a positive parameter; $\Delta u(t)=u(t+1)-u(t)$, $\Delta^2 u(t)=\Delta(\Delta u(t))$.

By a solution u of BVP (1.1), we mean a real sequence u which is defined on $\mathbf{Z}[0,T+1]$ and satisfies the difference equation as well as the boundary conditions in (1.1). A solution $\{u(t)\}_{t=0}^{T+1}$ of (1.1) is called to be positive if $u(t)>0$ for $t\in\mathbf{Z}[1,T]$.

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. Interesting examples and mathematical models coupled with this theory can be found in the classical monograph by S. Goldberg [15] and in the more recent books by R.P. Agarwal [1], V. Lakshmikantham and D. Trigiante [18] and S.N. Elaydi [19]. In recent years, the existence of solutions for nonlinear difference equations has been studied by many authors by using various methods and techniques,

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for example, fixed point theorems [5,8,13,17,22], the method of upper and lower solutions [3,4,6,16], coincidence degree theory [23], monotone iterative techniques [24] and critical point theory [2,7,9–12,14,20,21].

In this paper, the criteria of the existence, multiplicity and nonexistence of positive solutions for BVP (1.1) are established in terms of different values of λ via the fixed point index theory. Our results generalize and complement some previous findings of [5,6,25] and some other known results.

For convenience, we introduce the following notations:

$$f_0 = \liminf_{|x|\to 0} \min_{t\in \mathbf{Z}[1,T]} \frac{f(t,x)}{x}, \qquad f_\infty = \liminf_{|x|\to \infty} \min_{t\in \mathbf{Z}[1,T]} \frac{f(t,x)}{x},$$
$$f^0 = \limsup_{|x|\to 0} \max_{t\in \mathbf{Z}[1,T]} \frac{f(t,x)}{x}, \qquad f^\infty = \limsup_{|x|\to \infty} \max_{t\in \mathbf{Z}[1,T]} \frac{f(t,x)}{x}.$$

And we make the following assumptions:

(H_1) $f(t,x) > 0$ for any $t \in \mathbf{Z}[1,T]$ and $x > 0$;
(H_2) $f_0 = \infty$ and $f_\infty = \infty$;
(H_3) $f^0 = 0$ and $f^\infty = 0$;
(H_4) $p(0) \leqslant p(1) \leqslant \cdots \leqslant p(T)$.

Now we state the main results of this paper.

Theorem 1.1. *Assume that* (H_1), (H_2) *and* (H_4) *hold. Then there exists* $\lambda^* > 0$ *such that BVP* (1.1) *has at least two positive solutions for* $\lambda \in (0, \lambda^*)$, *at least one positive solution for* $\lambda = \lambda^*$ *and no positive solution for* $\lambda > \lambda^*$.

Theorem 1.2. *Assume that* (H_1), (H_3) *and* (H_4) *hold. Then there exists* $\lambda_* > 0$ *such that BVP* (1.1) *has at least two positive solutions for* $\lambda > \lambda_*$, *at least one positive solution for* $\lambda = \lambda_*$ *and no positive solution for* $\lambda \in (0, \lambda_*)$.

Theorem 1.3. *Let* ρ_1 *be the first eigenvalue of the linear boundary value problem*

$$\begin{cases} -\Delta\big[p(t-1)\Delta u(t-1)\big] + q(t)u(t) = \rho u(t), & t \in \mathbf{Z}[1,T], \\ u(0) = u(T), \qquad p(0)\Delta u(0) = p(T)\Delta u(T), \end{cases} \tag{1.2}$$

where $\rho_1 > 0$, *see Lemma* 2.3. *Set* $\frac{1}{0} := +\infty$ *and* $\frac{1}{+\infty} := 0$. *Then, the following hold true*:

(i) *If* $0 \leqslant f^\infty < f_0 \leqslant +\infty$, *then BVP* (1.1) *has at least one positive solution for any* $\lambda \in (\frac{\rho_1}{f_0}, \frac{\rho_1}{f^\infty})$.
(ii) *If* $0 \leqslant f^0 < f_\infty \leqslant +\infty$, *then BVP* (1.1) *has at least one positive solution for any* $\lambda \in (\frac{\rho_1}{f_\infty}, \frac{\rho_1}{f^0})$.

Corollary 1.1. *Assume that* $f : \mathbf{Z}[1,T] \times \mathbf{R} \to \mathbf{R}$ *is continuous, and* $xf(t,x) \geqslant 0$ *for any* $t \in \mathbf{Z}[1,T]$ *and* $x \in \mathbf{R}$. *Set* $\frac{1}{0} := +\infty$ *and* $\frac{1}{+\infty} := 0$. *Then, the following hold true*:

(i) *If* $0 \leqslant f^\infty < f_0 \leqslant +\infty$, *then BVP* (1.1) *has at least one positive solution and one negative solution for any* $\lambda \in (\frac{\rho_1}{f_0}, \frac{\rho_1}{f^\infty})$.
(ii) *If* $0 \leqslant f^0 < f_\infty \leqslant +\infty$, *then BVP* (1.1) *has at least one positive solution and one negative solution for any* $\lambda \in (\frac{\rho_1}{f_\infty}, \frac{\rho_1}{f^0})$.

Remark 1.1. Corollary 1.1 is more general than Theorem 1.1 in [25]. In our Corollary 1.1, we replace the nonlinearity $g(t)f(x)$ by the nonlinearity $f(t,x)$, and the nonlinearity $f(t,x)$ need not be strictly positive for all $x > 0$. Also, the case when $f^\infty = 0$, $f_0 = +\infty$, or $f^0 = 0$, $f_\infty = +\infty$ cannot be hold in Theorem 1.1 of [25], but it is allowed in our Corollary 1.1.

The rest of the paper is arranged as follows. Section 2 presents some preliminaries. The proofs of Theorems 1.1–1.3 and Corollary 1.1 are given in Sections 3 and 4, respectively.

2. Preliminaries

Denote by $\{\varphi(t)\}_{t=0}^{T+1}$ and $\{\psi(t)\}_{t=0}^{T+1}$ the solutions of the corresponding homogeneous equation,

$$-\Delta\big[p(t-1)\Delta u(t-1)\big] + q(t)u(t) = 0, \quad t \in \mathbf{Z}[1,T],$$

under the initial conditions,

$$\varphi(0) = \varphi(1) = 1; \qquad \psi(0) = 0, \qquad p(0)\psi(1) = 1.$$

Lemma 2.1. *(See [5].) For the solution $\{u(t)\}_{t=0}^{T+1}$ of the problem*

$$\begin{cases} -\Delta\big[p(t-1)\Delta u(t-1)\big]+q(t)u(t)=h(t), & t\in\mathbf{Z}[1,T],\\ u(0)=u(T), \qquad p(0)\Delta u(0)=p(T)\Delta u(T), \end{cases}$$

the formula

$$u(t)=\sum_{s=1}^{T}G(t,s)h(s), \quad t\in\mathbf{Z}[0,T+1]$$

holds, where

$$G(t,s)=\frac{\psi(T)}{D}\varphi(t)\varphi(s)-\frac{p(T)\Delta\varphi(T)}{D}\psi(t)\psi(s) + \begin{cases} \frac{p(T)\Delta\psi(T)-1}{D}\varphi(t)\psi(s)-\frac{\varphi(T)-1}{D}\varphi(s)\psi(t), & 0\leqslant s\leqslant t\leqslant T+1,\\ \frac{p(T)\Delta\psi(T)-1}{D}\varphi(s)\psi(t)-\frac{\varphi(T)-1}{D}\varphi(t)\psi(s), & 0\leqslant t\leqslant s\leqslant T+1, \end{cases}$$

and $D=\varphi(T)+p(T)\Delta\psi(T)-2>0$.

Lemma 2.2. *(See [5].) Green's function $G(t,s)$ possesses the following property,*

$$G(t,s)>0, \quad t,s\in\mathbf{Z}[0,T].$$

Now, let us set

$$M=\max_{t,s\in\mathbf{Z}[1,T]}G(t,s), \qquad m=\min_{t,s\in\mathbf{Z}[1,T]}G(t,s).$$

Consider T-dimensional Banach space

$$E=\big\{u=\{u(t)\}_{t=1}^{T}\colon u(t)\in\mathbf{R},\ t\in\mathbf{Z}[1,T]\big\}$$

with the norm $\|u\|=\max\{|u(t)|, t\in\mathbf{Z}[1,T]\}$ for all $u\in E$ and the cone P in E given by

$$P=\left\{u\in E\colon u(t)\geqslant 0,\ \min_{t\in\mathbf{Z}[1,T]}u(t)\geqslant\frac{m}{M}\|u\|\right\}.$$

For $u,v\in E$, we write $u\leqslant v$ if $u(t)\leqslant v(t)$ for any $t\in\mathbf{Z}[1,T]$. For any $r>0$, let $B_r=\{u\in E\colon \|u\|<r\}$ and $\partial B_r=\{u\in E\colon \|u\|=r\}$. We denote by θ the zero element of E.

Define operators $K,\mathbf{f},A:E\to E$, respectively, by

$$\begin{aligned} &(Ku)(t)=\sum_{k=1}^{T}G(t,k)u(k), \quad u\in E,\ t\in\mathbf{Z}[1,T];\\ &(\mathbf{f}u)(t)=f\big(t,u(t)\big), \quad u\in E,\ t\in\mathbf{Z}[1,T];\\ &A=K\mathbf{f}. \end{aligned} \tag{2.1}$$

From [5, Lemmas 3.1 and 3.2], we know that $A:E\to E$ is a completely continuous operator and $A(P)\subset P$.

Remark 2.1. By Lemma 2.1, it is easy to see that $u=\{u(t)\}_{t=1}^{T}\in E$ is a fixed point of the operator λA if and only if $\{u(t)\}_{t=0}^{T+1}$ is a solution of BVP (1.1), where $u(0)=u(T), u(T+1)=u(T)+\frac{p(0)}{p(T)}\Delta u(0)$.

K defined by (2.1) is an important operator in our later discussion. We present some properties of it as follows.

Lemma 2.3. *The spectral radius $r(K)>0$ and there exists $\xi\in E$ with $\xi>0$ on $\mathbf{Z}[1,T]$ such that $K\xi=r(K)\xi$ and $\sum_{t=1}^{T}\xi(t)=\frac{1}{r(K)}$. Moreover, $\rho_1=\frac{1}{r(K)}$ is the first positive eigenvalue of the linear BVP (1.2) and*

$$\sum_{t=1}^{T}(Ku)(t)\xi(t)=\frac{1}{\rho_1}\sum_{t=1}^{T}u(t)\xi(t), \quad \forall u\in E. \tag{2.2}$$

Proof. Define the cone $P_0 = \{u \in E\colon\ u(t) \geqslant 0,\ \forall t \in \mathbf{Z}[1,T]\}$. Then the cone P_0 is normal and has nonempty interiors $\operatorname{int} P_0$. It is clear that P_0 is also a total cone of E, that is, $E = \overline{P_0 - P_0}$, which means the set $P_0 - P_0 = \{u - v\colon\ u, v \in P_0\}$ is dense in E. It follows from Lemma 2.2 that K is strongly positive, that is, $K(u) \in \operatorname{int} P_0$ for $u \in P_0 \backslash \{\theta\}$. Obviously, $K(P_0) \subseteq P_0$. By the Krein–Rutman theorem ([26, Theorem 7.C]; [27, Theorem 19.3]), the spectral radius $r(K) > 0$ and there exists $\xi_0 \in E$ with $\xi_0 > 0$ on $\mathbf{Z}[1,T]$ such that $K\xi_0 = r(K)\xi_0$. Let $\xi = \frac{\xi_0}{r(K)\sum_{t=1}^{T}\xi_0(t)}$. Obviously, $\xi > 0$ on $\mathbf{Z}[1,T]$, $K\xi = r(K)\xi$ and $\sum_{t=1}^{T}\xi(t) = \frac{1}{r(K)}$.

Noticing that $K\xi = r(K)\xi$ is equivalent to the following BVP

$$\begin{cases} -\Delta\big[p(t-1)\Delta\xi(t-1)\big] + q(t)\xi(t) = \dfrac{1}{r(K)}\xi(t), & t \in \mathbf{Z}[1,T], \\ \xi(0) = \xi(T), \qquad p(0)\Delta\xi(0) = p(T)\Delta\xi(T), \end{cases}$$

we can obtain that $\rho_1 = \frac{1}{r(K)}$ is an eigenvalue of the linear BVP (1.2). From the strong positivity of K, we know that there exist $\eta \in P_0$ and a constant $c > 0$ such that $cK\eta \geqslant \eta$ on $\mathbf{Z}[1,T]$. Then ρ_1 is the first positive eigenvalue of the linear problem.

For $x, y : \mathbf{Z} \to \mathbf{R}$, a simple computation shows

$$\begin{aligned} &\sum_{t=1}^{T} y(t)\Delta\big[p(t-1)\Delta x(t-1)\big] - \sum_{t=1}^{T} x(t)\Delta\big[p(t-1)\Delta y(t-1)\big] \\ &\quad = p(T)x(T+1)y(T) + p(0)x(0)y(1) - p(T)x(T)y(T+1) - p(0)x(1)y(0). \end{aligned} \tag{2.3}$$

Since Ku is the unique solution of the following linear BVP

$$\begin{cases} -\Delta\big[p(t-1)\Delta w(t-1)\big] + q(t)w(t) = u(t), & t \in \mathbf{Z}[1,T], \\ w(0) = w(T), \qquad p(0)\Delta w(0) = p(T)\Delta w(T), \end{cases}$$

we have, by (2.3), $\xi(0) = \xi(T)$ and $p(0)\Delta\xi(0) = p(T)\Delta\xi(T)$,

$$\begin{aligned} \rho_1 \sum_{t=1}^{T}(Ku)(t)\xi(t) &= \sum_{t=1}^{T}(Ku)(t)\big\{-\Delta\big[p(t-1)\Delta\xi(t-1)\big] + q(t)\xi(t)\big\} \\ &= \sum_{t=1}^{T}\xi(t)\Delta\big[p(t-1)\Delta(Ku)(t-1)\big] + \sum_{t=1}^{T}\xi(t)q(t)(Ku)(t) = \sum_{t=1}^{T}u(t)\xi(t). \end{aligned}$$

Then (2.2) holds, and this completes the proof of the lemma. □

The proofs of the main theorems of this paper are based on the fixed point index theory. The following three well-known lemmas in [27,28] are needed in our argument.

Lemma 2.4. *Let E be a Banach space and $X \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E. Suppose that $A : X \cap \overline{\Omega} \to X$ is a completely continuous operator. If $\inf_{x \in X \cap \partial\Omega} \|Ax\| > 0$ and $\mu Ax \neq x$ for $x \in X \cap \partial\Omega$, $\mu \geqslant 1$, then the fixed point index $i(A, X \cap \Omega, X) = 0$.*

Lemma 2.5. *Let E be a Banach space and $X \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E. Suppose that $A : X \cap \overline{\Omega} \to X$ is a completely continuous operator. If there exists $x_0 \in X \backslash \{\theta\}$ such that $x - Ax \neq \mu x_0$ for all $x \in X \cap \partial\Omega$ and $\mu \geqslant 0$, then the fixed point index $i(A, X \cap \Omega, X) = 0$.*

Lemma 2.6. *Let E be a Banach space and $X \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E with $\theta \in \Omega$. Suppose that $A : X \cap \overline{\Omega} \to X$ is a completely continuous operator. If $Ax \neq \mu x$ for all $x \in X \cap \partial\Omega$ and $\mu \geqslant 1$, then the fixed point index $i(A, X \cap \Omega, X) = 1$.*

3. Proofs of Theorems 1.1 and 1.2

For convenience, we introduce the following notations.

$$\begin{aligned} &\Phi = \Big\{\big(\lambda, \{u(t)\}_{t=0}^{T+1}\big)\colon\ \lambda > 0,\ \{u(t)\}_{t=0}^{T+1} \text{ is a positive solution of BVP (1.1)}\Big\}; \\ &\Lambda = \Big\{\lambda > 0\colon \text{ there exists } \{u(t)\}_{t=0}^{T+1} \text{ such that } \big(\lambda, \{u(t)\}_{t=0}^{T+1}\big) \in \Phi\Big\}; \\ &\lambda^* = \sup \Lambda; \\ &\lambda_* = \inf \Lambda. \end{aligned}$$

Lemma 3.1. *Assume that $f_0=\infty$. Then $\Phi\neq\emptyset$.*

Proof. Let $R>0$ be fixed. Then we can choose $\lambda_0>0$ small enough such that $\lambda_0\sup_{u\in P\cap\overline{B}_R}\|Au\|<R$. It is easy to see that

$$\lambda_0 Au\neq\mu u,\quad \forall u\in P\cap\partial B_R,\ \mu\geqslant 1.$$

By Lemma 2.6, it follows that

$$i(\lambda_0 A, P\cap B_R, P)=1. \tag{3.1}$$

From $f_0=\infty$, it follows that there exists $r\in(0,R)$ such that

$$f(t,x)\geqslant\frac{\rho_1}{\lambda_0}x,\quad \forall x\in[0,r],\ t\in\mathbf{Z}[1,T], \tag{3.2}$$

where ρ_1 is given in Lemma 2.3. We may suppose that $\lambda_0 A$ has no fixed point on $P\cap\partial B_r$. Otherwise, the proof is finished. Now we shall prove

$$u\neq\lambda_0 Au+\mu\xi,\quad \forall u\in P\cap\partial B_r,\ \mu\geqslant 0, \tag{3.3}$$

where ξ is given in Lemma 2.3. Suppose the contrary, then there exist $u_1\in P\cap\partial B_r$ and $\mu_1\geqslant 0$ such that $u_1=\lambda_0 Au_1+\mu_1\xi$. Then $\mu_1>0$. Multiplying the equality $u_1=\lambda_0 Au_1+\mu_1\xi$ by ξ on its both sides, summing from 1 to T and using (2.2) and (3.2), it follows that

$$\begin{aligned}\sum_{t=1}^{T}u_1(t)\xi(t)&=\sum_{t=1}^{T}(\lambda_0 Au_1)(t)\xi(t)+\mu_1\sum_{t=1}^{T}\xi^2(t)=\frac{\lambda_0}{\rho_1}\sum_{t=1}^{T}f\big(t,u_1(t)\big)\xi(t)+\mu_1\sum_{t=1}^{T}\xi^2(t)\\&\geqslant\sum_{t=1}^{T}u_1(t)\xi(t)+\mu_1\sum_{t=1}^{T}\xi^2(t),\end{aligned}$$

which contradicts $\xi>0$ on $\mathbf{Z}[1,T]$. Thus, (3.3) holds. It follows from Lemma 2.5 that

$$i(\lambda_0 A, P\cap B_r, P)=0. \tag{3.4}$$

According to the additivity of the fixed point index, by (3.1) and (3.4), we have

$$i\big(\lambda_0 A, P\cap(B_R\backslash\overline{B}_r), P\big)=i(\lambda_0 A, P\cap B_R, P)-i(\lambda_0 A, P\cap B_r, P)=1,$$

which implies that the nonlinear operator $\lambda_0 A$ has one fixed point $u_0\in P\cap(B_R\backslash\overline{B}_r)$. Therefore, $(\lambda_0,\{u_0(t)\}_{t=0}^{T+1})\in\Phi$, where $u(0)=u(T), u(T+1)=u(T)+\frac{p(0)}{p(T)}\Delta u(0)$. The proof is complete. □

Lemma 3.2. *Assume that* (H_1) *and* (H_2) *hold. Then Λ is a bounded set.*

Proof. Let $(\lambda,\{u(t)\}_{t=0}^{T+1})\in\Phi$. It follows from (H_1) and (H_2) that there exists $C>0$ such that $f(t,x)\geqslant Cx$ for all $x\geqslant 0$ and $t\in\mathbf{Z}[1,T]$. By Lemma 2.3, we obtain that $K^{-1}:E\to E$, the inverse mapping of K, exists and $(K^{-1}u)(t)=\lambda(\mathbf{f}u)(t)$ for $t\in\mathbf{Z}[1,T]$. We assume that $u(t_0)=\|u\|=\max_{t\in\mathbf{Z}[1,T]}|u(t)|$ for $t_0\in\mathbf{Z}[1,T]$. Then,

$$\big\|K^{-1}\big\|\|u\|\geqslant\big\|K^{-1}u\big\|\geqslant\big|\big(K^{-1}u\big)(t_0)\big|=\lambda f\big(t_0,u(t_0)\big)\geqslant\lambda C\|u\|,$$

where $\|K^{-1}\|=\sup_{\|u\|=1}\|K^{-1}u\|$. By matrix theory, $\|K^{-1}\|=\rho_1$. Thus, $\lambda\leqslant\rho_1 C^{-1}$. This completes the proof of the lemma. □

Lemma 3.3. *Assume that* (H_1), (H_2) *and* (H_4) *hold. Then $(0,\lambda^*)\subset\Lambda$. Moreover, for any $\lambda\in(0,\lambda^*)$, BVP* (1.1) *has at least two positive solutions.*

Proof. For any fixed $\lambda\in(0,\lambda^*)$, we prove that $\lambda\in\Lambda$. By the definition of λ^*, there exists $\lambda_2\in\Lambda$ such that $\lambda<\lambda_2\leqslant\lambda^*$ and $(\lambda_2,\{u_2(t)\}_{t=0}^{T+1})\in\Phi$. Let $R<\min_{t\in\mathbf{Z}[1,T]}u_2(t)$ be fixed. From the proof of Lemma 3.1, we see that there exist $\lambda_1<\lambda$, $R>r$ and $u_1\in P\cap(B_R\backslash\overline{B}_r)$ such that $(\lambda_1,\{u_1(t)\}_{t=0}^{T+1})\in\Phi$. It is easy to see that $0<u_1(t)<u_2(t)$ for all $t\in\mathbf{Z}[1,T]$. Then by (H_1), we have

$$-\Delta\big[p(t-1)\Delta u_1(t-1)\big]+q(t)u_1(t)=\lambda_1 f\big(t,u_1(t)\big)<\lambda f\big(t,u_1(t)\big),\quad t\in\mathbf{Z}[1,T]$$

and

$$-\Delta\big[p(t-1)\Delta u_2(t-1)\big]+q(t)u_2(t)=\lambda_2 f\big(t,u_2(t)\big)>\lambda f\big(t,u_2(t)\big),\quad t\in\mathbf{Z}[1,T]. \tag{3.5}$$

Consider now the modified boundary value problem

$$\begin{cases} -\Delta\big[p(t-1)\Delta u(t-1)\big]+q(t)u(t)=\lambda f_1\big(t,u(t)\big), & t\in \mathbf{Z}[1,T],\\ u(0)=u(T), \qquad p(0)\Delta u(0)=p(T)\Delta u(T), \end{cases} \tag{3.6}$$

where

$$f_1(t,u)=\begin{cases} f(t,u_1(t)), & u(t)<u_1(t),\\ f(t,u(t)), & u_1(t)\leqslant u(t)\leqslant u_2(t),\\ f(t,u_2(t)), & u(t)>u_2(t). \end{cases}$$

Clearly, the function λf_1 is bounded for $t\in \mathbf{Z}[1,T]$ and $u\in \mathbf{R}$, and is continuous in u. Define the operator $A_1: E\to E$ by

$$(A_1u)(t)=\sum_{s=1}^{T}G(t,s)f_1\big(s,u(s)\big), \quad u\in E,\ t\in \mathbf{Z}[1,T].$$

Then $A_1: P\to P$ is completely continuous and $\{u(t)\}_{t=0}^{T+1}$ is a solution of (3.6) if and only if $u=\{u(t)\}_{t=1}^{T}\in E$ is a fixed point of operator λA_1. It is easy to see that there exists $r_0>\sqrt{T}\|u_2\|$ such that $\|\lambda A_1 u\|<r_0$ for any $u\in E$. It follows from Lemma 2.6 that

$$i(\lambda A_1, P\cap B_{r_0}, P)=1. \tag{3.7}$$

Put

$$U=\big\{u\in P\colon u_1(t)<u(t)<u_2(t),\ \forall t\in \mathbf{Z}[1,T]\big\}.$$

We claim that if $u\in P$ is a fixed point of operator λA_1, then $u\in U$. We first prove that $u(t)<u_2(t)$ on $\mathbf{Z}[1,T]$. Assume that $u(t)\geqslant u_2(t)$ for some $t\in \mathbf{Z}[1,T]$. By $u_2(0)=u_2(T)$ and $p(0)\Delta u_2(0)=p(T)\Delta u_2(T)$, we know that $u(t)-u_2(t)$ has a nonnegative maximum at some t_0 in $\mathbf{Z}[1,T]$. Then, $\Delta^2(u-u_2)(t_0-1)\leqslant 0$. Consequently, noticing the monotonicity of p, we have

$$\Delta\big[p(t_0-1)\Delta(u-u_2)(t_0-1)\big]=p(t_0)\Delta^2(u-u_2)(t_0-1)+\Delta p(t_0-1)\Delta(u-u_2)(t_0-1)\leqslant 0.$$

On the other hand, by (3.5), we have

$$\begin{aligned} -\Delta\big[p(t_0-1)\Delta(u-u_2)(t_0-1)\big]&\leqslant -\Delta\big[p(t_0-1)\Delta(u-u_2)(t_0-1)\big]+q(t_0)(u-u_2)(t_0)\\ &<\lambda f_1\big(t_0,u(t_0)\big)-\lambda f\big(t_0,u_2(t_0)\big)=0. \end{aligned}$$

This is a contradiction. It follows that $u(t)<u_2(t)$ on $\mathbf{Z}[1,T]$. In very much the same way, we can prove that $u(t)>u_1(t)$ on $\mathbf{Z}[1,T]$. The claim is thus proved. By virtue of the claim, the excision property of the fixed point index and (3.7), we obtain that

$$i(\lambda A_1, U, P)=i(\lambda A_1, P\cap B_{r_0}, P)=1.$$

From the definition of A_1, we know that $A_1=A$ on $\overline{U}$. Then,

$$i(\lambda A, U, P)=1. \tag{3.8}$$

Hence, the nonlinear operator λA has at least fixed point $v_1\in U$. Then $\{v_1(t)\}_{t=0}^{T+1}$ is one positive solution of BVP (1.1), where $v_1(0)=v_1(T)$, $v_1(T+1)=v_1(T)+\frac{p(0)}{p(T)}\Delta v_1(0)$. This gives $(\lambda,\{v_1(t)\}_{t=0}^{T+1})\in \Phi$ and $(0,\lambda^*)\subset \Lambda$.

We now find the second positive solution of BVP (1.1). By $f_\infty=\infty$ and the continuity of $f(t,x)$ with respect to x, there exists $C>0$ such that

$$f(t,x)\geqslant 2\rho_1\lambda^{-1}x-C, \quad \forall x\geqslant 0,\ t\in \mathbf{Z}[1,T]. \tag{3.9}$$

Set

$$\Omega=\{u\in P\colon u=\lambda Au+\tau\xi \text{ for some } \tau\geqslant 0\},$$

where ξ is given in Lemma 2.3. We claim that Ω is bounded in E. In fact, for any $u\in \Omega$, there exists $\tau\geqslant 0$ such that $u=\lambda Au+\tau\xi\geqslant \lambda Au$. Then, by (3.9), we have

$$u(t)\geqslant 2\rho_1(Ku)(t)-\lambda C(Kv_0)(t), \quad t\in \mathbf{Z}[1,T],$$

where $v_0(t)\equiv 1$. Multiplying the above inequality by $\xi(t)$ on both sides and summing from 1 to T, it follows from Lemma 2.3 that

$$\sum_{t=1}^{T}u(t)\xi(t)\geqslant 2\rho_1\sum_{t=1}^{T}(Ku)(t)\xi(t)-\lambda C\sum_{t=1}^{T}(Kv_0)(t)\xi(t)=2\sum_{t=1}^{T}u(t)\xi(t)-\lambda C.$$

This implies that $\sum_{t=1}^{T} u(t)\xi(t) \leqslant \lambda C$. Let $\delta = \min_{t\in\mathbf{Z}[1,T]} \xi(t) > 0$. Thus, $\|u\| \leqslant \lambda\delta^{-1}C$. Then we can conclude that Ω is bounded in E, proving our claim. Therefore there exists $R_1 > \sqrt{T}\|u_2\|$ such that

$$u \neq \lambda Au + \tau\xi, \quad \forall u \in P \cap \partial B_{R_1},\ \tau \geqslant 0.$$

This and Lemma 2.5 give

$$i(\lambda A, P \cap B_{R_1}, P) = 0. \tag{3.10}$$

Using a similar argument as in deriving (3.4), we have that

$$i(\lambda A, P \cap B_{r_1}, P) = 0,$$

where $0 < r_1 < \sqrt{T} \min_{t\in\mathbf{Z}[1,T]} u_1(t)$. Then according to the additivity of the fixed point index, by (3.8), (3.10), we have

$$i\big(\lambda A, P \cap \big(B_{R_1} \backslash (\overline{U} \cup \overline{B}_{r_1})\big), P\big) = i(\lambda A, P \cap B_{R_1}, P) - i(\lambda A, U, P) - i(\lambda A, P \cap B_{r_1}, P) = -1,$$

which implies that the nonlinear operator λA has at least one fixed point $v_2 \in P \cap (B_{R_1} \backslash (\overline{U} \cup \overline{B}_{r_1}))$. Thus, BVP (1.1) has another positive solution. The proof is complete. □

Lemma 3.4. *Assume that* (H_1), (H_2) *and* (H_4) *hold. Then* $\Lambda = (0, \lambda^*]$.

Proof. In view of Lemma 3.3, it suffices to prove that $\lambda^* \in \Lambda$. By the definition of λ^*, we can choose $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \geqslant \frac{\lambda^*}{2}$ $(n = 1, 2, \ldots)$ such that $\lambda_n \to \lambda^*$ as $n \to \infty$. By Lemma 3.3, one can choose $\{u_n\} \subset P \backslash \{\theta\}$ such that $(\lambda_n, \{u_n(t)\}_{t=0}^{T+1}) \in \Phi$. Then from (H_2) and the continuity of $f(t,x)$ with respect to x, we know that there exist $c > \frac{2\rho_1}{\lambda^*}$ and $d > 0$ such that $f(t,x) \geqslant cx - d$ for any $x \geqslant 0$ and $t \in \mathbf{Z}[1,T]$. Then

$$u_n(t) = (\lambda_n Au_n)(t) \geqslant \frac{c\lambda^*}{2}(Ku_n)(t) - \frac{d\lambda^*}{2}(Kv_0)(t), \quad t \in \mathbf{Z}[1,T],$$

where $v_0(t) \equiv 1$. Multiplying the above inequality by $\xi(t)$ on both sides and summing from 1 to T, it follows from Lemma 2.3 that

$$\sum_{t=1}^{T} u_n(t)\xi(t) \geqslant \frac{c\lambda^*}{2} \sum_{t=1}^{T} (Ku_n)(t)\xi(t) - \frac{d\lambda^*}{2} \sum_{t=1}^{T} (Kv_0)(t)\xi(t) = \frac{c\lambda^*}{2\rho_1} \sum_{t=1}^{T} u_n(t)\xi(t) - \frac{d\lambda^*}{2}. \tag{3.11}$$

We now show that $\{u_n\}$ is bounded. Suppose the contrary, then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $u_n(t_0) \to +\infty$ as $n \to \infty$ for some $t_0 \in \mathbf{Z}[1,T]$. By (3.11), we have

$$u_n(t_0)\xi(t_0)\left(\frac{c\lambda^*}{2\rho_1} - 1\right) \leqslant \left(\frac{c\lambda^*}{2\rho_1} - 1\right) \sum_{t=1}^{T} u_n(t)\xi(t) \leqslant \frac{d\lambda^*}{2},$$

which contradicts $u_n(t_0) \to +\infty$ as $n \to \infty$. Hence, $\{u_n\}$ is bounded. Since E is finite dimensional, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u^* \in P$ such that $u_n \to u^*$ as $n \to \infty$. By $u_n = \lambda_n Au_n$, letting $n \to \infty$, we obtain that $u^* = \lambda^* Au^*$. Since $u_n(t) = \lambda_n \sum_{s=1}^{T} G(t,s) f(s, u_n(s)) \geqslant \frac{\lambda^* m}{2} f(t, u_n(t))$ for all $t \in \mathbf{Z}[1,T]$, then by $\{u_n\} \subset P \backslash \{\theta\}$ we know that $1 \geqslant \frac{\lambda^* m}{2} \cdot \frac{f(t,u_n(t))}{u_n(t)}$ for all $t \in \mathbf{Z}[1,T]$. It follows from $f_0 = \infty$ that $u^* \in P \backslash \{\theta\}$. So, $\lambda^* \in \Lambda$. This completes the proof. □

Proof of Theorem 1.1. Theorem 1.1 readily follows from Lemmas 3.1–3.4. □

Lemma 3.5. *Assume that* (H_1) *and* (H_3) *hold. Then BVP* (1.1) *has at least one positive solution for* λ *large enough and has no positive solution for* λ *small enough.*

Proof. Let $r > 0$ be fixed. From (H_1) and the definition of cone P, it follows that there exists $C > 0$ such that $f(t, u(t)) \geqslant C$ for all $t \in \mathbf{Z}[1,T]$ and $u \in P \cap \partial B_r$. Then for $\lambda > \frac{r}{mTC}$ and $u \in P \cap \partial B_r$, one has

$$(\lambda Au)(t) = \lambda \sum_{s=1}^{T} G(t,s) f\big(s, u(s)\big) \geqslant \lambda mTC > r, \quad t \in \mathbf{Z}[1,T].$$

This gives that $\inf_{u\in P\cap\partial B_r} \|\lambda Au\| > 0$ and $\mu\lambda Au \neq u$ for $u \in P \cap \partial B_r$, $\mu \geqslant 1$. By Lemma 2.4, it follows that

$$i(\lambda A, P \cap B_r, P) = 0. \tag{3.12}$$

From $f^{\infty}=0$, there exists $R>r$ such that

$$f(t,x)\leqslant \frac{1}{2\lambda MT}x,\quad \forall x\in\left[\frac{m}{M}R,\infty\right),\ t\in\mathbf{Z}[1,T].$$

Then for $u\in P\cap\partial B_R$, by the definition of cone P, one has $\min_{t\in\mathbf{Z}[1,T]}u(t)\geqslant\frac{m}{M}\|u\|=\frac{m}{M}R$, and so

$$(\lambda Au)(t)=\lambda\sum_{s=1}^{T}G(t,s)f\big(s,u(s)\big)\leqslant\lambda MT\frac{1}{2\lambda MT}\|u\|<R,\quad t\in\mathbf{Z}[1,T].$$

It follows from Lemma 2.6 that

$$i(\lambda A,P\cap B_R,P)=1. \tag{3.13}$$

According to the additivity of the fixed point index, by (3.12) and (3.13), we have

$$i\big(\lambda A,P\cap(B_R\backslash\overline{B}_r),P\big)=i(\lambda A,P\cap B_R,P)-i(\lambda A,P\cap B_r,P)=1,$$

which implies that the nonlinear operator λA has at least one fixed point $u\in P\cap(B_R\backslash\overline{B}_r)$. Therefore, BVP (1.1) has at least one positive solution.

Next, we prove the nonexistence part. From (H_3) and the continuity of $f(t,x)$ with respect to x, there exists $C_1>0$ such that $f(t,x)\leqslant C_1x$ for any $t\in\mathbf{Z}[1,T]$ and $x\geqslant 0$. Assume that BVP (1.1) has one positive solution $\{u(t)\}_{t=0}^{T+1}$ for λ small enough such that $\lambda MTC_1<1$. Then

$$\|u\|=\left\|\lambda\sum_{s=1}^{T}G(t,s)f\big(s,u(s)\big)\right\|\leqslant\lambda MC_1\sum_{s=1}^{T}u(s)\leqslant\lambda MTC_1\|u\|<\|u\|,$$

which is a contradiction. The proof is complete. □

Lemma 3.6. *Assume that* (H_1), (H_3) *and* (H_4) *hold. Then* $0<\lambda_*<\infty$ *and* $(\lambda_*,+\infty)\subset\Lambda$. *Moreover, for any* $\lambda\in(\lambda_*,+\infty)$, *BVP* (1.1) *has at least two positive solutions.*

Proof. By virtue of Lemma 3.5, we can easily obtain that $0<\lambda_*<\infty$. For any fixed $\lambda\in(\lambda_*,+\infty)$, we prove that $\lambda\in\Lambda$. By the definition of λ_*, there exists $\lambda_1\in\Lambda$ such that $\lambda_*\leqslant\lambda_1<\lambda$ and $(\lambda_1,\{u_1(t)\}_{t=0}^{T+1})\in\Phi$. Let $r>\frac{M}{m}\|u_1\|$ be fixed. From the proof of Lemma 3.5, we see that there exist $\lambda_2>\lambda$, $R>r$ and $u_2\in P\cap(B_R\backslash\overline{B}_r)$ such that $(\lambda_2,\{u_2(t)\}_{t=0}^{T+1})\in\Phi$. By the definition of cone P, it is easy to see that $0<u_1(t)<u_2(t)$ for all $t\in\mathbf{Z}[1,T]$. Put

$$V=\big\{u\in P\colon u_1(t)<u(t)<u_2(t),\ \forall t\in\mathbf{Z}[1,T]\big\}.$$

An argument similar to the one used in deriving (3.8) in the proof of Lemma 3.3 yields

$$i(\lambda A,V,P)=1. \tag{3.14}$$

Hence, the nonlinear operator λA has at least fixed point $v_1\in V$. Then $\{v_1(t)\}_{t=0}^{T+1}$ is one positive solution of BVP (1.1), where $v_1(0)=v_1(T)$, $v_1(T+1)=v_1(T)+\frac{p(0)}{p(T)}\Delta v_1(0)$. This gives $(\lambda,\{v_1(t)\}_{t=0}^{T+1})\in\Phi$ and $(\lambda_*,+\infty)\subset\Lambda$.

We now find the second positive solution of BVP (1.1). From $f^0=0$, there exists $0<r_0<\sqrt{T}\min_{t\in\mathbf{Z}[1,T]}u_1(t)$ such that

$$f(t,x)\leqslant\frac{1}{2\lambda MT}x,\quad\forall x\in[0,r_0],\ t\in\mathbf{Z}[1,T].$$

Then for $u\in P\cap\partial B_{r_0}$, we have

$$(\lambda Au)(t)=\lambda\sum_{s=1}^{T}G(t,s)f\big(s,u(s)\big)\leqslant\lambda MT\frac{1}{2\lambda MT}\|u\|<r_0,\quad t\in\mathbf{Z}[1,T].$$

It follows from Lemma 2.6 that

$$i(\lambda A,P\cap B_{r_0},P)=1. \tag{3.15}$$

Using a similar argument as in deriving (3.13), we have that

$$i(\lambda A,P\cap B_{R_0},P)=1, \tag{3.16}$$

where $R_0 > \sqrt{T}\|u_2\|$. According to the additivity of the fixed point index, by (3.14)–(3.16), we have

$$i\big(\lambda A, P \cap \big(B_{R_0} \backslash (\overline{V} \cup \overline{B}_{r_0})\big), P\big) = i(\lambda A, P \cap B_{R_0}, P) - i(\lambda A, V, P) - i(\lambda A, P \cap B_{r_0}, P) = -1,$$

which implies that the nonlinear operator λA has at least one fixed point $v_2 \in P \cap (B_{R_0} \backslash (\overline{V} \cup \overline{B}_{r_0}))$. Thus, BVP (1.1) has another positive solution. The proof is complete. □

Lemma 3.7. *Assume that* (H_1), (H_3) *and* (H_4) *hold. Then* $\Lambda = [\lambda_*, +\infty)$.

The proof is similar to that of Lemma 3.4, so we omit it here.

Proof of Theorem 1.2. Theorem 1.2 readily follows from Lemmas 3.5–3.7. □

4. Proofs of Theorem 1.3 and Corollary 1.1

Proof of Theorem 1.3. (i) Fix $\lambda \in (\frac{\rho_1}{f_0}, \frac{\rho_1}{f^\infty})$. Then $f_0 > \frac{\rho_1}{\lambda}$ and $f^\infty < \frac{\rho_1}{\lambda}$. By $f_0 > \frac{\rho_1}{\lambda}$, there exists $r_1 > 0$ such that

$$f(t,x) \geqslant \frac{\rho_1}{\lambda} x, \quad \forall x \in [0, r_1],\ t \in \mathbf{Z}[1,T].$$

We may suppose that λA has no fixed point on $P \cap \partial B_{r_1}$. Otherwise, the proof of (i) is finished. As in deriving (3.4), we have that

$$i(\lambda A, P \cap B_{r_1}, P) = 0. \tag{4.1}$$

On the other hand, by $f^\infty < \frac{\rho_1}{\lambda}$ and the continuity of $f(t,x)$ with respect to x, there exists $\sigma \in (0,1)$ such that

$$f(t,x) \leqslant \frac{\rho_1 \sigma}{\lambda} x + C, \quad \forall x \in [0, +\infty),\ t \in \mathbf{Z}[1,T]. \tag{4.2}$$

Define

$$W = \big\{u \in P\colon\ u = s\lambda Au \text{ for some } s \in [0,1]\big\}.$$

We now show that W is bounded in E. For any $u \in W$, then there exists $s \in [0,1]$ such that $u = s\lambda Au$. By (4.2), we have $u = s\lambda Au \leqslant \rho_1 \sigma K u + \lambda C K v_0$, where $v_0(t) \equiv 1$, $t \in \mathbf{Z}[1,T]$. Thus

$$(I - K_1)u \leqslant CKv_0, \tag{4.3}$$

where $K_1 = \rho_1 \sigma K$ and I is the identity operator. Since $r(K_1) = \rho_1 \sigma r(K) < 1$, the inverse operator $(I - K_1)^{-1}$ exists and is given by $(I - K_1)^{-1} = I + K_1 + K_1^2 + \cdots$. This and $K_1(P) \subset P$ give $(I - K_1)^{-1}(P) \subset P$. Now, from (4.3), we have $u \leqslant (I - K_1)^{-1} CKv_0$. Hence, W is bounded. Then there exists $R_1 > \max\{r_1, \sup_{u \in W} \|u\|\}$ such that

$$u \neq s\lambda Au, \quad \forall u \in P \cap \partial B_{R_1},\ s \in [0,1].$$

This and Lemma 2.6 give $i(\lambda A, P \cap B_{R_1}, P) = 1$. Taking (4.1) into account, we have $i(\lambda A, P \cap (B_{R_1} \backslash \overline{B}_{r_1}), P) = 1$, which implies that λA has at least one fixed point in $P \cap (B_{R_1} \backslash \overline{B}_{r_1})$. That is, BVP (1.1) has at least positive solution.

(ii) Fix $\lambda \in (\frac{\rho_1}{f_\infty}, \frac{\rho_1}{f^0})$. Then $f^0 < \frac{\rho_1}{\lambda}$ and $f_\infty > \frac{\rho_1}{\lambda}$. By $f^0 < \frac{\rho_1}{\lambda}$, there exist $\varepsilon \in (0,1)$ and $r_2 > 0$ such that

$$f(t,x) \leqslant \frac{\rho_1}{\lambda}(1 - \varepsilon)x, \quad \forall x \in [0, r_2],\ t \in \mathbf{Z}[1,T]. \tag{4.4}$$

Now we prove

$$\lambda Au \neq \mu u, \quad \forall u \in P \cap \partial B_{r_2},\ \mu \geqslant 1. \tag{4.5}$$

If (4.5) does hold, there exist $\mu_0 \geqslant 1$ and $u_0 \in P \cap \partial B_{r_2}$ such that $\lambda Au_0 = \mu_0 u_0$. Then, by (4.4), we have

$$u_0(t) \leqslant (\lambda Au_0)(t) \leqslant \lambda \sum_{k=1}^{T} G(t,k) f\big(k, u_0(k)\big) \leqslant \rho_1(1 - \varepsilon) \sum_{k=1}^{T} G(t,k) u_0(k), \quad t \in \mathbf{Z}[1,T].$$

This gives $\rho_1(1 - \varepsilon)Ku_0 \geqslant u_0$. Multiplying this inequality by ξ, and summing from 1 to T, it follows from (2.4) that

$$(1 - \varepsilon) \sum_{t=1}^{T} u_0(t)\xi(t) = \rho_1(1 - \varepsilon) \sum_{t=1}^{T} (Ku_0)(t)\xi(t) \geqslant \sum_{t=1}^{T} u_0(t)\xi(t).$$

This together with $\sum_{t=1}^{T} u_0(t)\xi(t) > 0$ implies that $\varepsilon \leqslant 0$, which contradicts the choice of ε, and so (4.5) holds. It follows from Lemma 2.6 that

$$i(\lambda A, P \cap B_{r_2}, P) = 1. \tag{4.6}$$

By $f_\infty > \frac{\rho_1}{\lambda}$, using a similar argument as in deriving (3.10), we see that there exists $R_2 > r_2$ such that

$$i(\lambda A, P \cap B_{R_2}, P) = 0.$$

Then taking (4.6) into account, we have $i(\lambda A, P \cap (B_{R_2} \backslash \overline{B}_{r_2}), P) = -1$, which implies λA has at least one fixed point in $P \cap (B_{R_2} \backslash \overline{B}_{r_2})$. Therefore, BVP (1.1) has at least one positive solution. The proof is complete. □

Proof of Corollary 1.1. (i) Fix $\lambda \in (\frac{\rho_1}{f_0}, \frac{\rho_1}{f^\infty})$. In view of the fact that $xf(t,x) \geqslant 0$ for any $t \in \mathbf{Z}[1,T]$ and $x \in \mathbf{R}$, we know that $A(P) \subset P$. It follows from Theorem 1.3 that BVP (1.1) has at least one positive solution.

Put $f_2(t,x) = -f(t,-x)$, $\forall (t,x) \in \mathbf{Z}[1,T] \times \mathbf{R}$. Define operators $\mathbf{f}_2, A_2 : E \to E$, respectively, by

$$(\mathbf{f}_2 u)(t) = f_2\big(t, u(t)\big), \quad u \in E,\ t \in \mathbf{Z}[1,T];$$
$$A_2 = K\mathbf{f}_2.$$

Obviously, $A_2(P) \subset P$. From the proof of Theorem 1.3, it is easy to see that λA_2 has at least one fixed point $u_0 \in P \backslash \{\theta\}$. Then, $\lambda A(-u_0) = \lambda K\mathbf{f}(-u_0) = \lambda K(-\mathbf{f}_2)u_0 = -(\lambda A_2)u_0 = -u_0$. That is, $\lambda A(-u_0) = -u_0$. Hence, BVP (1.1) has at least one negative solution.

The proof of (ii) is similar and omitted. □

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