Wild $p$-Cyclic Actions on Smooth Projective Surfaces
with $p_g = q = 0$

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0. INTRODUCTION

Throughout this paper we assume that the ground field $k$ is an algebraically closed field of positive characteristic $p$.

Let $X$ be a smooth projective surface, not necessarily minimal, with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ defined over $k$. In this paper we study automorphisms of order $p$ of $X$ and prove the following.

MAIN THEOREM. Let $X$ be a smooth projective surface, not necessarily minimal, with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ defined over a field $k$ of characteristic $p > 0$. Suppose that $X$ admits an automorphism $g$ of order $p$. Then we have the following:

1. The set of fixed points $X^g$ of $g$ is nonempty and connected, i.e., either a point or a connected curve.

2. The quotient variety $Y = X/(\langle g \rangle)$ has at most rational singularities, and its minimal resolution $\tilde{Y}$ has $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = H^2(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$.

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Note first that the theorem covers smooth rational surfaces, Enriques surfaces, some elliptic surfaces, and surfaces of general type with geometric genus \( p_g = 0 \).

For K3 surfaces, a similar result was obtained by Dolgachev and Keum [DK]. More precisely, it is proved that if \( g \) is an automorphism of order \( p \) of a K3 surface \( X \) in positive characteristic \( p \), then the set \( X^g \) is empty, a point, two points, or a connected curve. The reader is referred to [DK] for details.

We remark that if \( G \) is a finite abelian group of prime power order acting on a smooth projective variety defined over an algebraically closed field of characteristic \( p \geq 0 \) and if \( p \) does not divide \( |G| \), then either \( X^G = \emptyset \) or \( X^G \) contains at least two points (Assadi et al. [ABK, Corollary 3.2]). For example, over the field of complex numbers an involution acting on a Godeaux surface, a surface of general type with \( p_g = 0 \) and \( K^2 = 1 \), must have \( X^g \) consisting of five points and at most five mutually disjoint smooth irreducible curves [KL].

In Sections 1 and 2 we recall some general results about wild cyclic actions developed in [DK], including Grothendieck’s spectral sequence for cohomology of \( G \)-equivariant sheaves from [G]. Section 3 is devoted to the proof of our Main Theorem. In Section 4 we apply our result to \( p \)-torsions of rational elliptic surfaces in characteristic \( p \), proving for example that there is no \( p \)-torsion if \( p > 5 \). At the end of Section 3 we also consider the case of wild \( p \)-cyclic actions on surfaces with arbitrary \( p_g \) and \( q \); see Theorem 3.8 and Example 3.9.

1. GROUP COHOMOLOGY SHEAVES

Let \( G \) be a group acting on a topological space \( X \), and let \( Y = X/G \) be the quotient space and \( \pi: X \to Y \) the quotient map. Consider the category \( \mathcal{A}(X, G) \) of abelian \( G \)-sheaves on \( X \) and the category \( \mathcal{A} \) of abelian groups. The functor

\[
\mathcal{A}(X, G) \to \mathcal{A}, \quad \mathcal{F} \mapsto \Gamma(X, \mathcal{F})^G
\]

can be represented as a composition of functors in two different ways,

\[
\mathcal{A}(X, G) \to \mathcal{A}(Y) \to \mathcal{A}, \quad \mathcal{F} \mapsto \pi_*^G \mathcal{F} \mapsto \Gamma(Y, \pi_*^G \mathcal{F}),
\]

\[
\mathcal{A}(X, G) \to \mathcal{A} \to \mathcal{A}, \quad \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) \mapsto \Gamma(X, \mathcal{F})^G,
\]

where \( \pi_*^G \mathcal{F} \) is the sheaf \( U \mapsto \Gamma(\pi^{-1}(U), \mathcal{F})^G \). This gives two spectral sequences for the compositions of functors (see [G], Theorem 5.2.1),

\[
'\text{E}^{p, q}_2(\mathcal{F}) = H^p(Y, \mathcal{A}^q(G, \mathcal{F})) \Rightarrow H^n, \quad (1.1)
\]
Here the sheaves $H^q(G, \mathcal{F})$ can be defined on open subsets of $Y$ by
\[
H^q(G, \mathcal{F})(U) = H^q(G, \mathcal{F} \mid \pi^{-1}(U)).
\] (1.3)

We will apply this to the situation:

- $X$ is an irreducible algebraic variety over a field $k$,
- $G$ is a finite group of its automorphisms, and
- $\mathcal{F}$ is the structure sheaf $\mathcal{O}_X$.

**Proposition 1.1 [DK].** Let $G$ be a finite group acting on an irreducible variety $X$. Then the sheaves $H^q(G, \mathcal{O}_X)$, $q > 0$, are equal to zero over the quotient of the open subset of $X$ where $G$ acts freely. In particular, they are torsion $\mathcal{O}_Y$-modules.

Let $G$ be a cyclic group of order $n$. Then we have the isomorphisms of $\mathcal{O}_Y$-modules
\[
\begin{align*}
H^{2i-1}(G, \mathcal{O}_X) &= \text{Ker } T/\text{Im}(g - 1), \\
H^{2i}(G, \mathcal{O}_X) &= \text{Ker}(g - 1)/\text{Im } T, \quad i > 0, \\
H^0(G, \mathcal{O}_X) &= \mathcal{O}_Y,
\end{align*}
\] (1.4)

where $T = 1 + g + \cdots + g^{n-1}$ and $g - 1$, being elements of the group algebra $\mathbb{Z}[G]$, are the homomorphisms of the $\mathcal{O}_Y$-modules
\[
T : \pi_*(\mathcal{O}_X) \to \mathcal{O}_Y, \quad g - 1 : \pi_*(\mathcal{O}_X) \to \pi_*(\mathcal{O}_X).
\]

**Proposition 1.2 [DK].** Let $G = (g)$ be a cyclic group acting on a Cohen–Macaulay algebraic variety $X$.

1. Assume that the quotient $Y = X/G$ is also Cohen–Macaulay (e.g., $\dim X = 2$). Then we have an exact sequence of sheaves on $Y$,
\[
0 \to \omega_Y \to (\pi_* \omega_X)^G \to \mathcal{E}xt^1_{\mathcal{O}_Y}(H^2(G, \mathcal{O}_X), \omega_Y) \to 0,
\]

where $\omega_X$ and $\omega_Y$ are the canonical dualizing sheaves of $X$ and $Y$, respectively.

2. Assume additionally that $g$ acts freely outside a closed subset of codimension $\geq 2$ or that $n$ is invertible in $k$. Then
\[
\omega_Y \cong (\pi_* \omega_X)^G.
\]

3. $\chi(Y, H^1(G, \mathcal{O}_X)) = \chi(Y, H^2(G, \mathcal{O}_X))$. 

\[\text{"E}^q_{\pi}(\mathcal{F}) = H^p(G, H^q(X, \mathcal{F})) \Rightarrow H^n.\] (1.2)
2. ARTIN–SCHREIER COVERINGS

Now let us recall some facts about the Artin–Schreier coverings of algebraic surfaces (see [DK] or [T]).

Let $\pi: X \to Y = X/G$ be as in Section 1 with $G = (g)$ of order $p = \text{char}(k)$.

Recall first a well-known fact from algebra: Any cyclic extension of degree $p$ of a field $K$ of characteristic $p > 0$ is the splitting field of an equation $x^p - x = a$ for some $a \in K$. Globalizing this fact, we get the following.

**Proposition 2.1** [DK]. There is a canonical filtration of $\mathcal{O}_Y$-modules for $X$:

$$\pi_*(\mathcal{O}_X) = \mathcal{F}_{p-1} \supset \cdots \supset \mathcal{F}_1 \supset \mathcal{F}_0 = \mathcal{O}_Y,$$

whose quotients $\mathcal{L}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$, $i = 1, \ldots, p-1$, are ideal sheaves in $\mathcal{O}_Y$ satisfying

$$\mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots, \quad \text{Im } T = \mathcal{L}_{p-1}.$$

All $\mathcal{F}_i$ and $\mathcal{L}_i$ are locally free over the nonsingular locus of $Y$.

**Corollary 2.2** [DK]. Outside a finite set $S$ of points in $Y$ the sheaves $\mathcal{L}_i$ are locally free and

$$\mathcal{L}_i = \mathcal{L}_1^{a_i}.$$

There exists an open affine covering $\{U_i\}$ of $Y \setminus S$ such that

$$\pi^{-1}(U_i) \cong \text{Spec}(\mathcal{O}_Y(U_i)[t_i]/(t_i^p - a_it_i - b_i))$$

for some $a_i, b_i \in \mathcal{O}_Y(U_i)$. The elements $a_i$ define a section of $\mathcal{L}_1^{p+1}$ which is a $(p-1)$th power of a section $s$ of $\mathcal{L}_1^{-1}$. The group $G$ acts on $X$ by the formula $t_i \to t_i + s_i$, where $s_i = s|_{U_i}$.

The open subset $Y \setminus S$ is the smooth locus of $Y$ where the primary components of the ideal sheaves $\mathcal{L}_i$ are all of codimension 1.

Over $Y \setminus S$, $X$ lives in the total space of $\mathcal{L}_1^{(p-1)}$. Define

$$\mathcal{L} = (\mathcal{L}_1)^{**} = \text{Hom}_{\mathcal{O}_Y}(\text{Hom}_{\mathcal{O}_Y}(\mathcal{L}_1, \mathcal{O}_Y), \mathcal{O}_Y).$$

This is a reflexive sheaf of rank 1 on $Y$, hence locally free on the smooth locus of $Y$. Let $B$ be the non-negative Weil divisor corresponding to
\( \mathcal{L}^{-1} = \mathcal{L}^* \) so that
\[
\mathcal{L} = \mathcal{O}_Y(-B), \quad \mathcal{L}^{-1} = \mathcal{O}_Y(B).
\]
We shall call it the \textit{branch divisor} of the cover \( \pi: X \to Y \).

**Corollary 2.3 [DK].** Outside a finite set of points on \( Y \), we have
\[
\mathcal{R}^2(G, \mathcal{O}_X) = \mathcal{O}_Y/\text{Im}(T) \cong \mathcal{O}_{(p-1)B}.
\]

**Proof.** Take an open subset \( Y' \subset Y \setminus S \) where \( \mathcal{L}_1 = \mathcal{L} \).

**Corollary 2.4 [DK].** We have the formula for the canonical divisor \( K_X \) of \( X \),
\[
K_X = \pi^*(K_Y + (p - 1)B),
\]
where the pull back of a Weil divisor is the closure of the pull back of its restriction to the open subset \( Y \setminus S \).

Note that this adjunction formula is different from the usual one in characteristic 0.

**Example 2.5.** Take \( p = \text{char}(k) = 3 \) and consider the \( \mathbb{Z}/3 \)-action on \( X = \text{Spec } k[u, v] \) generated by the automorphism
\[
g(u) = u, \quad g(v) = u + v.
\]
Let \( w = v g(v) g^2(v) = v^3 - u^2 v \). Then \( u \) and \( w \) are algebraically independent elements of the invariant subring \( k[u, v]^f \) and
\[
\pi^*(du \wedge dw) = -u^2 du \wedge dv.
\]

**Remark 2.6.** This section holds true for higher dimensions if we replace “outside a finite set” with “outside a codimension \( \geq 2 \) set.”

### 3. The Proof of Main Theorem

In this section we prove the main theorem.

Let \( X \) be a smooth projective surface, not necessarily minimal, with \( H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0 \) defined over a field \( k \) of characteristic \( p > 0 \). Assume that \( X \) admits an automorphism \( g \) of order \( p \). As before \( Y \) denotes the quotient \( X/(g) \) and \( \sigma: \tilde{Y} \to Y \) its minimal resolution.

**Lemma 3.1.**
\[
H^2(Y, \mathcal{O}_Y) = 0.
\]
Proof. Since $\omega_Y \rightarrow (\pi_* \omega_X)^G$ is injective and
\[ H^0(Y, (\pi_* \omega_X)^G) = H^0(X, \omega_X)^G = 0, \]
we have
\[ H^0(Y, \omega_Y) = H^2(Y, \mathcal{O}_Y) = 0. \]

Lemma 3.2. The target cohomologies $H^1$ and $H^2$ in the spectral sequences (1.1) and (1.2) are computed as
\[ H^1 \cong H^2 \cong k. \]

Proof. Consider the second spectral sequence (1.2). Since $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$,
\[ E_2^{p,q} = H^p(G, H^q(X, \mathcal{O}_X)) = 0, \quad q \neq 0. \]
This immediately yields that
\[ H^n \cong E_2^{n,0}. \]
Since $G$ acts trivially on $H^0(X, \mathcal{O}_X) = k$ and $T(k) = 0$, we get
\[ E_2^{2,0} = H^2(G, H^0(X, \mathcal{O}_X)) = k/T(k) = k. \]
Finally, it is easy to see that
\[ E_2^{1,0} = H^1(G, H^0(X, \mathcal{O}_X)) = H^1(G, k) = \text{Hom}(G, k) = k. \]
This completes the proof.

Recall from (1.4) that $H^2(G, \mathcal{O}_X) = \mathcal{O}_Y/\text{Im} T$. Let $Z$ be the closed subscheme of $Y$ defined by the ideal sheaf $\text{Im}(T)$. Then $H^2(G, \mathcal{O}_X) = \mathcal{O}_Z$.

Lemma 3.3.
\[ Z_{\text{red}} = \pi(X^g). \]

Proof. The inclusion $Z_{\text{red}} \subset \pi(X^g)$ follows from Proposition 1.1.
Let $x \in X^g$; then $g$ acts on the local ring $\mathcal{O}_{X, x}$, sending its maximal ideal $\mathfrak{m}$ to itself. Since $T$ kills constants, the image of $T$ on $\mathcal{O}_{X, x}$ is contained in $\mathfrak{m} \cap \mathcal{O}_{Y, \pi(x)}$, the maximal ideal of $\mathcal{O}_{Y, \pi(x)}$. Thus $\pi(x)$ belongs to $Z_{\text{red}}$. \qed
LEMMA 3.4. The fixed point set $X^g$ is non-empty and connected.

Proof. If $X^g$ is empty, then the algebraic Euler characteristic $\chi(X)$ must be divisible by $p$, the order of the group $G$, which is absurd.

By Lemma 3.3, $X^g$ is connected if and only if $Z$ is connected. So it is enough to show that $H^0(Y, R^i G, \mathcal{O}_X) \cong k$. To do this, consider the first spectral sequence (1.1). By Proposition 1.1 and Lemma 3.1, for any $q \geq 0$, $p > 1$,

\[ E_2^{p,q} = H^p(Y, R^q G, \mathcal{O}_X) = 0. \]

So, we can apply Proposition XV.5.5 from [CE] to find an exact sequence

\[ 0 \to E_1^{1,1} \to H^2 \to E_2^{0,2} \to 0. \]

It is easy to see that

\[ E_1^{0,2} = E_2^{0,2}. \]

So, by Lemma 3.2,

\[ \dim E_2^{0,2} \leq 1. \]

On the other hand,

\[ \dim E_2^{0,2} = \dim H^0(Y, \mathcal{O}_Z) \geq 1. \]

This proves the assertion. 

This proves the first part of our Main Theorem. To prove the second part, we divide it into two cases:

(i) $X^g$ is a point.

(ii) $X^g$ is a connected curve.

LEMMA 3.5. If $X^g$ is a point, then

\[ H^0(Y, R^i G, \mathcal{O}_X) \cong H^0(Y, R^2 G, \mathcal{O}_X) \cong k \]

and

\[ H^1(Y, \mathcal{O}_Y) = 0. \]

Proof. Recall that every spectral sequence gives the standard five term sequence

\[ 0 \to E_1^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2. \]

(3.1) Apply it to the first spectral sequence (1.1). By Proposition 1.1, for $i > 0$ the sheaf $R^i G, \mathcal{O}_X$ is concentrated at a point, so that

\[ E_2^{p,q} = H^p(Y, R^q G, \mathcal{O}_X) = 0 \quad \text{when} \quad q > 0, \ p > 0 \text{ or } p > 2, q = 0. \]
By [CE, Proposition XV.5.9], we have the exact sequence

\[
0 \to H^1(Y, \mathscr{O}_Y) \to H^1 \to H^0(Y, \mathscr{F}^1(G, \mathscr{O}_X)) \to H^2(Y, \mathscr{O}_Y) \\
\to H^2 \to H^0(Y, \mathscr{F}^2(G, \mathscr{O}_X)) \to H^2(Y, \mathscr{F}^1(G, \mathscr{O}_X)) = 0. \tag{3.2}
\]

By Proposition 1.2(3), we also have

\[
\dim_k H^0(Y, \mathscr{F}^1(G, \mathscr{O}_X)) = \dim_k H^0(Y, \mathscr{F}^2(G, \mathscr{O}_X)). \tag{3.3}
\]

Now the result follows from (3.2), (3.3), and Lemmas 3.1 and 3.2.

**Lemma 3.6.** If \( X^s \) is a point, then

1. the fixed point gives a rational singularity on \( Y \).
2. the minimal resolution \( \tilde{Y} \) has

\[
H^1(\tilde{Y}, \mathscr{O}_{\tilde{Y}}) = H^2(\tilde{Y}, \mathscr{O}_{\tilde{Y}}) = 0.
\]

**Proof.** Note first that if \( X^s \) is finite, then for \( i = 1, 2 \),

\[
H^0(Y, \mathscr{F}^i(G, \mathscr{O}_X)) \cong \bigoplus_{x \in X^s} H^i(G, \mathscr{O}_{X, x}). \tag{3.4}
\]

So, by Lemma 3.5, \( H^i(G, \mathscr{O}_{X, x}) \) is a one-dimensional vector space over \( k \).

Now (1) follows from a result of Peskin [P, Theorem 6].

Now \( \sigma: \tilde{Y} \to Y \) is a resolution of the rational singularity. Thus

\[
R^1 \sigma_* \mathscr{O}_{\tilde{Y}} = 0.
\]

Since \( H^q(Y, R^q \sigma_* \mathscr{O}_{\tilde{Y}}) = 0 \) for \( q = 1, 2 \), we have

\[
0 \to H^1(Y, \mathscr{O}_Y) \to H^1(\tilde{Y}, \mathscr{O}_{\tilde{Y}}) \to H^0(Y, R^1 \sigma_* \mathscr{O}_{\tilde{Y}}) \to H^2(Y, \mathscr{O}_Y) \\
\to H^2(\tilde{Y}, \mathscr{O}_{\tilde{Y}}) \to H^0(Y, R^2 \sigma_* \mathscr{O}_{\tilde{Y}}).
\]

Now (2) follows from Lemmas 3.1 and 3.5.

**Lemma 3.7.** If \( X^s \) is a connected curve, then \( Y \) has at most rational singularities and the minimal resolution \( \tilde{Y} \) has

\[
H^1(\tilde{Y}, \mathscr{O}_{\tilde{Y}}) = H^2(\tilde{Y}, \mathscr{O}_{\tilde{Y}}) = 0.
\]

**Proof.** Let \( \sigma: \tilde{Y} \to Y \) be a resolution of singularities and \( \tilde{X} \) be the normalization of \( \tilde{Y} \) in the function field of \( X \). Let \( Z \to \tilde{X} \) be its resolution of singularities. Then the composition \( f: Z \to \tilde{X} \to X \) is a resolution of singularities of a nonsingular surface. This easily implies that \( R^1 f_* \mathscr{O}_Z = 0 \),
and this gives immediately that

\[ H^1(Z, \mathcal{O}_Z) \cong H^1(X, \mathcal{O}_X) = H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0. \]

Now the group \( G = (g) \) acts on \( \tilde{X} \) with the quotient equal to \( Y \). Consider the exact sequence

\[ 0 \to \mathcal{O}_Y \to \pi_* (\mathcal{O}_{\tilde{X}}) \to \text{Im} (g - 1) \to 0, \quad (3.5) \]

where \( g - 1: \pi_* (\mathcal{O}_X) \to \pi_* (\mathcal{O}_{\tilde{X}}) \) is the \( \mathcal{O}_Y \)-module homomorphism. Since \( \pi: \tilde{X} \to Y \) is a finite morphism, we have \( H^1(Y, \pi_* (\mathcal{O}_X)) = H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0 \), and hence we infer from (3.5) that \( H^1(Y, \mathcal{O}_Y) = 0 \) if \( H^0(Y, \text{Im}(g - 1)) = 0 \). Applying Proposition 2.1 we see that \( \text{Im}(g - 1) \) admits a filtration with quotients isomorphic to ideal sheaves of \( \mathcal{O}_Y \) and \( \mathcal{O}_Y \). So, it is enough to show that \( H^0(Y, \mathcal{O}_Y) = 0 \). But this is easy. Over a complement to a finite set of points, \( \mathcal{L}_i = \mathcal{O}_Y(-B) \) for some non-negative Weil divisor \( B \), the branch divisor of \( \pi: \tilde{X} \to Y \). Since \( X^\delta \) is one-dimensional, \( B \) is positive and hence \( \mathcal{L}_i \) cannot have nonzero sections. This proves that \( H^1(Y, \mathcal{O}_Y) = 0 \).

Next, Lemma 3.1 and the Leray spectral sequence for \( \sigma: \tilde{Y} \to Y \) imply that the singularities on \( Y \) are all rational and that \( H^2(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0 \).

Now the second part of our Main Theorem follows from Lemmas 3.6 and 3.7.

In the rest of this section we consider the case of wild \( p \)-cyclic actions on surfaces with arbitrary \( p_g := \dim_k H^2(X, \mathcal{O}_X) \) and \( q := \dim_k H^1(X, \mathcal{O}_X) \).

**Theorem 3.8.** Let \( X \) be a smooth projective surface, not necessarily minimal, defined over a field \( k \) of characteristic \( p > 0 \). Suppose that \( X \) admits an automorphism \( g \) of order \( p \) with \( X^\delta \) finite. Then

\[ |X^\delta| \leq 1 + q + p_g. \]

**Proof.** Consider the second spectral sequence (1.2). It is easy to see that

\[ E_2^{1,0} = H^1(G, H^0(X, \mathcal{O}_X)) = H^1(G, k) = \text{Hom}(G, k) = k \]

and that

\[ \dim_k E_2^{0,1} = \dim_k H^0(G, H^1(X, \mathcal{O}_X)) \leq q. \]

This implies that

\[ \dim_k H^1 \leq 1 + q. \]
As in the proof of Lemma 3.1, we have
\[ \dim_k H^2(Y, \mathcal{O}_Y) \leq \dim_k H^0(Y, (\pi_* \omega_Y)^G) = \dim_k H^0(X, \omega_X)^G \leq p_g. \]
From the standard exact sequence
\[ 0 \to H^1(Y, \mathcal{O}_Y) \to H^1 \to H^0(Y, \mathcal{X}^1(G, \mathcal{O}_X)) \to H^2(Y, \mathcal{O}_Y) \to H^2, \]
we see that
\[ \dim_k H^0(Y, \mathcal{X}^1(G, \mathcal{O}_X)) \leq 1 + q + p_g. \]
Now the result follows from the formula
\[ H^0(Y, \mathcal{X}^i(G, \mathcal{O}_X)) \equiv \bigoplus_{x \in X^g} H^i(G, \mathcal{O}_{X,x}), \]
and the fact that for any isolated fixed point \( x \in X \) the cohomology \( H^1(G, \mathcal{O}_{X,x}) \) is a non-trivial finite-dimensional vector space over \( k \).

Also, for a smooth projective surface \( X \) with arbitrary \( p_g \) and \( q \), \( X^g \) may be a non-connected curve, as we show by the following example.

**Example 3.9.** Let \( C_1 \) and \( C_2 \) be smooth projective curves defined over a field \( k \) of characteristic \( p > 0 \). Suppose that \( C_i \) admits an automorphism \( g_i \) of order \( p \). Let \( X = C_1 \times C_2 \). Then we have the following:

1. \( X^{g_1 \times g_2} \) is finite and \( |X^{g_1 \times g_2}| = |C_1^g| |C_2^g| \).
2. \( X^{g_1 \times \text{id}} \) is a disjoint union of \( |C_1^g| \) irreducible curves.

Note that \( q(X) = g(C_1) + g(C_2) \) and \( p_g(X) = g(C_1) g(C_2) \). There are indeed plenty of such curves \( C_i \). We give a few in characteristic 3.

(i) Let \( C \) be the normalization of the curve
\[ ((x^3 - y^2 x)(y^2 - z^2) + z^5 = 0) \subset \mathbb{P}^2 \]
and \( g \) the automorphism of order 3,
\[ g(x) = x + y, \quad g(y) = y, \quad g(z) = z. \]
Then \( C \) has genus 4 and \( |C^g| = 3 \).

(ii) Let \( C \) be the normalization of the curve
\[ ((x^3 - y^2 x)(y^2 + z^2) + z^5 = 0) \subset \mathbb{P}^2 \]
and \( g \) be the same as in (i). Then \( C \) has genus 2 and \( |C^g| = 2 \).
Let $C$ be the normalization of the curve

$$(x(x + y)(x - y)(y^4 - z^4) + z^7 = 0) \subset \mathbb{P}^2$$

and $g$ be the same as in (i). Then $C$ has genus 8 and $|C^g| = 5$.

4. $p$-TORSIONS OF RATIONAL ELLIPTIC SURFACES IN CHARACTERISTIC $p$

Let $S$ be an elliptic surface defined over $k$, algebraically closed and of characteristic $p > 0$. Namely, $S$ is a smooth projective surface with a relatively minimal elliptic fibration $\phi: S \to \mathbb{P}^1$, all defined over $k$. We always assume that $\phi$ admits a section $O: \mathbb{P}^1 \to S$ and that $\phi$ has a $p$-torsion section $P: \mathbb{P}^1 \to S$. Denote by $\chi(S)$ the arithmetic genus or the algebraic Euler characteristic of $S$. We have the following useful formulas.

**Height Paring Formula** (Shioda [S, Theorem 8.6]). The height paring $\langle -, - \rangle$ was defined by Shioda on the Mordell–Weil group of an elliptic surface. Since $P$ is a torsion section, $\langle P, P \rangle = 0$. Thus the explicit formula for the height paring has the form

$$2\chi(S) + 2P \cdot O = \sum \text{contr}_v(P),$$

where the summation is taken over all critical values of $\phi$ and $\text{contr}_v(P)$ is a rational number determined by the incidence relation among the divisor $P$ and the irreducible components of the singular fiber $\phi^{-1}(v)$; e.g., it is equal to 0 if $P$ meets the identity component of $\phi^{-1}(v)$.

**Noether Formula.**

$$12\chi(S) = c_2(S).$$

In particular, if $S$ is rational, $c_2(S) = 12$.

**Picard Number Formula.** For $S$ rational, the Picard number

$$\rho(S) = 10.$$
Case (i): $p \geq 5$. In this case, the number $\text{contr}_v(P)$ is not zero only if the singular fiber $\phi^{-1}(v)$ is of type $I_{a,p}$. Let $I_{a,p}, \ldots, I_{a,p}$ be all singular fibers of $\phi$ of such type. Then by the formula for $\text{contr}_v(P)$ [S, p. 229] or [CZ], we have

$$\sum_v \text{contr}_v(P) = \sum_{v=1}^r \frac{i_v(p - i_v)a_v}{p} < \sum_{v=1}^r \frac{pa_v}{4} \leq 3\chi(S), \quad (4.1)$$

where we assume that $P$ meets the $i_v, a_v$th component of $\phi^{-1}(v)$ counted from the identity component, and use the formula for the second Chern number of the surface $S$,

$$12\chi(S) = c_2(S) \geq \sum_{v=1}^r pa_v.$$

This proves that $2P \cdot O < \chi(S)$ and hence that $P \cdot O = 0$.

Case (ii): $p = 3$. In this case, the number $\text{contr}_v(P)$ is not zero only if the singular fiber $\phi^{-1}(v)$ is of type $I_{3,a}$, $IV$, or $IV^*$, and hence

$$\sum_v \text{contr}_v(P) = \frac{2s}{3} + \frac{4t}{3} + \sum_{v=1}^r \frac{2a_v}{3},$$

where $s$ (resp. $t$) is the number of fibers of $\phi$ of type $IV$ (resp., type $IV^*$). Combining this with the inequality

$$12\chi(S) = c_2(S) \geq 4s + 8t + \sum_{v=1}^r 3a_v,$$

(which holds even in the case of wild ramification), we have

$$2\chi(S) + 2P \cdot O \leq \frac{8\chi(S)}{3} - \frac{2s}{9} - \frac{4t}{9} < \frac{8\chi(S)}{3},$$

proving that $P \cdot O = 0$.

Case (iii): $p = 2$. In this case, the number $\text{contr}_v(P)$ is not zero only if the singular fiber $\phi^{-1}(v)$ is of type $I_{2,a}$, $I_{2,a}^*$, $III$, or $III^*$, and hence

$$2\chi(S) + 2P \cdot O = \sum_v \text{contr}_v(P) = \frac{s}{2} + \frac{3t}{2} + \sum_{v=1}^r \frac{a_v}{2} + \sum_{u=1}^m \left(1 + \frac{b_u}{4}\right),$$

where $s$ (resp. $t$) is the number of fibers of $\phi$ of type $III$ (resp., type $III^*$) and $I_{b,u}^*, \ldots, I_{b,u}^*$ are all singular fibers of type $I_{b,u}^*$. Applying the inequality

$$12\chi(S) = c_2(S) \geq 3s + 9t + \sum_{v=1}^r 2a_v + \sum_{u=1}^m (b_u + 6)$$

proving that $P \cdot O = 0$. 

In all cases, we have proven the desired inequalities.
(which holds even in the case of wild ramification), we have

\[ 8P \cdot O + 2m + s + 3t \leq 4\chi(S), \]

proving that \( P \cdot O = 1 \) only if \( \chi(S) = 2 \) and \( m = s = t = 0 \); \( P \cdot O = 0 \) otherwise.

From now on, we assume in addition that \( \chi(S) = 1 \); i.e., \( S \) is a rational elliptic surface admitting a \( p \)-torsion section \( P \). An immediate consequence of Proposition 4.1 is that the fibration \( \phi: S \to \mathbb{P}^1 \) has no fiber which is supersingular. We apply our Main Theorem together with Proposition 4.1 to this situation and get the following further information.

**Proposition 4.2.** Let \( \phi: S \to \mathbb{P}^1 \) be a rational elliptic surface defined over \( k \), algebraically closed, and of characteristic \( p > 0 \). Assume that \( \phi \) admits a \( p \)-torsion section \( P \). Then:

1. If \( \phi \) has a fiber of multiplicative type \( I_n \), then \( p \) divides \( n \).
2. The fibration \( \phi \) has exactly one fiber of additive type \( II, III, IV, I_n^*, IV^*, III^*, \) or \( II^* \).

**Proof.** Let \( g_p \) be the automorphism of \( S \) defined by the translation by the \( p \)-torsion section \( P \).

1. Let \( F \) be a singular fiber of \( \phi \) of type \( I_n \). Suppose that \( p \) does not divide \( n \). Then the curve \( P \) must meet the identity component of \( F \). Since the multiplicative group \( \mathbb{G}_m = k^* \) has no \( p \)-torsion elements, \( P \) and \( O \) meet in the identity component, a contradiction to Proposition 4.1.

2. Let \( F \) be a singular fiber of additive type. Then the automorphism \( g_p \) has at least one fixed point on \( F \). By our Main Theorem the number of singular fibers of additive type is at most 1. Since \( P \cdot O = 0 \), no point of a smooth fiber or of a fiber of multiplicative type can be fixed by \( g_p \), and hence the fixed locus of \( g_p \) is concentrated in the union of additive type fibers. This proves that \( \phi \) has at least one additive type fiber.

**Corollary 4.3.** Assume \( p = \text{char}(k) > 5 \). Then an elliptic fibration on a rational surface has no \( p \)-torsion sections.

**Proof.** Let \( \phi: S \to \mathbb{P}^1 \) be a rational elliptic surface admitting a \( p \)-torsion section \( P \). Assume that \( p > 5 \). If \( \phi \) has no singular fiber of multiplicative type, then by Proposition 4.2 \( \phi \) has only one singular fiber, which is of additive type. This is absurd. (No single fiber can satisfy both \( c_2(S) = 12 \) and \( \rho(S) = 10 \).) Thus, \( \phi \) has at least one singular fiber of multiplicative type \( I_{ap} \). Then the Picard number condition implies that \( p \leq 7 \). It remains to rule out the possibility that \( p = 7 \). But in this case it is easy to see that no combination of types of singular fibers containing at least one \( I_7 \) and exactly one additive type can satisfy \( c_2 = 12 \).
Remark 4.4. For rational elliptic surfaces in arbitrary characteristic, Oguiso and Shioda [OS] classified all pairs \((T, E(K))\), where \(T\) is the trivial lattice, or equivalently the types of the singular fibers, and \(E(K)\) is the Mordell–Weil group of the elliptic fibration. There are 74 different cases, some of them having nontrivial torsion sections. According to our result, some cases from the list cannot occur if \(p = \text{char}(k)\) divides the order of the torsion subgroup of \(E(K)\). For example, Case 73 cannot occur in characteristic \(p = 2\), as it has two fibers of additive type. Similarly, one can rule out the following cases.

Case 41 in \(\text{char}(k) = 2\), Case 42 in \(\text{char}(k) = 2\), Case 58 in \(\text{char}(k) = 2\),
Case 59 in \(\text{char}(k) = 2\), Case 60 in \(\text{char}(k) = 2\), Case 61 in \(\text{char}(k) = 3\),
Case 66 in \(\text{char}(k) = 2\), Case 66 in \(\text{char}(k) = 3\), Case 68 in \(\text{char}(k) = 3\),
Case 74 in \(\text{char}(k) = 2\).

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