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ScienceDirect
Journal of Approximation Theory 157 (2009) 70–88
Journal of Approximation Theory
www.elsevier.com/locate/jat

An operator approach to multipoint Padé approximations

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Received 20 February 2008; received in revised form 2 June 2008; accepted 4 July 2008
Available online 5 November 2008

Communicated by Guillermo López Lagomasino

Abstract

First, an abstract scheme of constructing biorthogonal rational systems related to some interpolation problems is proposed. We also present a modification of the famous step-by-step process of solving the Nevanlinna–Pick problems for Nevanlinna functions. The process in question gives rise to three-term recurrence relations with coefficients depending on the spectral parameter. These relations can be rewritten in the matrix form by means of two Jacobi matrices. As a result, a convergence theorem for multipoint Padé approximants to Nevanlinna functions is proved.
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Keywords: Nevanlinna–Pick problem; Biorthogonal rational functions; R_{II}-fraction; Multipoint Padé approximant; Jacobi matrix; Generalized eigenvalue problem; Markov function

1. Introduction

Moment problems as well as interpolation problems arise in a wide range of mathematical and physical sciences (see [1,5,21]). The classical power moment problem can be formulated as follows.

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doi:10.1016/j.jat.2008.07.002

The Hamburger moment problem. Given is a sequence $\{c_j\}_{j=0}^{\infty}$ of real numbers. Find a positive Borel measure $\mathrm{d}\sigma$ on $\mathbb{R}$ such that

$$c_j = \int_{\mathbb{R}} t^j \mathrm{d}\sigma(t), \quad j = 0, 1, \ldots.$$

In a view of the Hamburger–Nevanlinna theorem (see [1]), the moment problem is equivalent to the problem of finding the Nevanlinna function $\varphi(\lambda)\left(= \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{t-\lambda}\right)$ having the following asymptotic expansions

$$\varphi(\lambda) = -\frac{c_0}{\lambda} - \frac{c_1}{\lambda^2} - \cdots - \frac{c_{2n}}{\lambda^{2n+1}} + o\left(\frac{1}{\lambda^{2n+1}}\right) \quad (\lambda = \mathrm{i}y, y \to +\infty)$$

for all $n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. The latter problem can be solved by means of the Schur algorithm. This algorithm leads to the J-fraction

$$-\cfrac{1}{\lambda - a_0 - \cfrac{b_0^2}{\lambda - a_1 - \cfrac{b_1^2}{\ddots}}}, \tag{1.1}$$

where a_j are real numbers, b_j are positive numbers [1] (see also [14]). Recall that the theory of J-fractions is also related to the Padé approximation theory and the theory of orthogonal polynomials. Under some natural conditions it is possible to say that all these theories (J-fractions, Padé approximation and orthogonal polynomials) are equivalent to one another. On the other hand, the J-fraction (1.1) generates the following infinite Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} a_0 & b_0 & & \\ b_0 & a_1 & b_1 & \\ & b_1 & a_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

In fact, the Jacobi matrix $\mathcal{J}$ is a key tool for analyzing the moment problem as well as the Nevanlinna function φ via operator methods. For example, using Jacobi matrices techniques one can prove convergence results for Padé approximants to Nevanlinna functions (see, for instance, [5,27]; see also [11] where generalized Jacobi matrices associated to indefinite moment problems for generalized Nevanlinna functions are considered).

The main goal of the present paper is to generalize the above-described scheme to the case of Nevanlinna–Pick problems in the class of Nevanlinna functions. Our approach is based on the relations between the theory of multipoint Padé approximants (Padé interpolants), the theory of biorthogonal rational functions, and generalized eigenvalue problem for two Jacobi matrices [31,28,29,32].

In theory of biorthogonal rational functions, the so-called continued fractions of R_{II} type appear. These continued fractions were introduced and studied by Ismail and Masson [17]. Nevertheless, note that continued fractions of the same type were considered earlier in problems connected with rational interpolation problems (see, e.g. [30,10]). It appears that the continued fractions of the R_{II} type are closely connected with the diagonal Padé interpolation problem from one side and with the theory of generalized eigenvalue problem for two Jacobi matrices on the other side [31]. In turn, both problems are equivalent (under some natural conditions) to theory of the biorthogonal rational functions (BRF) [29,32]. Note that theory of *orthogonal* rational functions studied and developed in [9] can be considered as a special case of theory of BRF (for details see, e.g. [32]).

2. Padé interpolation and biorthogonality

In this section we present basic facts concerning Padé interpolation and corresponding biorthogonal rational functions. We follow mostly [31,32] but some of the results appear to be new.

Let monic polynomials $P_n(z) = z^n + O(z^{n-1})$ satisfy the R_{II} type recurrence relation

$$P_{n+1}(z) + (\alpha_n z + \beta_n) P_n(z) + r_n(z - a_n)(z - b_n) P_{n-1}(z) = 0 \tag{2.1}$$

with initial conditions

$$P_0 = 1, \qquad P_1 = z - \beta_0.$$

Monicity of polynomials $P_n(z)$ assumes the restriction upon the recurrence coefficients

$$\alpha_0 = -1, \qquad \alpha_n + r_n + 1 = 0, \quad n = 1, 2, \ldots. \tag{2.2}$$

In what follows we will assume that $r_n \neq 0,\ n = 1, 2, \ldots$ (nondegeneracy).

Introduce the polynomials

$$A_0 = B_0 = 1, \qquad A_n(z) = \prod_{k=1}^{n} (z - a_k), \qquad B_n(z) = \prod_{k=1}^{n} (z - b_k).$$

As shown by Ismail and Masson [17] there exists a linear functional σ defined on all rational functions (without a polynomial part) with the prescribed poles $a_1, b_1, a_2, b_2, \ldots$ by the moments

$$c_{nm} = \sigma \left\{ \frac{1}{A_n(z) B_m(z)} \right\}, \quad n, m = 0, 1, 2, \ldots \tag{2.3}$$

such that the orthogonality relation

$$\sigma \left\{ \frac{P_n(z) q_j(z)}{A_n(z) B_n(z)} \right\} = 0, \quad j = 0, 1, \ldots, n-1, \tag{2.4}$$

holds, where $q_j(z)$ is any polynomial of degree not exceeding j and

$$\sigma \left\{ \frac{P_n(z) z^n}{A_n(z) B_n(z)} \right\} = \kappa_n \neq 0.$$

The normalization coefficients κ_n satisfy the recurrence relation [17]

$$\kappa_{n+1} + \alpha_n \kappa_n + r_n \kappa_{n-1} = 0. \tag{2.5}$$

It is important to note that, in contrast to the case of the ordinary orthogonal polynomials, we can take *two* first coefficients κ_0, κ_1 as arbitrary parameters. Then all further coefficients $\kappa_2, \kappa_3, \ldots$ are determined uniquely through (2.5).

Note also that if for some $n = n_0 > 1$ we have $\kappa_{n_0} = \kappa_{n_0-1}$ then from (2.5) and (2.2) it follows that $\kappa_{n_0+1} = \kappa_{n_0} = \kappa_{n_0-1}$ and hence we then have $\kappa_n = \kappa_{n_0}$ for all $n \geq n_0 - 1$. Moreover, we also have from (2.5)

$$r_{n_0-1}(\kappa_{n_0-1} - \kappa_{n_0-2}) = 0.$$

Due to our assumption $r_n \neq 0$ we have $\kappa_{n_0-2} = \kappa_{n_0-1} = \kappa_{n_0}$. Repeating this process we arrive at condition

$$\kappa_1 = \kappa_0. \tag{2.6}$$

We thus have

Proposition 2.1. *Condition $\kappa_{n_0} = \kappa_{n_0-1}$ for some $n_0 > 1$ is equivalent to the condition $\kappa_0 = \kappa_1$. In this case we have $\kappa_n \equiv const$ for all $n = 0, 1, 2, \ldots$.*

This case will be considered as a degeneration and in what follows we will assume that $\kappa_1 \neq \kappa_0$. Then from this proposition it follows $\kappa_n \neq \kappa_{n-1}$ for $n = 2, 3, \ldots$. Moreover we will assume that $\kappa_n \neq 0$ for all $n = 0, 1, 2, \ldots$.

Introduce the rational functions [31]

$$R_n^{(1)}(z) = \frac{P_n(z)}{A_n(z)}, \qquad R_n^{(2)}(z) = \frac{P_n(z)}{B_n(z)}. \tag{2.7}$$

It is assumed that zeros of polynomials $P_n(z)$ do not coincide with points a_i, b_j, so rational functions $R_n^{(1)}(z)$ and $R_n^{(2)}(z)$ have the $[n/n]$ type. Rational functions $R_n^{(1)}(z)$ have prescribed poles $a_1, a_2, \ldots, a_n$ and rational functions $R_n^{(2)}(z)$ have prescribed poles $b_1, b_2, \ldots, b_n$.

These functions satisfy obvious recurrence relations

$$(z - a_{n+1})R_{n+1}^{(1)}(z) + (\alpha_n z + \beta_n)R_n^{(1)}(z) + r_n(z - b_n)R_{n-1}^{(1)}(z) = 0 \tag{2.8}$$

and

$$(z - b_{n+1})R_{n+1}^{(2)}(z) + (\alpha_n z + \beta_n)R_n^{(2)}(z) + r_n(z - a_n)R_{n-1}^{(2)}(z) = 0. \tag{2.9}$$

On the other hand, these recurrence relations can be rewritten in terms of the generalized eigenvalue problem (GEVP) [31]

$$J_1 \vec{R}^{(1)} = z J_2 \vec{R}^{(1)}$$

and

$$J_3 \vec{R}^{(2)} = z J_2 \vec{R}^{(2)}$$

where $\vec{R}^{(1)}$ is an infinite-dimensional vector with components $\{R_0^{(1)}, R_1^{(1)}, \ldots\}$ (as well as $\vec{R}^{(2)}$) and J_1, J_2, J_3 are 3-diagonal (Jacobi) matrices which entries are obvious from the above recurrence relations for $R_n^{(1)}, R_n^{(2)}$. As was shown in [31] the GEVP leads naturally to theory of biorthogonal rational functions associated with the polynomials $P_n(z)$ of the R_{II}-type. Here we propose a more simple scheme of construction of the pair of biorthogonal rational functions.

Introduce the rational functions $U_n(z)$ and $V_n(z)$ by the formulas:

$$U_n(z) = R_n^{(1)}(z) - \xi_n R_{n-1}^{(1)}(z), \qquad V_n(z) = R_n^{(2)}(z) - \xi_n R_{n-1}^{(2)}(z) \tag{2.10}$$

where $\xi_n = \kappa_n/\kappa_{n-1}$ (it assumed that $\xi_0 = 0$ so that $U_0 = V_0 = 1$). Clearly, the rational functions $U_n(z)$ have the poles $a_1, a_2, \ldots, a_n$ and the rational functions $V_n(z)$ have the poles $b_1, b_2, \ldots, b_n$.

We have

Theorem 2.2. *The rational functions (2.10) form a biorthogonal system with respect to the functional* σ:

$$\sigma\{U_n(z)V_m(z)\} = h_n\delta_{nm}, \quad n, m = 0, 1, \dots \tag{2.11}$$

where the normalization coefficients are

$$h_n = \frac{\kappa_n}{\kappa_{n-1}}(\kappa_{n-1} - \kappa_n).$$

The proof of this theorem is direct by using orthogonality relations (2.4).

Note that the normalization coefficient is nonzero $h_n \neq 0$ due to our assumptions on nondegeneracy $\kappa_0 \neq \kappa_1$ and $\kappa_n \neq 0$.

We can give an equivalent definition of the functions $U_n(z)$ and $V_n(z)$ using the determinant expressions:

$$U_n(z) = \frac{P_n(a_n)}{\Delta_n}\begin{vmatrix} c_{00} & c_{10} & \dots & c_{n,0} \\ c_{01} & c_{11} & \dots & c_{n,1} \\ \dots & \dots & \dots & \dots \\ c_{0,n-1} & c_{1,n-1} & \dots & c_{n,n-1} \\ 1 & A_1^{-1}(z) & \dots & A_n^{-1}(z) \end{vmatrix}, \tag{2.12}$$

$$V_n(z) = \frac{P_n(b_n)}{\Delta_n}\begin{vmatrix} c_{00} & c_{01} & \dots & c_{0,n} \\ c_{10} & c_{11} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots \\ c_{n-1,0} & c_{n-1,1} & \dots & c_{n-1,n} \\ 1 & B_1^{-1}(z) & \dots & B_n^{-1}(z) \end{vmatrix}, \tag{2.13}$$

where

$$\Delta_n = \begin{vmatrix} c_{00} & c_{01} & \dots & c_{0,n-1} \\ c_{10} & c_{11} & \dots & c_{1,n-1} \\ \dots & \dots & \dots & \dots \\ c_{n-1,0} & c_{n-1,1} & \dots & c_{n-1,n-1} \end{vmatrix}. \tag{2.14}$$

(It is assumed that $\Delta_0 = 1$). In what follows we will assume that

$$\Delta_n \neq 0, \quad n = 1, 2, 3, \dots$$

(this is another nondegeneracy condition).

Formulas (2.12), (2.13) follow directly from definition of moments (2.3). In order to obtain appropriate coefficients in front of determinantal expressions (2.12), (2.13) we can present expression for the rational function $U_n(z)$ in the following form

$$U_n(z) = \sum_{k=0}^{n} \frac{\gamma_{nk}}{A_k(z)}.$$

The leading term in this sum is

$$\gamma_{nn} = U_n(z)A_n(z)|_{z=a_n}.$$

On the other hand we have from the explicit expression (2.10)

$$U_n(z)A_n(z)\Big|_{z=a_n} = P_n(a_n)$$

whence we obtain the factor $\frac{P_n(a_n)}{\Delta_n}$ in front of determinant of the formula (2.12). Similarly we obtain the factor $\frac{P_n(b_n)}{\Delta_n}$ in front of determinant of the formula (2.13).

Note also that from the determinantal formulas (2.12), (2.13) it follows directly that

$$\sigma\{U_n(z)V_m(z)\} = \frac{\Delta_{n+1}}{\Delta_n} P_n(a_n)P_n(b_n)\delta_{nm}.$$

Comparing with (2.11) we obtain an interesting relation

$$P_n(a_n)P_n(b_n) = \frac{\Delta_n}{\Delta_{n+1}}\kappa_n(1-\kappa_n/\kappa_{n-1}). \tag{2.15}$$

From this relation it follows that condition

$$P_n(a_n)P_n(b_n) \neq 0 \tag{2.16}$$

guarantees nondegeneracy conditions $\kappa_n \neq 0$, $\kappa_n \neq \kappa_{n-1}$ and $\Delta_n \neq 0$. Thus we will assume that condition (2.16) holds. It is instructive to consider what happens when condition (2.16) does not hold. For example assume that $P_n(a_n) = 0$ for some n. Then the rational function $R_n^{(1)}(z) = P_n(z)/A_n(z)$ has the order $[n-1/n-1]$, i.e. it has poles $a_1, a_2, \dots, a_{n-1}$. Corresponding rational function $U_n(z)$ constructed by (2.10) will also have poles $a_1, a_2, \dots, a_{n-1}$ which means a degeneration (absence of the pole a_n).

We can present rational functions $U_n(z)$ and $V_n(z)$ in the form

$$U_n(z) = \frac{S_n(z)}{(1-\xi_n)A_n(z)}, \qquad V_n(z) = \frac{T_n(z)}{(1-\xi_n)B_n(z)}, \tag{2.17}$$

where $S_n(z) = z^n + O(z^{n-1})$ and $T_n(z) = z^n + O(z^{n-1})$ are monic polynomials of degree n. Polynomials $S_n(z)$, $T_n(z)$ are expressed in terms of polynomials $P_n(z)$:

$$S_n(z) = \frac{P_n(z) - \xi_n(z-a_n)P_{n-1}(z)}{1-\xi_n}, \qquad T_n(z) = \frac{P_n(z) - \xi_n(z-b_n)P_{n-1}(z)}{1-\xi_n}. \tag{2.18}$$

Moreover $S_0 = T_0 = 1$.

We have

Proposition 2.3. *Polynomials $S_n(z)$, $T_n(z)$ satisfy a system of first-order recurrence relations*

$$\begin{aligned} S_{n+1}(z) &= \nu_n^{(1)}(z-b_n)S_n(z) + \nu_n^{(2)}(z-a_n)T_n(z) \\ T_{n+1}(z) &= \nu_n^{(3)}(z-b_n)S_n(z) + \nu_n^{(4)}(z-a_n)T_n(z), \quad n = 1, 2, \dots \end{aligned} \tag{2.19}$$

where

$$\nu_n^{(1)} = \frac{\xi_n\beta_n - \xi_n\xi_{n+1}a_{n+1} - r_n a_n}{r_n(b_n - a_n)}$$

$$\nu_n^{(2)} = \frac{\xi_n\beta_n - \xi_n\xi_{n+1}a_{n+1} - r_n b_n}{r_n(a_n - b_n)}$$

$$\nu_n^{(3)} = \frac{\xi_n\beta_n - \xi_n\xi_{n+1}b_{n+1} - r_n a_n}{r_n(b_n - a_n)}$$

$$\nu_n^{(4)} = \frac{\xi_n \beta_n - \xi_n \xi_{n+1} b_{n+1} - r_n b_n}{r_n(a_n - b_n)}.$$

Note that $\nu_n^{(1)} + \nu_n^{(2)} = \nu_n^{(3)} + \nu_n^{(4)} = 1$ *providing monicity condition for polynomials* $S_{n+1}(z), T_{n+1}(z)$.

Proof. It is sufficient to write down

$$U_{n+1}(z) = \frac{P_{n+1}(z)}{A_{n+1}(z)} - \xi_{n+1} \frac{P_n(z)}{A_n(z)} = \frac{S_{n+1}(z)}{(1 - \xi_{n+1})A_{n+1}} \tag{2.20}$$

and express $P_{n+1}(z)$ in terms of $P_n(z)$ and $P_{n-1}(z)$ using recurrence relation (2.1). This allows one to obtain an expression of $P_n(z)$ in terms of polynomials $S_n(z)$, $S_{n+1}(z)$:

$$P_n(z) = \zeta_n^{(1)} \left(S_{n+1}(z) - (z - b_n) S_n(z) \right), \tag{2.21}$$

where

$$\zeta_n^{(1)} = \frac{r_n(1 - \xi_n)}{r_n(b_n - a_{n+1}) - \xi_n(\beta_n + \alpha_n a_{n+1})}.$$

Analogously

$$P_n(z) = \zeta_n^{(2)} \left(T_{n+1}(z) - (z - a_n) T_n(z) \right), \tag{2.22}$$

where

$$\zeta_n^{(2)} = \frac{r_n(1 - \xi_n)}{r_n(a_n - b_{n+1}) - \xi_n(\beta_n + \alpha_n b_{n+1})}.$$

Then we arrive at relations (2.19). □

Note that relations (2.19) (as well as (2.21), (2.22)) do not hold for $n = 0$ because coefficients $\nu_0^{(i)}$ as well as a_0, b_0 are not defined. However, relations (2.19) will be valid for $n = 0$ if we put

$$S_0 = T_0 = 1$$

and

$$\xi_0 = \frac{r_0 \kappa_0}{\kappa_0 - \kappa_1} \tag{2.23}$$

whereas a_0, b_0 and r_0 can be *arbitrary* parameters. Indeed it is elementary verified that in this case we have from relations (2.19) for $n = 0$

$$S_1(z) = z + \frac{a_1 \kappa_1 - \beta_0 \kappa_0}{\kappa_0 - \kappa_1}, \qquad T_1(z) = z + \frac{b_1 \kappa_1 - \beta_0 \kappa_0}{\kappa_0 - \kappa_1} \tag{2.24}$$

which is compatible with expression for $S_1(z)$, $T_1(z)$ obtained from (2.18) for $n = 1$. Thus we can assume that relations (2.19) are valid for $n = 0, 1, 2, \ldots$ under condition (2.23). Note that this condition is formally equivalent to condition

$$\kappa_{-1} = \frac{\kappa_0 - \kappa_1}{r_0}$$

obtained from recurrence relation (2.5) if one puts $n = 0$ (with arbitrary nonzero r_0). Equivalently, we can assume that for $n = 0$ coefficients $\nu_n^{(i)}$ take the values

$$\nu_0^{(1)} = \frac{\kappa_0(\beta_0 - a_0) + \kappa_1(a_0 - a_1)}{(b_0 - a_0)(\kappa_0 - \kappa_1)}, \qquad \nu_0^{(2)} = \frac{\kappa_0(\beta_0 - b_0) + \kappa_1(b_0 - a_1)}{(a_0 - b_0)(\kappa_0 - \kappa_1)}$$

and

$$\nu_0^{(3)} = \frac{\kappa_0(\beta_0 - a_0) + \kappa_1(a_0 - b_1)}{(b_0 - a_0)(\kappa_0 - \kappa_1)}, \qquad \nu_0^{(4)} = \frac{\kappa_0(\beta_0 - b_0) + \kappa_1(b_0 - b_1)}{(a_0 - b_0)(\kappa_0 - \kappa_1)}.$$

Vice versa, one can show that starting from the system (2.19) with $b_n \neq a_n,\ n = 0, 1, \ldots$ and with initial conditions $T_0 = S_0 = 1$ one construct a pair of biorthogonal functions $U_n(z), V_n(z)$ by (2.17) [29].

The Padé interpolation problem [6] (sometimes called the Cauchy–Jacobi, Newton–Padé or multipoint Padé approximation problem [24,5]) consists in finding a pair of polynomials $P_n(z), Q_m(z)$ such that

$$Y_s P_n(z_s) = Q_m(z_s), \quad s = 0, 1, 2, \ldots, n + m, \tag{2.25}$$

where Y_s and z_s are two given complex sequences ($s = 0, 1, 2, \ldots$). The degrees of polynomials $P_n(z), Q_m(z)$ are less than or equal to n and m correspondingly. The rational function

$$r_{mn}(z) = \frac{Q_m(z)}{P_n(z)}$$

is called the rational interpolant corresponding to the sequences Y_s and z_s.

We will consider only the so-called normal case of the Padé interpolation problem [6] meaning that the degrees of polynomials $P_n(z), Q_m(z)$ are exactly n and m and there are no common zeros of polynomials $P_n(z)$ and $Q_m(z)$. In the normal case we have for every pair (n, m) the conditions [6]

$$r_{m,n+1}(z) \neq r_{mn}(z) \neq r_{m+1,n}(z).$$

In practice, it is assumed that $Y_s = F(z_s)$ for some given function $F(z)$ of the complex argument z. In this case formula (2.25) gives a rational interpolant $r_{mn}(z)$ of the function $F(z)$ for the given sequence z_s of interpolation points. Note that when all interpolation points coincide $z_s = z_0, s = 0, 1, 2, \ldots$, then we have the ordinary Padé approximation in the point z_0. The set $r_{mn}(z), m, n = 0, 1, 2, \ldots$ is called the Padé interpolation table for the function $F(z)$.

Consider the so-called diagonal string [29,32] in the Padé interpolation table, i.e. the set $r_{n-1,n}(z),\ n = 1, 2, \ldots$. This means that we are seeking a solution of the problem

$$F(z_s) = \frac{Q_{n-1}(z_s)}{P_n(z_s)}, \quad s = 0, 1, 2, \ldots 2n - 1. \tag{2.26}$$

Padé interpolants for the diagonal string satisfy simple orthogonality properties [24,29]

$$[z_0, z_1, \ldots z_{2n-1}] \left\{ z^j P_n(z) \right\} = 0, \quad j = 0, 1, \ldots, n - 1, \tag{2.27}$$

where

$$[z_0, z_1, \ldots, z_{2n-1}]\{f(z)\} \equiv \int_\Gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z_0)(\zeta - z_1)\ldots(\zeta - z_{2n-1})} \tag{2.28}$$

is the divided difference of the order $2n - 1$ from the function $f(z)$. It is assumed that the integration contour Γ avoids all singularity points of the function $f(z)$. Note that formula (2.28) is called the Hermite form of the divided difference operation [5].

Orthogonality relation (2.27) can be extended to biorthogonality relation for two rational functions $U_n(z), V_n(z)$ as follows. Consider the diagonal Padé interpolation problem for the same function $F(z)$ but with slightly modified interpolation sequence

$$F(z_s) = \frac{\tilde{Q}_{n-1}(z_s)}{\tilde{P}_n(z_s)}, \quad s = 0, 1, 2, \dots, 2n-2, 2n \tag{2.29}$$

(i.e. for the given n we have $2n$ interpolation points as in the previous scheme (2.26), but the final point z_{2n-1} is replaced by z_{2n}). Construct the rational functions

$$U_n(z) = \frac{P_n(z)}{(z-z_1)(z-z_3)\dots(z-z_{2n-1})},$$
$$V_n(z) = \frac{\tilde{P}_n(z)}{(z-z_2)(z-z_4)\dots(z-z_{2n})}. \tag{2.30}$$

Then the biorthogonality relation

$$[z_0, z_1, \dots z_{2n-1}]\left\{\frac{U_n(z)V_m(z)}{z-z_0}\right\} = h_n\delta_{nm}, \quad n, m = 0, 1, 2, \dots \tag{2.31}$$

holds with some normalization constant $h_n \neq 0$ [29,32]. It is easily verified that polynomials $P_n(z)$ and $\tilde{P}_n(z)$ satisfy the R_{II} recurrence relations (2.1) whereas the rational functions $U_n(z), V_n(z)$ satisfy the generalized eigenvalue problem of type (2.8). Thus the generalized eigenvalue problem for two Jacobi matrices is related with the diagonal Padé interpolation problem. For further development and generalizations of this subject see [29,32,23].

3. Nevanlinna–Pick problems

In this section we propose a modification of the famous step-by-step process of solving the Nevanlinna–Pick problem in the class of Nevanlinna functions [1,2].

First, let us recall that a Nevanlinna function is a function which is holomorphic in the open upper half plane $\mathbb{C}_+$ and has a nonnegative imaginary part in $\mathbb{C}_+$. Let $\mathbf{N}[\alpha, \beta]$ denote a class of all functions φ having the representation

$$\varphi(\lambda) = \int_\alpha^\beta \frac{d\sigma(t)}{t-\lambda}, \tag{3.1}$$

where $d\sigma(t)$ is a finite measure. A function of the class $\mathbf{N}[\alpha, \beta]$ is called a Markov function. Clearly, a Markov function is also a Nevanlinna function. Moreover, if the singularities of the Nevanlinna function φ are contained in $[\alpha, \beta]$ then $\varphi \in \mathbf{N}[\alpha, \beta]$ (see, for instance, [1]). Let us consider the following Nevanlinna–Pick problem.

Problem $\mathbf{NP}[\alpha, \beta]$. Given are two infinite sequences $\{z_k\}_{k=0}^\infty$, $\{w_k\}_{k=0}^\infty$ ($z_k \in \mathbb{C}_+$). Find a function $\varphi \in \mathbf{N}[\alpha, \beta]$ such that

$$\varphi(z_k) = w_k, \quad k = 0, 1, 2, \dots.$$

As is known (see [2]), the problem $\mathbf{NP}[\alpha, \beta]$ is solvable if and only if the Hermitian forms

$$\sum_{j,k=0}^N \frac{w_j(z_j-\alpha) - \overline{w}_k(\overline{z}_k-\alpha)}{z_j-\overline{z}_k}\xi_j\overline{\xi}_k, \qquad \sum_{j,k=0}^N \frac{w_j(\beta-z_j) - \overline{w}_k(\beta-\overline{z}_k)}{z_j-\overline{z}_k}\xi_j\overline{\xi}_k \tag{3.2}$$

are nonnegative definite for all $N \in \mathbb{Z}_+$.

It is also natural to consider the truncated Nevanlinna–Pick problem.

Problem $\mathbf{NP}([\alpha, \beta], \mathbf{n})$. Given are two finite sequences $\{z_k\}_{k=0}^n$, $\{w_k\}_{k=0}^n$ ($z_k \in \mathbb{C}_+$). Describe all functions $\varphi \in \mathbf{N}[\alpha, \beta]$ satisfying the property

$$\varphi(z_k) = w_k, \quad k = 0, 1, \dots, n.$$

Note that the problem $\mathbf{NP}([\alpha, \beta], \mathbf{n})$ is solvable if and only if the Hermitian forms (3.2) are nonnegative definite for $N = 0, 1, \dots, n$.

The algorithm of solving the Nevanlinna–Pick problems in question is based on the subsequent statement.

Lemma 3.1. *Let $\varphi \in \mathbf{N}[\alpha, \beta]$ and let $z \in \mathbb{C}_+$ be a fixed number. Then there exist numbers $a^{(1)}$, $a^{(2)} \in \mathbb{R}$ and $b > 0$ such that the function τ defined by the equality*

$$\varphi(\lambda) = -\frac{1}{a^{(2)}\lambda - a^{(1)} + b^2(\lambda - z)(\lambda - \overline{z})\tau(\lambda)} \tag{3.3}$$

belongs to $\mathbf{N}^0[\alpha, \beta] := \mathbf{N}[\alpha, \beta] \cup \{0\}$.

Proof. Setting $\Phi(\lambda) := -\frac{1}{\varphi(\lambda)}$, define the function

$$\Psi(\lambda) = \frac{\Phi(\lambda) - \Phi(z)}{\Phi(\lambda) - \overline{\Phi(z)}} : \frac{\lambda - z}{\lambda - \overline{z}}. \tag{3.4}$$

Due to the Schwartz lemma, we have that

$$|\Psi(\lambda)| \le 1, \quad \operatorname{Im}\lambda > 0.$$

So, the function $\widetilde{\Psi}_1$ defined from the relation

$$\Psi(\lambda) = \frac{\widetilde{\Psi}_1(\lambda) - \mathrm{i}}{\widetilde{\Psi}_1(\lambda) + \mathrm{i}} \tag{3.5}$$

is a Nevanlinna function. Plugging (3.5) into (3.4), one obtains

$$\widetilde{\Psi}_1(\lambda) = -\mathrm{i}\frac{\Phi(\lambda)(2\lambda - z - \overline{z}) - \overline{\Phi(z)}(\lambda - z) - \Phi(z)(\lambda - \overline{z})}{\Phi(\lambda)(z - \overline{z}) - \Phi(z)(\lambda - \overline{z}) + \overline{\Phi(z)}(\lambda - z)}. \tag{3.6}$$

Now, let us consider the following function

$$\Psi_1(\lambda) := \widetilde{\Psi}_1(\lambda) + \mathrm{i}\frac{2\lambda - (z + \overline{z})}{z - \overline{z}} = \widetilde{\Psi}_1(\lambda) + \frac{2\lambda - (z + \overline{z})}{2\operatorname{Im} z}.$$

Obviously, Ψ_1 is a Nevanlinna function. Taking into account (3.6), Ψ_1 admits the following representation

$$\Psi_1(\lambda) = -\frac{\operatorname{Im}\Phi(z)}{\operatorname{Im} z}\frac{(\lambda - z)(\lambda - \overline{z})}{\Phi(\lambda) - \frac{\operatorname{Im}\Phi(z)}{\operatorname{Im} z}\lambda + \frac{\operatorname{Im}\Phi(z)\overline{z}}{\operatorname{Im} z}}. \tag{3.7}$$

Finally, introducing

$$\tau(\lambda) = -\frac{1}{\Psi_1(\lambda)} \in \mathbf{N}, \qquad b = \frac{\operatorname{Im}\Phi(z)}{\operatorname{Im} z} > 0, \qquad a^{(2)} = \frac{\operatorname{Im}\Phi(z)}{\operatorname{Im} z} \in \mathbb{R},$$
$$a^{(1)} = -\frac{\operatorname{Im}\Phi(z)\overline{z}}{\operatorname{Im} z} \in \mathbb{R},$$

one can easily transform (3.7) into (3.3). To complete the proof, it is sufficient to observe that, due to (3.1) and (3.3), all singularities of τ are contained in $[\alpha, \beta]$. $\square$

Remark 3.2. The transformation (3.3) could be viewed as a substitute for the Schwartz lemma. A transformation for Caratheodory functions similar to (3.3) was proposed in [10].

Remark 3.3. Substituting λ for z and $\overline{z}$ in (3.3) we get

$$\varphi(z) = -\frac{1}{a^{(2)}z - a^{(1)}}, \qquad \varphi(\overline{z}) = -\frac{1}{a^{(2)}\overline{z} - a^{(1)}}.$$

Expressing from the above relations $a^{(1)}$ and $a^{(2)}$, one can obtain the following formulas

$$a^{(2)} = -\frac{\operatorname{Im} \frac{1}{\varphi(z)}}{\operatorname{Im} z}, \qquad a^{(1)} = -\frac{\operatorname{Im} \frac{1}{\varphi(z)}}{\operatorname{Im} z} z + \frac{1}{\varphi(z)}. \tag{3.8}$$

It is easy to see that the numbers $a^{(1)}$, $a^{(2)}$ are uniquely determined by (3.8). Further, equality (3.3) can be rewritten as follows

$$b^2\tau(\lambda) = -\frac{\frac{1}{\varphi(\lambda)} + a^{(2)}\lambda - a^{(1)}}{(\lambda - z)(\lambda - \overline{z})}. \tag{3.9}$$

In fact, the number b can be chosen arbitrary. So, to be definite we always choose $b > 0$ in the following way

$$b^2 = \int_\alpha^\beta d\sigma(t).$$

In this case, the function τ possesses the integral representation (3.1) with a probability measure.

Remark 3.4. It also easily follows from the theory of generalized Nevanlinna functions (see [12, 13,20]) that the right-hand side of (3.9) is a Nevanlinna function.

Remark 3.5. By comparing the first terms in asymptotic expansions of the right-hand side and left-hand side of (3.3), we see that

$$a^{(2)} = \left(\int_\alpha^\beta d\sigma(t)\right)^{-1} + b^2. \tag{3.10}$$

Now, we are in a position to solve the problem $\mathbf{NP}([\alpha, \beta], \mathbf{n})$. Let the given problem $\mathbf{NP}([\alpha, \beta], \mathbf{n})$ be solvable and let φ be a solution of the problem $\mathbf{NP}([\alpha, \beta], \mathbf{n})$. Due to Lemma 3.1, $\varphi_0 := \varphi$ admits the following representation

$$\varphi(\lambda) = -\frac{1}{a_0^{(2)}\lambda - a_0^{(1)} + b_0^2(\lambda - z_0)(\lambda - \overline{z}_0)\varphi_1(\lambda)}, \tag{3.11}$$

where $\varphi_1 \in \mathbf{N}^0[\alpha, \beta]$. From (3.11) we see that

$$\varphi_1(\lambda) = -\frac{\frac{1}{\varphi(\lambda)} + a_0^{(2)}\lambda - a_0^{(1)}}{b_0^2(\lambda - z_0)(\lambda - \overline{z}_0)}.$$

So, if $\varphi_1 \not\equiv 0$ then it is a solution of the problem $\mathbf{NP}([\alpha,\beta],\mathbf{n-1})$ with the sequences $\{z_k\}_{k=1}^n$ and $\{w_k^{(1)}\}_{k=1}^n$, where

$$w_k^{(1)} = \varphi_1(z_k) = -\frac{\frac{1}{w_k} + a_0^{(2)} z_k - a_0^{(1)}}{b_0^2 (z_k - z_0)(z_k - \overline{z}_0)}.$$

Therefore, the original problem $\mathbf{NP}([\alpha,\beta],\mathbf{n})$ is reduced to the problem $\mathbf{NP}([\alpha,\beta],\mathbf{n-1})$. Similarly, the problem $\mathbf{NP}([\alpha,\beta],\mathbf{n-1})$ can be reduced to the problem $\mathbf{NP}([\alpha,\beta],\mathbf{n-2})$ and so on. Finally, one has a sequence of the linear fractional transformations

$$\varphi_j(\lambda) = -\frac{1}{a_j^{(2)}\lambda - a_j^{(1)} + b_j^2(\lambda - z_j)(\lambda - \overline{z}_j)\varphi_{j+1}(\lambda)} \quad (j = 0, 1, \ldots, n)$$

having the following matrix representations

$$\mathcal{W}_j(\lambda) = \begin{pmatrix} 0 & -\dfrac{1}{b_j(\lambda - \overline{z}_j)} \\ b_j(\lambda - z_j) & \dfrac{a_j^{(2)}\lambda - a_j^{(1)}}{b_j(\lambda - \overline{z}_j)} \end{pmatrix} \quad (j = 0, 1, \ldots, n). \tag{3.12}$$

If the above-described algorithm consists of exactly $n+1$ steps then we say that the problem $\mathbf{NP}([\alpha,\beta],\mathbf{n})$ is *nondegenerate*. So, we have proved the following theorem which gives the complete solution of the problem $\mathbf{NP}([\alpha,\beta],\mathbf{n})$.

Theorem 3.6 (*[1]*). *Any solution φ of the nondegenerate problem* $\mathbf{NP}([\alpha,\beta],\mathbf{n})$ *admits the following representation*

$$\varphi(\lambda) = \frac{w_{11}(\lambda)\tau(\lambda) + w_{12}(\lambda)}{w_{21}(\lambda)\tau(\lambda) + w_{22}(\lambda)}, \tag{3.13}$$

where $\tau \in \mathbf{N}^0[\alpha,\beta]$ and

$$\mathcal{W}_{[0,n]}(\lambda) = (w_{ij}(\lambda))_{i,j=1}^2 := \mathcal{W}_0(\lambda)\mathcal{W}_1(\lambda)\ldots\mathcal{W}_n(\lambda). \tag{3.14}$$

It should be also remarked that $\mathcal{W}_j$ is the Blaschke–Potapov factor [3,26].

4. Rational systems related to Nevanlinna–Pick problems

Let us suppose that the given Markov function has the integral representation (3.1) with a probability measure $d\sigma$ which support contains infinitely many points, i.e.

$$\int_\alpha^\beta d\sigma(t) = 1.$$

Let us also suppose that for the given sequence $\{z_k\}_{k=0}^\infty$ there exists $\delta > 0$ such that

$$\operatorname{Im} z_k > \delta, \quad k = 0, 1, 2, \ldots. \tag{4.1}$$

In this case, it follows from the uniqueness theorem for analytic functions that the numbers z_k and $w_k := \varphi(z_k)$ ($k \in \mathbb{Z}_+$) uniquely determine the function φ. So, the Nevanlinna–Pick problem with the data $\{z_k\}_{k=0}^\infty$, $\{w_k\}_{k=0}^\infty$ has a unique solution.

Since φ is not rational the given data give rise to infinitely many steps of the step-by-step process. So, we have infinitely many linear fractional transformations of the form (3.3) which lead to the following continued fraction

$$-\cfrac{1}{a_0^{(2)}\lambda - a_0^{(1)} - \cfrac{b_0^2(\lambda-z_0)(\lambda-\overline{z}_0)}{a_1^{(2)}\lambda - a_1^{(1)} - \cfrac{b_1^2(\lambda-z_1)(\lambda-\overline{z}_1)}{\ddots}}}$$

$$= -\frac{1}{a_0^{(2)}\lambda - a_0^{(1)}} - \frac{b_0^2(\lambda-z_0)(\lambda-\overline{z}_0)}{a_1^{(2)}\lambda - a_1^{(1)}} - \frac{b_1^2(\lambda-z_1)(\lambda-\overline{z}_1)}{a_2^{(2)}\lambda - a_2^{(1)}} - \cdots. \tag{4.2}$$

Continued fraction (4.2) is an R_{II}-fraction (see [17]). Consider the $(n+1)$th convergent of continued fraction (4.2)

$$R_n(\lambda) := -\frac{1}{a_0^{(2)}\lambda - a_0^{(1)}} - \frac{b_0^2(\lambda-z_0)(\lambda-\overline{z}_0)}{a_1^{(2)}\lambda - a_1^{(1)}} - \cdots - \frac{b_{n-1}^2(\lambda-z_{n-1})(\lambda-\overline{z}_{n-1})}{a_n^{(2)}\lambda - a_n^{(1)}}.$$

It is obvious that R_n is a solution of the problem $\mathbf{NP}([\alpha, \beta], \mathbf{n})$, i.e. the following equality holds true

$$R_n(z_k) = w_k = \varphi(z_k), \quad k = 0, 1, \ldots, n.$$

Definition 4.1. The $[L/M]$ multipoint Padé approximant for a function φ at the points $\{\alpha_k\}_{k=1}^{\infty}$ is defined as a ratio

$$f^{[L/M]}(\lambda) = \frac{A^{[L/M]}(\lambda)}{B^{[L/M]}(\lambda)}$$

of two polynomials $A^{[L/M]}$, $B^{[L/M]}$ of formal degree L and M, respectively, such that

$$f^{[L/M]}(\alpha_k) = \varphi(\alpha_k), \quad k = 1, \ldots, L + M + 1.$$

Since R_n is real, the rational function R_n is the $[n/n]$ multipoint Padé approximant for φ at the points $\{\infty, z_0, \overline{z}_0, \ldots, z_n, \overline{z}_n, \ldots\}$.

It is well known that to every continued fraction there corresponds a recurrence relation. In particular, for continued fraction (4.2) a recurrence relation takes the following form

$$u_{j+1} - (a_j^{(2)}\lambda - a_j^{(1)})u_j + b_{j-1}^2(\lambda - z_{j-1})(\lambda - \overline{z}_{j-1})u_{j-1} = 0 \quad (j \in \mathbb{N}). \tag{4.3}$$

Define polynomials of the first kind $P_j(\lambda)$ as solutions $u_j = P_j(\lambda)$ of the system (4.3) with the initial conditions

$$u_0 = 1, \qquad u_1 = a_0^{(2)}\lambda - a_0^{(1)}. \tag{4.4}$$

Similarly, the polynomials of the second kind $Q_j(\lambda)$ are defined as solutions $u_j = Q_j(\lambda)$ of the system (4.3) subject to the following initial conditions

$$u_0 = 0, \qquad u_1 = -1. \tag{4.5}$$

Note that in our setting (2.4) is transformed into the following orthogonality relations (see also [15])

$$\int_\alpha^\beta t^j P_{n+1}(t)\frac{\mathrm{d}\sigma(t)}{|t-z_0|^2\dots|t-z_n|^2}=0,\quad j=0,1,\dots,n. \tag{4.6}$$

It follows from the theory of continued fractions that $R_n(\lambda)=\frac{Q_{n+1}(\lambda)}{P_{n+1}(\lambda)}$ (see, for details, [18]).

Recurrence relation (4.3) can be renormalized to the following one

$$b_j(z_j-\lambda)\widehat{u}_{j+1}-(a_j^{(2)}\lambda-a_j^{(1)})\widehat{u}_j+b_{j-1}(\overline{z}_{j-1}-\lambda)\widehat{u}_{j-1}=0\quad (j\in\mathbb{N}), \tag{4.7}$$

where

$$\widehat{u}_0=u_0,\qquad \widehat{u}_j=\frac{u_j}{b_0\dots b_{j-1}(z_0-\lambda)\dots(z_{j-1}-\lambda)}\quad (j\in\mathbb{N}).$$

Relation (4.6) implies that

$$\int_\alpha^\beta \widehat{P}_{n+1}(t)\frac{1}{t-\overline{z}_j}\mathrm{d}\sigma(t)=0,\quad j=0,\dots,n. \tag{4.8}$$

Now, setting

$$\xi_0=0,\qquad \xi_j=\left(\int_\alpha^\beta\frac{t^jP_j(t)\mathrm{d}\sigma(t)}{|t-z_0|^2\dots|t-z_{j-1}|^2}\right)\left(\int_\alpha^\beta\frac{t^{j-1}P_{j-1}(t)\mathrm{d}\sigma(t)}{|t-z_0|^2\dots|t-z_{j-2}|^2}\right)^{-1}$$
$$(j\in\mathbb{N})$$

one can see that the simple linear combinations $\widehat{P}_j-\xi_j\widehat{P}_{j-1}$ ($j\in\mathbb{Z}_+$) give orthogonalization of the system

$$\left\{1,\frac{1}{\lambda-z_0},\frac{1}{\lambda-z_1},\dots\right\}$$

of rational functions (see Theorem 2.2, see also [7]). It should be also noted here that systems of orthogonal rational functions related to Nevanlinna–Pick problems were proposed in [8,22,25] (see also [9]).

Further, relation (4.7) can be rewritten as follows

$$z_jb_j\widehat{u}_{j+1}+a_j^{(1)}\widehat{u}_j+\overline{z}_{j-1}b_{j-1}\widehat{u}_{j-1}=\lambda(b_j\widehat{u}_{j+1}+a_j^{(2)}\widehat{u}_j+b_{j-1}\widehat{u}_{j-1})\quad (j\in\mathbb{N}). \tag{4.9}$$

The system (4.9) gives us the possibility to rewrite the Cauchy problem (4.3) and (4.4) in the matrix form

$$J^{(1)}_{[0,\infty)}\pi(\lambda)=\lambda J^{(2)}_{[0,\infty)}\pi(\lambda),$$

where $\pi(\lambda)=\left(\widehat{P}_0(\lambda),\widehat{P}_1(\lambda),\dots,\widehat{P}_j(\lambda),\dots\right)^\top$ and

$$J^{(1)}_{[0,\infty)}=\begin{pmatrix}a_0^{(1)}&\overline{z}_0b_0&&\\ z_0b_0&a_1^{(1)}&\overline{z}_1b_1&\\ &z_1b_1&a_2^{(1)}&\ddots\\ &&\ddots&\ddots\end{pmatrix}\qquad J^{(2)}_{[0,\infty)}=\begin{pmatrix}a_0^{(2)}&b_0&&\\ b_0&a_1^{(2)}&b_1&\\ &b_1&a_2^{(2)}&\ddots\\ &&\ddots&\ddots\end{pmatrix}.$$

We denote by $\ell^2_{[0,n]}$ the space of $(n+1)$ vectors with the usual inner product. Define a standard basis in $\ell^2_{[0,n]}$ by the equalities

$$e_j = \{\delta_{l,k}\}_{k=0}^n, \quad j = 0, 1, \ldots, n.$$

Let $J^{(1)}_{[j,k]}$ ($J^{(2)}_{[j,k]}$) be a submatrix of $J^{(1)}_{[0,\infty)}$ ($J^{(2)}_{[0,\infty)}$), corresponding to the linear subspace spanned by the vectors $e_l, \ldots, e_k (0 \le j \le k \le n)$, that is,

$$J^{(1)}_{[j,k]} = \begin{pmatrix} a^{(1)}_j & \overline{z}_j b_j & \mathbf{0} \\ z_j b_j & \ddots & \\ \mathbf{0} & & a^{(1)}_k \end{pmatrix}, \qquad J^{(2)}_{[j,k]} = \begin{pmatrix} a^{(2)}_j & b_j & \mathbf{0} \\ b_j & \ddots & \\ \mathbf{0} & & a^{(2)}_k \end{pmatrix}.$$

Proposition 4.2. *The matrix $J^{(2)}_{[0,n]}$ is positive definite for all $n \in \mathbb{Z}_+$.*

Proof. Let us consider the Hermitian form

$$\left(J^{(2)}_{[0,n]}\xi, \xi\right) = a^{(2)}_0|\xi_0|^2 + b_0\overline{\xi}_0\xi_1 + b_0\xi_0\overline{\xi}_1 + a^{(2)}_1|\xi_1|^2 + \cdots + a^{(2)}_n|\xi_n|^2. \tag{4.10}$$

Due to (3.10) and our assumptions, we have that $a^{(2)}_j = 1 + b_j^2$. Therefore, one can rewrite form (4.10) in the following manner

$$\left(J^{(2)}_{[0,n]}\xi, \xi\right) = |\xi_0|^2 + |b_0\xi_0 + \xi_1|^2 + \cdots + |b_{n-1}\xi_{n-1} + \xi_n|^2 + |b_n\xi_n|^2 \ge 0.$$

Thus, $J^{(2)}_{[0,n]}$ is a positive definite matrix. $\square$

Finally, we should note that for the matrix $J^{(2)}_{[0,\infty)}$ the following factorization holds true

$$J^{(2)}_{[0,\infty)} = \begin{pmatrix} a^{(2)}_0 & b_0 & & \\ b_0 & a^{(2)}_1 & b_1 & \\ & b_1 & a^{(2)}_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & b_0 & & \\ 0 & 1 & b_1 & \\ & 0 & 1 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & & \\ b_0 & 1 & 0 & \\ & b_1 & 1 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

5. *m*-functions of linear pencils

In this section we give a matrix representation of multipoint Padé approximants for Markov functions.

Definition 5.1. The function

$$m_{[j,n]}(\lambda) = \left((J^{(1)}_{[j,n]} - \lambda J^{(2)}_{[j,n]})^{-1} e_j, e_j\right) \tag{5.1}$$

will be called the m-function of the linear pencil $J^{(1)}_{[j,n]} - \lambda J^{(2)}_{[j,n]}$.

To see the correctness of the above definition it is sufficient to rewrite (5.1) in the following form

$$m_{[j,n]}(\lambda) = \left((J^{(2)}_{[j,n]})^{-1}(J^{(1)}_{[j,n]}(J^{(2)}_{[j,n]})^{-1} - \lambda)^{-1} e_j, e_j\right). \tag{5.2}$$

From (5.2) one can conclude that $m_{[j,n]}$ is a Nevanlinna function.

Proposition 5.2. *The m-functions $m_{[j,n]}$ and $m_{[j+1,n]}$ are related by the equality*

$$m_{[j,n]} = -\frac{1}{a_j^{(2)}\lambda - a_j^{(1)} + b_j^2(\lambda - z_j)(\lambda - \overline{z}_j)m_{[j+1,n]}(\lambda)}. \tag{5.3}$$

Proof. Consider the following block representation of the matrix $J^{(1)}_{[j,n]} - \lambda J^{(2)}_{[j,n]}$

$$J^{(1)}_{[j,n]} - \lambda J^{(2)}_{[j,n]} = \begin{pmatrix} a_j^{(1)} - a_j^{(2)}\lambda & B \\ B^* & J^{(1)}_{[j+1,n]} - \lambda J^{(2)}_{[j+1,n]} \end{pmatrix},$$

where $B = (b_j(\overline{z}_j - \lambda), 0, \dots, 0)$. According to the Frobenius formula [16, Section 0.7.3] the matrix $(J^{(1)}_{[j,n]} - \lambda J^{(2)}_{[j,n]})^{-1}$ has the following block representation

$$(J^{(1)}_{[j,n]} - \lambda J^{(2)}_{[j,n]})^{-1} = \begin{pmatrix} \left(a_j^{(1)} - a_j^{(2)}\lambda - B^*(J^{(1)}_{[j+1,n]} - \lambda J^{(2)}_{[j+1,n]})^{-1}B\right)^{-1} & * \\ * & * \end{pmatrix}. \tag{5.4}$$

Plugging (5.4) into (5.1), one obtains (5.3). □

Corollary 5.3. *The following equalities hold true*

$$m_{[0,n]}(\lambda) = R_n(\lambda) = \frac{Q_{n+1}(\lambda)}{P_{n+1}(\lambda)} \in \mathbf{N}[\alpha, \beta]. \tag{5.5}$$

Proof. Relation (5.3) implies that the rational functions $m_{[0,n]}$ and R_n have the same expansions into R_{II}-fractions. So, $m_{[0,n]} = R_n$. By using standard argumentation, from (4.6) one can conclude that *all the zeros of P_{n+1} are contained in* $[\alpha, \beta]$ (see [1,15]). The latter means that the Nevanlinna function $m_{[0,n]}$ belongs to $\mathbf{N}[\alpha, \beta]$. □

So, now one can say that R_n is a solution of $\mathbf{NP}([\alpha, \beta], \mathbf{n})$. By using standard argumentation, from (5.5) we can conclude the following result.

Corollary 5.4. *The zeros of P_{n+1} and Q_{n+1} are interlace.*

Below, we will need the following statement.

Corollary 5.5. *The spectrum $\sigma\left(J^{(1)}_{[0,n]}(J^{(2)}_{[0,n]})^{-1}\right)$ of the matrix $J^{(1)}_{[0,n]}(J^{(2)}_{[0,n]})^{-1}$ is contained in* $[\alpha, \beta]$.

Proof. From the formula for calculation of inverse matrices, (5.1), and (5.5) one can see that

$$m_{[0,n]}(\lambda) = \frac{\det(J^{(1)}_{[1,n]} - \lambda J^{(2)}_{[1,n]})}{\det(J^{(1)}_{[0,n]}(J^{(2)}_{[0,n]})^{-1} - \lambda)\det(J^{(2)}_{[0,n]})} = \frac{Q_{n+1}(\lambda)}{P_{n+1}(\lambda)}.$$

So, the statement immediately follows from Corollary 5.4 and the fact that all the zeros of P_{n+1} are contained in $[\alpha, \beta]$. □

Remark 5.6. It should be remarked that, for the case of the Laurent orthogonal polynomials, a similar scheme with two matrices and m-functions were considered in [4].

6. A convergence result for multipoint Padé approximants

The goal of this section is to prove an analog of Markov's convergence theorem by making use of the operator representation of multipoint Padé approximants.

We begin with an auxiliary statement.

Lemma 6.1. *The following inequalities hold true*

$$\left((J^{(2)}_{[0,n]})^{-1}e_0, e_0\right) \leq 1 \quad (n \in \mathbb{Z}_+).$$

Proof. The proof is by induction. First, note that

$$\left((J^{(2)}_{[n,n]})^{-1}e_n, e_n\right) = \frac{1}{a^{(2)}_n} = \frac{1}{1+b_n^2} \leq 1 \quad (n \in \mathbb{Z}_+).$$

Suppose that $\left((J^{(2)}_{[k+1,n]})^{-1}e_{k+1}, e_{k+1}\right) \leq 1$. It follows from the Ricatti equation [14, formula (2.15)] (see also (5.3)) that

$$\begin{aligned}\left((J^{(2)}_{[k,n]})^{-1}e_k, e_k\right) &= \frac{1}{a^{(2)}_k - b_k^2\left((J^{(2)}_{[k+1,n]})^{-1}e_{k+1}, e_{k+1}\right)}\\ &= \frac{1}{1+b_k^2 - b_k^2\left((J^{(2)}_{[k+1,n]})^{-1}e_{k+1}, e_{k+1}\right)} \leq 1. \quad \square\end{aligned}$$

Now, we are ready to prove the main result of this section.

Theorem 6.2 (*[15]*). *Let $\varphi \in \mathbf{N}[\alpha,\beta]$ and let the sequence $\{z_k\}_{k=1}^{\infty}$ satisfy the condition (4.1). Then the sequence $f^{[n/n]} = R_n$ converges to φ locally uniformly in $\mathbb{C} \setminus [\alpha,\beta]$.*

Proof. We first recall the well-known estimate for the resolvent of self-adjoint operator J (for instance see [19, Theorem V.3.2])

$$\|(J-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(J))}. \tag{6.1}$$

Next, observe that the operator $J^{(1)}_{[0,n]}(J^{(2)}_{[0,n]})^{-1}$ is self-adjoint with respect to the following inner product

$$\left((J^{(2)}_{[0,n]})^{-1}x, y\right) \quad x, y \in \mathbb{C}^{n+1}.$$

Taking into account representations (5.5) and (5.2), the Cauchy–Schwartz inequality, (6.1), Corollary 5.5, and Lemma 6.1, we obtain

$$\begin{aligned}|R_n(\lambda)| &= \left|\left((J^{(2)}_{[0,n]})^{-1}(J^{(1)}_{[0,n]}(J^{(2)}_{[0,n]})^{-1}-\lambda)^{-1}e_0, e_0\right)\right|\\ &\leq \frac{\left((J^{(2)}_{[0,n]})^{-1}e_0, e_0\right)}{\operatorname{dist}(\lambda,[\alpha,\beta])} \leq \frac{1}{\operatorname{dist}(\lambda,[\alpha,\beta])}.\end{aligned} \tag{6.2}$$

It follows from (6.2) and Montel's theorem that the family $\{R_n\}$ is precompact in the topology of locally uniform convergence in $\mathbb{C} \setminus [\alpha,\beta]$. Note that

$$R_n(z_k) = \varphi(z_k), \quad n \geq k.$$

Thus, applying the Vitali theorem completes the proof. □

Remark 6.3. Theorem 6.2 was proved in [15] by means of another method. The rates of convergence of multipoint Padé approximants was also given in [15]. The operator interpretation of the rates of convergence and a more detailed analysis of the underlying linear pencil will be given in the forthcoming paper.

Acknowledgments

MD expresses his gratitude to Professor V.A. Derkach for carefully reading the manuscript and giving many helpful comments. The authors also thank the referees for helpful suggestions.

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