Finite groups whose all irreducible character degrees are Hall-numbers

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Abstract

We call \( m \) is a Hall-number for \( G \) if \( m \) is the order of a Hall subgroup of \( G \), that is, \( \gcd(|G|/m, m) = 1 \). The aim of this paper is to investigate the structure of the finite group \( G \) whose all irreducible character degrees are Hall-numbers for \( G \).

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1. Introduction

Throughout the following, all groups are assumed to be finite. For a group \( G \), we call the set \( \text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\} \) the irreducible character degrees of \( G \). And the set \( \text{cd}(G) \) is very important in studying \( G \). Many results have been obtained about the relationship between the set \( \text{cd}(G) \) and the structure of \( G \) (see [8,10–12] for a few examples). The aim of this paper is to investigate the structure of \( G \) whose every irreducible character degree \( m \) is a Hall-number for \( G \), that is, \( \gcd(|G|/m, m) = 1 \) for all \( m \in \text{cd}(G) \).

The following are our main theorems.

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Theorem 1.1. Let $G$ be a solvable group. Then every irreducible character degree of $G$ is a Hall-number for $G$ if and only if one of the following holds:

(i) $G$ is an abelian group;
(ii) $G$ is a semi-direct product of a cyclic Hall subgroup $M > 1$ acting on a normal abelian Hall subgroup $F$. And if $[F, P] > 1$, then $P$ acts fixed-point freely on $[F, P]$ for every $P \in \text{Syl}_p(M)$;
(iii) $G$ is a semi-direct product of a cyclic Hall subgroup $L$ acting on a normal Hall subgroup $H$, where the order of $L$ is square-free, $H$ is a group satisfying the properties of (ii).

Theorem 1.2. Let $G$ be a non-solvable group. Then every irreducible character degree of $G$ is a Hall-number for $G$ if and only if $G$ has normal Hall subgroups $M$ and $L$ that satisfy:

(i) $|G : M|$ is square-free;
(ii) $L \cong L_2(2^f)$ with $f \geq 2$;
(iii) $M = N \times L$ where $N = C_G(L)$;
(iv) every degree in $\text{cd}(N)$ is a Hall-number.

Furthermore, if $G$ is such a group, then $M$ has a complement $D$ which is isomorphic to a subgroup of the outer automorphism group of $L$. In particular, $D \cong G/M$ is cyclic and $|D| = |G : M|$ divides $f$.

2. Proof of Theorem 1.1

Lemma 2.1. [10] Let $V$, $N$ be normal subgroups of $G$ and $V < N$ such that $G/N$ (respectively $N/V$) is cyclic of order $a$ (respectively $b$). Moreover, let $V$ be elementary abelian and suppose that both $G/V$ and $N$ are Frobenius groups with kernel $N/V$ respectively $V$. Then $\text{cd}(G) \cup \{ab\} = \{1, a\} \cup \{ib \mid i \text{ divides } a\}$.

Lemma 2.2. Let $N$ be a normal subgroup of $G$. If every element of $\text{cd}(G)$ is a Hall-number for $G$, then every element of $\text{cd}(N)$ (respectively $\text{cd}(G/N)$) is a Hall-number for $N$ (respectively $G/N$).

Proof. We can conclude Lemma 2.2 by [6, Theorem 6.2] and [6, Theorem 11.29].

Proof of Theorem 1.1. ($\Leftarrow$) Suppose $G$ satisfies the properties of (ii). By induction on $|G|$, we may conclude that $m$ is a Hall-number for every $m \in \text{cd}(G)$: Take $\chi \in \text{Irr}(G)$ such that $\chi(1) = m$, let $M = P \times Q$ be a direct-product of a Sylow $p$-subgroup $P > 1$ and a Hall $p'$-subgroup $Q$. If $P < M$, let $\theta_1$ (respectively $\theta_2$) be an irreducible constituent of the restriction of $\chi$ on $P F$ (respectively $Q F$), then by induction we conclude that gcd$(\theta_1(1), |PF|/\theta_1(1)) = 1 = \text{gcd}(\theta_2(1), |QF|/\theta_2(1))$. This implies that gcd$(\chi(1), |G|/\chi(1)) = 1$. Therefore we can assume that $P = M$ is a Sylow $p$-subgroup of $G$, we easily see that $[F, P] = G'$ is a splitting normal abelian subgroup of $G$. So every irreducible character of $[F, P]$ is extendible to its inertia group in $G$. Let $\lambda$ be an irreducible constituent of $\chi_{[F, P]}$. If $\lambda = 1_{[F, P]}$, then $\lambda$ is extendible to $G$, by [6, Theorem 6.17] we see that $\chi(1) = 1$ is a Hall-number. Suppose $\lambda \neq 1_{[F, P]}$. As $P$ acts fixed-point freely on $[F, P]$, it follows that $P$ acts fixed-point freely on $\text{Irr}([F, P])$. So $I_G(\lambda)$
the inertia group of $\lambda$ in $G$ is $F$, we also conclude that $\chi(1) = |P|$ is a Hall-number by Clifford theorem [6, Theorem 6.2].

Suppose $G$ satisfies the properties of (iii). Let $\chi \in \text{Irr}(G)$, $\theta$ be an irreducible constituent of $\chi_H$, and $M = 1_G(\theta)$ the inertia group of $\theta$ in $G$. As $H$ is a Hall subgroup of $G$, $\theta$ is extendible to $M$, and $\chi(1)/\theta(1) = |G:M|$. Furthermore by previous argument and $H$ is a Hall subgroup of $G$ satisfying the properties of (ii), we can conclude that $\theta(1)$ is a Hall-number for $G$. And the order of $L$ is a square-free Hall-number for $G$, $M \geq H$, so we can conclude that $|G:M|$ is Hall-number for $G$. Hence $\chi(1) = \theta(1)(\chi(1)/\theta(1))$ is a Hall-number for $G$.

($\Rightarrow$) Obviously, by Lemma 2.2 if the character degrees of $G$ are Hall-numbers for $G$, then the character degrees of $M/N$ are Hall-numbers for $M/N$, where $M/N$ is a section of $G$ ($M$ be a subnormal subgroup of $G$, $N$ be a normal subgroup of $M$).

**Step 1.** Every nilpotent section $M/N$ of $G$ is an abelian group.

By induction, we can suppose that $G = M/N$. Note that since $(\chi(1))^2$ divides $|G:Z(G)|$ to every non-linear irreducible character $\chi$ of $G$. It follows that $\chi(1)$ is a Hall-number for $G$, so nilpotent group $G$ is an abelian group.

**Step 2.** Let $F = F_1$ be the Fitting subgroup of $G$, and $F_i \triangleleft G$ such that $F_{i+1}/F_i = \text{Fit}(G/F_i)$ for $i \geq 1$. We claim that $F_{i+1}/F_i$ is cyclic and isomorphic to some Hall subgroup of $G$ for $i \geq 1$.

Now we can assume that $F_m = G$ and $F_m-1 < G$. It suffices to show that $G/F_m-1$ is a cyclic group and isomorphic to some Hall subgroup of $G$ by induction and Lemma 2.2.

We know that $G/F_m-1$ is an abelian group by Step 1, so $|G:F_{m-1}| \in \text{cd}(G)$, we conclude that $G/F_m-1$ is isomorphic to some Hall subgroup of $G$ by $|G:F_m-1|$ is a Hall-number for $G$.

The following we prove $G/F_m-1$ is a cyclic group. We can assume that $G/F_m-1$ is a $p$-group by induction, where $p$ is a prime. Let $\Phi$ be a pre-image of the Frattini subgroup of $G/F_{m-2}$ in $G$. Note that $F_{m-1}/\Phi$ is a faithful and completely reducible $G/F_{m-1}$-module, so there is a chief factor $F_{m-1}/E$ of $G$ such that $G/F_{m-1}$ non-trivially acts on $F_{m-1}/E$. By investigating the factor group $G/E$, it is easy to conclude that $G/E$ is a normal abelian and $F_{m-1}/E$ is a normal abelian Sylow $q$-subgroup of $G/E$ (prime $q \neq p$). Now we consider the minimal non-abelian factor subgroup $G/B$ of $G/E$, by [6, Theorem 12.3] and Step 1, we conclude that $G/B$ is a Frobenius group, Frobenius kernel written as $A/B$ and $A$ is a cyclic group, $|G/A| \in \text{cd}(G/B) \subseteq \text{cd}(G)$. Note that $G/E$ is a $\{p,q\}$-group and the Sylow $q$-subgroup of $G/E$ is a normal abelian subgroup of $G/E$, we see that $A/B$ is a $q$-group, and $A/G$ is a cyclic $p$-group. By the hypotheses of the theorem we conclude that $|G/A|$ is the order of a Sylow $p$-subgroup of $G$, so $G/A$ is isomorphic to some cyclic Sylow $p$-subgroup of $G$. Now we have proved that $p$-group $G/F_m-1$ is cyclic.

**Step 3.** The last step.

By Step 2, $F_2/F$ is a cyclic group, so $G/F_2$ is an abelian group, and $F_3(G) = G$.

Now we assume that $F_2 = G$. Let $P$ be a Sylow $p$-subgroup of $G$ and $P \neq F$, we claim that $P$ acts fixed-point freely on $[F, P]$. In fact, by induction we can assume that $G = P[F]$. Note that $F$ is an abelian Hall subgroup of $G$, so we have $F = C_F(P) \times [F, P]$, and $G = P[F] = P[F, P] \times C_F(P)$. Also by induction we can assume that $C_F(P) = 1$, so $F = [F, P] = G'$. 
In this case, we let $\lambda \in \text{Irr}(F)$ be an arbitrary non-principal irreducible character of $F$. We know that all irreducible constituents of $\lambda^G$ are non-linear (otherwise, $C_{\text{Irr}(F)}(P) > 1$, so $C_F(P) > 1$ and this contradicts $C_F(P) = 1$), so it follows that for any irreducible constituent $\chi$ of $\lambda^G$ we have $\chi(1) > 1$ and $\chi(1)$ is a Hall-number for $G$, and thus $\chi(1) = |P|$. Now we have proved that $\lambda^G$ is irreducible for every non-principal $\lambda \in \text{Irr}(F)$, so $P$ acts fixed-point freely on $\text{Irr}(F)$, and $P$ acts fixed-point freely on $F$. We can conclude that $G$ satisfies (ii).

The following we assume that $F_2 < G$ and $F_3 = G$. We claim that $|G/F_2|$ is square-free. By induction we can assume that $G/F_2$ is a $p$-group, and by Step 2, $G$ is the semi-direct product of $P$ acting on $F_2$ where $P$ is a Sylow $p$-subgroup of $G$. Let $A \cong G/F$ be a Hall subgroup of $G$, $P \leq A$ and $B$ be a Hall $p'$-subgroup of $A$, so $A = P[B]$, $G = (PB)F$, where $B \cong F_2/F$.

By the last paragraph we conclude that $A = P[B]$, $G = (PB)F$, note that $P[B]F < G$, so by induction we can assume that $G = P[B, P]F$, that is $B = [B, P]$. Note that $|B, P| = B > 1$, by the previous argument we know that $P[B]$ is a Frobenius group with kernel $B$. As $B$ acts non-trivially on $F/E$ and $F/E$ is a chief factor, we have $BF = F_2$, $(BF/E)' = F/E$. Because the irreducible character degrees of $F_2/E$ are Hall-numbers for $F_2/E$, it is easy to conclude that $F_2/E$ is a Frobenius group with kernel $F/E$. So $G/E$ is a 2-Frobenius group (so-called), by Lemma 2.1 we have $p'|B| \in \text{cd}(G)$ for all $i$, so that $p'| |P|$. Therefore, by the hypothesis that all character degrees in $\text{cd}(G)$ are Hall-numbers, we see that $|P| = p$, so $|G/F_2|$ is square-free, and thus, $G$ satisfies (iii). \[ \square \]

3. Proof of Theorem 1.2

At first, we consider the simple groups. Let $G$ be a non-abelian simple group. By the classification theorem for finite simple groups we know that $G$ is one of the following: a sporadic simple group, an alternating group of degree at least 5, and a simple group of Lie type.

Now we consider $G$ is an alternating simple group. We refer to [7] for details about partitions, Young diagrams and hooks. Consider a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of the integer $n$. Thus $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$. We call the $\lambda_i$ ($i = 1, 2, \ldots, m$) the part of $\lambda$ and $m$ the length of $\lambda$. Moreover, for $i \geq 1$, $m_i = m_1(\lambda)$ denotes the number of parts equal to $i$ in $\lambda$. Thus $m = \sum_{i \geq 1} m_i$. The Young diagram of $\lambda$ consists of $n$ nodes (boxes) with $\lambda_i$ nodes in the $i$th row. We refer to the nodes in matrix notation, i.e. the $(i, j)$-node is the $j$th node in the $i$th row. The $(i, j)$-hook consists of the nodes in the Young diagram to the right and below the $(i, j)$-node, and including this node. The number of nodes in this hook is its hooklength, denote by $h_{ij}$.

The degree $f_{\lambda}$ of $\lambda$ is

$$f_{\lambda} = n! / \left( \prod_{i,j} h_{ij} \right).$$

It is known that this is the degree of the complex irreducible representation of the symmetric group $S_n$ labeled by $\lambda$.

We denote the another partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$ the conjugate (associated) partition of $\lambda$, where $\lambda'_i = \sum_{j: \lambda_j \geq i} 1$. So the irreducible representation of $S_n$ labeled by $\lambda$ remains irreducible when restricted to $A_n$ if and only if $\lambda \neq \lambda'$. If $\lambda = \lambda'$, then the restriction is a sum of two irreducible representations of the same degree.
Lemma 3.1. Let $G$ be an alternating simple group and every irreducible character degree of $G$ is a Hall-number for $G$. Then $G \cong A_5$.

Proof. If $G \cong A_5$, then $cd(G) = \{1, 3, 4, 5\}$, so every irreducible character degree of $A_5$ is a Hall-number for $A_5$.

For $n > 5$, let us investigate the characters corresponding to the following non-self-associated partition of $n$ where: $\lambda = (n - 2, 1, 1)$. Then

$$f_\lambda = (n - 1)(n - 2)/2$$

is an irreducible character degree of $A_n$,

$$|A_n|/f_\lambda = n!(n - 1)(n - 2) = n(n - 3)!.$$ 

If either $n \geq 7$ is odd or $n \geq 6$ is even, then

$$\text{gcd}(n(n - 3)!, (n - 1)(n - 2)/2) > 1.$$ 

So $\text{gcd}(|A_n|/f_\lambda, f_\lambda) > 1$ when $n > 5$, a contradiction. □

Next we consider the simple groups of Lie type. For notation and basic properties of finite groups of Lie type, we refer to [1]. We will denote by $L$ a finite simple group of Lie type over a field of $q$ elements, where $q = p^f$ is a power of a prime $p$. If we denote by $S$ a simple linear algebraic group of adjoint type, and by $\sigma$ an endomorphism of $S$, then the set $S_\sigma$ of fixed points is finite and the derived subgroup of $S_\sigma$ is isomorphic to $L$. The order of out automorphism group of $L$ is a product of $d$ (diagonal automorphisms), $f$ (field automorphisms), and $g$ (graph automorphisms). Note that the degrees of the unipotent characters of $S_\sigma$ and $L$ are the same and that $|L| = |S_\sigma|/d$. Thus, if $\psi$ is some unipotent character of $L$, then for any irreducible constituent $\chi$ of $\psi^{\text{aut}(L)}$, $\chi(1)$ divides $gf\psi(1)$, and we know $\psi(1) | \chi(1)$ by Clifford theorem [6, Theorem 6.2].

Lemma 3.2. Let $G$ be a simple group of Lie type and every irreducible character degree of $G$ is a Hall-number for $G$. Then $G \cong L_2(2^f)$ where $f \geq 2$.

Proof. We know that $G$ is one of the following types: type $A_n(q) \ (n \geq 1)$; type $2A_n(q^2) \ (n \geq 2)$; type $B_n(q) \ (n \geq 2)$; type $C_n(q) \ (n \geq 3)$; type $D_n(q) \ (n \geq 4)$; type $2D_n(q^2) \ (n \geq 4)$; and exceptional type.

- For type $A_n(q), n \geq 1$.

If $n \geq 2$, then the unipotent character $\psi^{(1,n)}$ has degree $m = q(q^n - 1)/(q - 1)$, where $q = p^f$ for some prime $p$,

$$|G| = q^{n(n+1)/2} \prod_{i=1}^{n}(q^i + 1)/\text{gcd}(n+1, q-1).$$ 

So $\text{gcd}(|G|/m, m) > 1$ for $n \geq 2$, a contradiction.
Suppose that $n = 1$ and $q$ is odd. Let $m = q - 1$ when $q = 3 \pmod{4}$, and $m = q + 1$ when $q = 1 \pmod{4}$, by [5, Chapter XI], we conclude that $m \in \text{cd}(G)$ and $\gcd(|G|/m, m) > 1$, a contradiction. Therefore $n = 1$, $q$ is even, and $G \cong L_2(2^f)$.

- For type $^2A_n(q^2)$, $n \geq 2$.

  The unipotent character $\psi^{(1,n)}$ has degree $m = q(q^n - (-1)^n)/(q + 1)$, where $q = p^f$ for some prime $p$,

  $$|G| = q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1} - (-1)^{i+1})/\gcd(n + 1, q + 1).$$

  So $\gcd(|G|/m, m) > 1$ for $n \geq 2$, a contradiction.

- For type $B_n(q)$ ($n \geq 2$) or $C_n(q)$ ($n \geq 3$).

  The unipotent character $\psi^\alpha$ corresponding to the symbol $\alpha = (1^n)$ has degree $m = (q^n - q)(q^n + 1)/2(q - 1)$,

  $$|G| = q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)/\gcd(2, q - 1).$$

  So $\gcd(|G|/m, m) > 1$ for $n \geq 2$ except for $B_2(2)$, but $B_2(2) \cong S_6$ is not a simple group, a contradiction.

- For type $D_n(q)$.

  Corresponding to the symbol $\alpha_1 = (1^n)$, the unipotent character $\psi^{\alpha_1}$ of $D_n(q)$ has degree $m_1 = q^2(q^n - 1)/(q^2 - 1)$,

  $$|G| = q^{n(n-1)}(q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1)/\gcd(4, q^n - 1).$$

  So $\gcd(|G|/m_1, m_1) > 1$ for $n \geq 4$, a contradiction.

- For $^2D_n(q^2)$ ($n \geq 4$).

  Corresponding to the symbol $\alpha_2 = (1^{n-1})$, the unipotent character $\psi^{\alpha_2}$ of $^2D_n(q^2)$ has degree $m_2 = q(q^{n-2} - 1)(q^n + 1)/(q^2 - 1)$,

  $$|G| = q^{n(n-1)}(q^n + 1) \prod_{i=1}^{n-1} (q^{2i} - 1)/\gcd(4, q^n + 1).$$

  So $\gcd(|G|/m_2, m_2) > 1$ for $n \geq 4$, a contradiction.
Now suppose that $G$ is of exceptional type. Checking the unipotent characters of $G$ listed on [1, pp. 477–490], we can easily conclude that there is some unipotent character $\psi^\alpha$ of $G$ such that $\gcd(|G|/\psi^\alpha(1), \psi^\alpha(1)) > 1$, a contradiction. □

**Lemma 3.3.** Let $G$ be a non-abelian simple group, and every irreducible character degree of $G$ is a Hall-number for $G$. Then $G \cong L_2(2^f)$ where $f \geq 2$.

**Proof.** If $G$ is one of the sporadic simple groups, then by [2] there is $m \in \mathrm{cd}(G)$ such that $\gcd(|G|/m, m) > 1$, a contradiction. Thus by Lemmas 3.1 and 3.2, we can conclude that $G \cong L_2(2^f)$ where $f \geq 2$. □

**Lemma 3.4.** Let $S = L_2(2^f)$, and $S \leq G \leq \Aut(S)$. If $p$ is an odd prime divisor of $|G:S|$, then there is a character $\chi \in \Irr(S)$ such that $\chi(1) = 2^f + 1$, and its stabilizer $H$ in $G$ satisfies $|H:S| = p$.

**Proof.** Let $v$ be an element of order $2^f - 1$ in the field $F$ with $2^f$ elements. By [3] we know that we can associate the characters $\chi_i$ of degree $2^f + 1$ for $1 \leq i \leq 2^f - 1 - 1$ with $\{v^i, v^{-i}\}$. We can identify the outer automorphisms of $S$ with the Galois automorphisms of $F$. Let $\phi$ be the Frobenius automorphism of $F$, and recall that the outer automorphisms of $S$ are generated by $\phi$. Hence, we can view the action of $\phi$ on the $\chi_i$’s where $\chi_i$ is mapped to $\chi_{2i^*}$ where $2i^*$ is the unique integer between 1 and $2^f - 1 - 1$ that is congruent to $\pm 2i$ modulo $2^f - 1$. Then $H$ is generated by $S$ and an element $h$ where the action of $h$ on $S$ corresponds to the action of $\phi^{f/p}$. Take $j = (2^f - 1)(2^f/p - 1)$, and we claim that $v^j$ generates the subfield of $F$ fixed by $\phi^{f/p}$. Then one can show that $H$ will be the stabilizer of $\chi_j$ in $G$ as desired. □

**Proof of Theorem 1.2.** ($\Leftarrow$) Consider $m \in \mathrm{cd}(G)$ and $\chi \in \Irr(G)$ so that $\chi(1) = m$. Let $\theta$ be an irreducible consistent of $\chi_M$. By Lemma 3.3, we know that all the degrees in $\mathrm{cd}(L)$ are Hall-numbers. Combining this with the facts that all degrees in $\mathrm{cd}(N)$ are Hall-numbers and $\gcd(|N|, |L|) = 1$, it follows that all degrees in $\mathrm{cd}(M)$ and particularly $\theta(1)$ are Hall-numbers for $M$. Let $T$ be the stabilizer of $\theta$ in $G$. Since $M$ is a Hall subgroup of $G$, it follows that $\theta$ will be extendible to $T$. Since $G/M$ is cyclic, we determine that $\chi(1) = |G:T|\theta(1)$, and as $|G:M|$ is square-free, we conclude that $m = \chi(1)$ is a Hall-number for $G$.

($\Rightarrow$) Conversely, suppose that $G$ is non-solvable and every degree in $\mathrm{cd}(G)$ is a Hall-number. We take $N$ normal in $G$ and maximal with respect to $G/N$ is non-solvable. Then $G/N$ possesses a unique minimal normal subgroup $M/N$. By Lemma 2.2, every degree in $\mathrm{cd}(N)$ is a Hall-number.

**Step 1.** $M/N \cong L_2(2^f)$ where $f \geq 2$.

Let $M/N = M_1/N \times \cdots \times M_s/N$ where all of the subgroups $M_i/N$ are isomorphic to some non-abelian simple group $S$. By Lemma 2.2, every degree in $\mathrm{cd}(M_i/N)$ is a Hall-number for $M_i/N$. Applying Lemma 2.2 again, every degree in $\mathrm{cd}(M_1/N) = \mathrm{cd}(S)$ is a Hall-number for $S$, and by Lemma 3.3, $S \cong L_2(2^f)$ where $f \geq 2$. If $s > 1$ and $\phi \in \Irr(M_1/N)$ is any non-principal character, then $\phi \times 1 \times \cdots \times 1 \in \Irr(M/N)$ and $\phi \times 1 \times \cdots \times 1(1) = \phi(1)$ is not a Hall-number for $M/N$. Therefore, we must have $s = 1$ and $M/N \cong S \cong L_2(2^f)$.

**Step 2.** If $A$ and $B$ are normal in $G$ such that $A/B \cong L_2(2^f)$, then $\gcd(|B|, |L_2(2^f)|) = 1$. 
By Lemma 2.2, we know that every degree in cd(A) is a Hall-number for A. Since cd(L_2(2^f)) = \{1, 2^f + 1, 2^f, 2^f - 1\} and |A : B| = |L_2(2^f)| = (2^f + 1)2^f(2^f - 1), it follows that |B| and |L_2(2^f)| are relatively prime.

**Step 3.** If A and B are normal in G such that A/B \cong L_2(2^f), then A = B \times L, where L \cong L_2(2^f).

Notice that by Step 2, B is a normal Hall subgroup of A, so by the Schur–Zassenhaus theorem, we know that B has a complement L in A. We work to prove that L is normal in A. We work by induction on |B|. If B = 1, then the result is trivial. Thus, we may assume that B > 1.

Suppose now that B is a minimal normal subgroup of G. Let P be a Sylow 2-subgroup of A. We know that 2 does not divide |B|, so P is isomorphic to a Sylow 2-subgroup of A/B \cong L_2(2^f). This implies that P is an elementary abelian 2-group of order 2^f. Since B is solvable and minimal normal, it must be abelian. Now, P acts coprimely on B, so by Fitting’s lemma, we have B = CB(P) \times [B, P].

Suppose B > CB(P). Now, as f ≥ 2, we see that P is not cyclic, and so, [B, P]P is not a Frobenius group. Thus, there is a character \( \lambda \in \text{Irr}([B, P]) \) so that \( CP(\lambda) > 1 \). Let \( v = \lambda \times 1_{CB(P)} \), and write T for the stabilizer of v in A. Observe that \( CP(\lambda)B \leq T \). Since B is a Hall subgroup of T, we see that v is extendible to \( \psi \in \text{Irr}(T) \), and \( \psi^A \in \text{Irr}(A) \). We have \( \psi^A(1) = |A:T|v(1) = |A:T| \). Since \( \psi^A(1) \) is a Hall-number, we determine that T is a Hall subgroup of M.

Because \( CP(\lambda) \leq T \), we see that 2 divides |T|, and so, T contains a Sylow 2-subgroup \( P_1 \) of A with \( CP(\lambda) \leq P_1 \). This implies that \( P_1B/B \cap P\ B/B > 1 \), and so \( PB = P_1B \) by [4, Satz III, Theorem 8.2]. Hence \( P \leq T \), and thus, \( \lambda \) is invariant in T. This implies that \( C_{BP}(P) > 1 \), which contradicts Fitting’s lemma.

We now have B = CB(P). This implies that \( BP \leq CA(B) \), and thus, B < CA(B). Since A/B is simple and CA(B) is normal in A, we deduce that \( A = CA(B) \). Since B is now central in A, this proves that L is normal in this case.

We now assume that B is not a minimal normal subgroup of G. Let E be a minimal normal subgroup of G contained in B. Notice that G/E satisfies the hypotheses of the theorem. Hence, we may apply the inductive hypothesis to obtain \( A/E \cong B/E \times LE/E \). Now, LE and E are normal subgroup of G, E is minimal normal in G, and LE/E \cong L \cong L_2(2^f). Thus, we can use the previous argument to see that \( LE = L \times E \). Since L is a Hall subgroup of LE, this implies that L is normal in A, and so A = B \times L as desired.

**Step 4.** The last step.

Applying Steps 2 and 3 to M and N, we deduce that M = N \times L where L \cong L_2(2^f) and gcd(|N|, |L|) = 1. This implies that L is a normal subgroup of G. Let C = CG(L), and observe that \( C \cap L = 1 \) and \( N \leq C \). On the other hand, \( CL/C \cong L \), so G/C is non-solvable. By the maximality of N, we have N = C. This implies that G/N is isomorphic to a subgroup of the automorphism group of M/N. Since the outer automorphism group of L is cyclic of order f, we conclude that G/M is cyclic and \( |G:M| \) divides f.

We can find a character of degree \( 2^f - 1 \) in \( \text{Irr}(M/N) \) that induces irreducibly to G (see [9, Theorem 2.7 and Remark 2.8]). This implies that \( |G:M|(2^f - 1) \) is in cd(G), and hence, a Hall-number for G. It follows that \( |G:M| \) is relatively prime to \( |N|, 2^f, \) and \( 2^f + 1 \), and in particular, \( |G:M| \) is odd. If \( f = 2 \), then this implies that \( G = M \), and we are done in this case. Thus, we
may assume that \( f \geq 3 \). With \( f \geq 3 \), we can find a character of degree \( 2^f + 1 \) in \( \text{Irr}(M/N) \) that induces irreducibly to \( G/N \) (see [9, Theorem 27]). Hence, \(|G:M|(2^f + 1)\) is a Hall-number for \( G \). We deduce that \(|G:M|\) is relatively prime to \( 2^f - 1 \). Now, \( M \) is a Hall subgroup of \( G \).

We now work to show that \(|G:M|\) is square-free. Let \( p \) be a prime divisor of \(|G:M|\) (and note by the previous paragraph that \( p \) must be odd). There is a unique subgroup \( K \supset M \) such that \(|K:M| = p\). We can find a character \( \theta \in \text{Irr}(M/N) \) (with \( \theta(1) = 2^f - 1 \) or \( \theta(1) = 2^f + 1 \)) such that \( K \) is a stabilizer of \( \theta \) in \( G \) (see Lemma 3.4). Since \( M \) is a Hall subgroup, \( \theta \) is extendible to \( K \), and so \(|G:K|\theta(1) \in \text{Irr}(G)\). Now, \(|G:K|\theta(1)\) is a Hall-number for \( G \). Because \( p \) divides \(|K|\), we deduce that \( p \) does not divide \(|G:K|\), and hence, \( p^2 \) does not divide \(|G:M|\). We conclude that \(|G:M|\) is square-free. \( \square \)

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