Controllability Result for Nonlinear Evolution Integrodifferential Systems

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Abstract—in this paper, we establish sufficient conditions for the controllability of nonlinear evolution integrodifferential systems in a Banach space. The results are obtained by using the resolvent operators and the Schaefer fixed-point theorem. An example is provided to illustrate the theory. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Controllability of nonlinear systems represented by ordinary differential equations in infinite-dimensional spaces has been extensively studied by several authors [1-5]. Most of the controllability results for nonlinear infinite-dimensional control systems concern the so-called semilinear control system that consists of a linear part and nonlinear part. Klamka [6] discussed the controllability of nonlinear systems by using the Schauder fixed-point theorem. Constrained controllability of nonlinear systems in abstract spaces has been studied by Papageorgiou [7] and Bian [8]. Zhang [9] established the local exact controllability for semilinear evolution systems by means of the contraction mapping principle. Chukwu and Lenhart [10] discussed the constrained exact controllability for nonlinear time-independent dynamical system in a Banach space. Naito [11] established the controllability for nonlinear Volterra integrodifferential systems. Balasubramaniam et al. [12], discussed the local controllability of functional integrodifferential systems in Banach spaces by using the fractional powers of operators and Banach fixed-point theorem. Recently, Dauer and Mahmudov [13] studied the approximate and complete controllability for semilinear functional differential systems in Hilbert spaces by using the Banach and Schauder fixed-point...
theorem. The purpose of this paper is to study the controllability of nonlinear evolution integro-differential systems in Banach spaces by using the resolvent operators and the Schaefer fixed-point theorem. The nonlinear evolution integro-differential systems with resolvent operators considered here serves as an abstract formulation of partial integro-differential equations which arises in heat flow in material with memory and many other physical phenomena [14–16].

2. PRELIMINARIES

Consider the nonlinear evolution integro-differential system of the form

\[
\frac{d}{dt}[x(t) - g(t, x(t))] = A(t)x(t) + \int_0^t B(t, s)x(s) \, ds + (Gu)(t) + f(t, x(t)), \quad t \in J = [0, b],
\]

\[x(0) = x_0,
\]

where the state \(x(\cdot)\) takes values in a Banach space \(X\) with the norm \(\| \cdot \|\), and the control function \(u(\cdot)\) is given in \(L^2(J, U)\), a Banach space of admissible control functions with \(U\) as a Banach space. Here, \(A(t)\) and \(B(t, s)\) are closed linear operators on \(X\) with dense domain \(D(A)\) which is independent of \(t\), \(G\) is a bounded linear operator from \(U\) into \(X\), \(g : J \times X \rightarrow X\) and \(f : J \times X \rightarrow X\) are continuous functions.

In this section, we collect some basic results about resolvent operators [14]. We shall make the following assumptions

(I) \(A(t)\) generates a strongly continuous semigroup of evolution operators in the Banach space \(X\).

(II) Suppose \(Y\) is the Banach space formed from \(D(A)\) with the graph norm. \(A(t)\) and \(B(t, s)\) are closed operators. It follows that \(A(t)\) and \(B(t, s)\) are in the set of bounded operators from \(Y\) to \(X\), \(B(Y, X)\), for \(0 < t < b\) and \(0 \leq s \leq t \leq b\), respectively. Further, \(A(t)\) and \(B(t, s)\) are continuous on \(0 < t < b\) and \(0 \leq s \leq t \leq b\), respectively, into \(B(Y, X)\).

**DEFINITION 1.** A resolvent operator for (1) is a bounded operator valued function \(R(t, s) \in B(X), 0 \leq s \leq t \leq b\), the space of bounded linear operators on \(X\), having the following properties.

(a) \(R(t, s)\) is strongly continuous in \(s\) and \(t\), \(R(s, s) = I, 0 \leq s \leq b, \|R(t, s)\| \leq Me^{\beta(t-s)}\) for some constants \(M\) and \(\beta\).

(b) \(R(t, s)Y \subset Y, R(t, s)\) is strongly continuous in \(s\) and \(t\) on \(Y\).

(c) For each \(x \in D(A)\), \(R(t, s)x\) is strongly continuously differentiable in \(t\) and \(s\) and

\[
\frac{\partial R}{\partial t}(t, s)x = A(t)R(t, s)x + \int_s^t B(t, r)R(r, s)x \, dr,
\]

\[
\frac{\partial R}{\partial s}(t, s)x = -R(t, s)A(s)x - \int_s^t R(t, r)B(r, s)x \, dr,
\]

with \(\frac{\partial R}{\partial t}(t, s)x\) and \(\frac{\partial R}{\partial s}(t, s)x\) strongly continuous on \(0 \leq s \leq t \leq b\). Here, \(R(t, s)\) can be extracted from the evolution operator of the generator \(A(t)\). The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Banach space. A number of results follow directly from the definition of the resolvent operator.

**DEFINITION 2.** A function \(C([0, b], X)\) is a mild solution of problem (1) if the following holds

\[x(0) = x_0; \text{ for each } 0 \leq t < b \text{ and } s \in [0, t), \text{ the function } A(s)R(t, s)g(s, x(s)) \text{ is integrable and the integral equation}
\]

\[x(t) = R(t, 0) [x_0 - g(0, x_0)] + g(t, x(t)) + \int_0^t R(t, s)A(s)g(s, x(s)) \, ds
\]

\[+ \int_0^t R(t, s) \int_0^s B(s, \tau)g(\tau, x(\tau)) \, d\tau \, ds + \int_0^t R(t, s) [(Gu)(s) + f(s, x(s))] \, ds
\]

is satisfied.
SCHAEFER'S THEOREM. (See [17].) Let $E$ be a normed linear space. Let $F : E \to E$ be a completely continuous operator, i.e., it is continuous and the image of any bounded set is contained in a compact set, and let

$$\zeta(F) = \{x \in E; x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then, either $\zeta(F)$ is unbounded or $F$ has a fixed point.

DEFINITION 3. System (1) is said to be controllable on the interval $J$ if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(t)$ of (1) satisfies $x(b) = x_1$.

Further, we assume the following hypotheses.

(i) The resolvent operator $R(t, s)$ is compact and there exist constants $M_i > 0$, $i = 1, 2, 3$ such that $\|R(t, s)\| \leq M_1$, $\|R(t, s)A(s)\| \leq M_2$, and $\|B(t, s)\| \leq M_3$.

(ii) The linear operator $W : L^2(J, U) \to X$, defined by

$$Wu = \int_0^b R(b, s)Gu(s) \, ds,$$

has an induced inverse operator $\bar{W}^{-1}$, which takes values in $L^2(J, U)/\ker W$, and there exist positive constants $M_4, M_5$ such that $\|G\| \leq M_4$ and $\|\bar{W}^{-1}\| \leq M_5$ (see [18]).

(iii) The function $g : J \times X \to X$ is completely continuous and for any bounded set $D$ in $C([0, b], X)$ the set $\{t \to g(t, x(t)) : x \in D\}$ is equicontinuous in $C([0, b], X)$ and there exists a constant $L > 0$ such that

$$\|g(t, x)\| \leq L, \quad t \in J, \quad x \in X.$$

(iv) For each $t \in J$, the function $f(t, \cdot) : X \to X$ is continuous and for each $x \in X$, the function $f(\cdot, x) : J \to X$ is strongly measurable.

(v) For every positive integer $k$, there exists $\alpha_k \in L^1(0, b)$ such that for a.e. $t \in J$

$$\sup_{\|x\| \leq k} \|f(t, x)\| \leq \alpha_k(t).$$

(vi) There exists an integrable function $m : J \to [0, \infty)$ such that

$$\|f(t, x)\| \leq m(t)\Omega(\|x\|), \quad t \in J, \quad x \in X,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(vii)

$$M_1 \int_0^b m(s) \, ds < \int_c^\infty \frac{ds}{\Omega(s)},$$

where $c = M_1 (\|x_0\| + L) + L + M_2Lb + M_1M_3Lb^2 + M_1Nb$ and

$$N = M_4M_5 \left[ \|x_1\| + M_1 (\|x_0\| + L) + L + M_2Lb + M_1M_3Lb^2 + M_1 \int_0^b m(s)\Omega(\|x(s)\|) \, ds \right].$$

Then, system (1) has a mild solution of the following form

$$x(t) = R(t, 0)[x_0 - g(0, x_0)] + g(t, x(t)) + \int_0^t R(t, s)A(s)g(s, x(s)) \, ds$$

$$+ \int_0^t R(t, s)B(s, \tau)g(\tau, x(\tau)) \, d\tau + \int_0^t R(t, s)[(Gu)(s) + f(s, x(s))] \, ds.$$  \hspace{1cm} (3)
In order to study the controllability problem of \( (1) \), we introduce a parameter \( \lambda \in (0, 1) \) and consider the following system

\[
\frac{d}{dt} [x(t) - \lambda g(t, x(t))] = A(t)x(t) + \lambda \int_0^t B(t, s)x(s)\,ds + \lambda (Gu)(t) + \lambda f(t, x(t)), \quad t \in J, \tag{4}
\]

\( x(0) = \lambda x_0, \)

Then, for system \( (4) \), there exists a mild solution of the following form

\[
x(t) = \lambda R(t, 0) [x_0 - g(0, x_0)] + \lambda g(t, x(t)) + \lambda \int_0^t R(t, s)A(s)g(s, x(s))\,ds
+ \lambda \int_0^t R(t, s) \int_0^s B(s, \tau)g(\tau, x(\tau))\,d\tau\,ds + \lambda \int_0^t R(t, s) [(Gu)(s) + f(s, x(s))]\,ds.
\]

### 3. CONTROLLABILITY RESULT

**THEOREM.** If Hypotheses (i)-(vii) are satisfied, then system \( (1) \) is controllable on \( J \).

**PROOF.** Consider the space \( C = C(J, X) \), the Banach space of all continuous functions from \( J \) into \( X \) with sup norm.

Using Hypothesis (ii) for an arbitrary function \( x(\cdot) \), define the control

\[
u(t) = W^{-1} \left[ x_1 - R(b, 0) [x_0 - g(0, x_0)] - g(b, x(b)) - \int_0^b R(b, s)A(s)g(s, x(s))\,ds
- \int_0^b R(b, s) \int_0^s B(s, \tau)g(\tau, x(\tau))\,d\tau\,ds - \int_0^b R(b, s)f(s, x(s))\,ds \right] (t).
\]

We shall now show that when using this control the operator \( F : C \rightarrow C \) defined by

\[
(Fx)(t) = R(t, 0) [x_0 - g(0, x_0)] + g(t, x(t)) + \int_0^t R(t, s)A(s)g(s, x(s))\,ds
+ \int_0^t R(t, s) \int_0^s B(s, \tau)g(\tau, x(\tau))\,d\tau\,ds + \int_0^t R(t, s) [(Gu)(s) + f(s, x(s))]\,ds
\]

has a fixed point. This fixed point is then a solution of equation \( (3) \).

Clearly, \( x(b) = x_1 \), which means that the control \( u \) steers the integrodifferential system \( (1) \) from the initial state \( x_0 \) to \( x_1 \) in time \( b \), provided we can obtain a fixed point of the nonlinear operator \( F \).

First, we obtain a priori bounds for the following equation

\[
x(t) = \lambda R(t, 0) [x_0 - g(0, x_0)] + \lambda g(t, x(t)) + \lambda \int_0^t R(t, s)A(s)g(s, x(s))\,ds
+ \lambda \int_0^t R(t, s) \int_0^s B(s, \tau)g(\tau, x(\tau))\,d\tau\,ds + \lambda \int_0^t R(t, \eta)GW^{-1}
\times \left[ x_1 - R(b, 0)(x_0 - g(0, x_0)) - g(b, x(b)) - \int_0^b R(b, s)A(s)g(s, x(s))\,ds
- \int_0^b R(b, s) \int_0^s B(s, \tau)g(\tau, x(\tau))\,d\tau\,ds - \int_0^b R(b, s)f(s, x(s))\,ds \right] (\eta)\,d\eta
+ \lambda \int_0^t R(t, s)f(s, x(s))\,ds.
\]
We have, from the assumptions,

\[
\|x(t)\| \leq M_1 (\|x_0\| + L) + L + M_2 L b + M_1 M_3 L b^2 + \int_0^t \|R(t, \eta)\| M_4 M_5 \\
\times \left[\|x_1\| + M_1 (\|x_0\| + L) + L + M_2 L b + M_1 M_3 L b^2 + M_1 \int_0^b m(\eta) \Omega (\|x(\eta)\|) \, d\eta \right] \, d\eta \\
+ M_1 \int_0^t m(s) \Omega (\|x(s)\|) \, ds \\
\leq M_1 (\|x_0\| + L) + L + M_2 L b + M_1 M_3 L b^2 + M_1 N b + M_1 \int_0^t m(s) \Omega (\|x(s)\|) \, ds.
\]

Denoting by \(v(t)\) the right-hand side of the above inequality, we have \(c = v(0) = M_1 (\|x_0\| + L) + L + M_2 L b + M_1 M_3 L b^2 + M_1 N b\), \(\|x(t)\| \leq v(t)\), and

\[
v' (t) = M_1 m(t) \Omega (\|x(t)\|) \\
\leq M_1 m(t) \Omega (v(t)).
\]

This implies

\[
\int_{0}^{v(t)} \frac{ds}{\Omega(s)} \leq M_1 \int_{0}^{b} m(s) \, ds < \int_{c}^{\infty} \frac{ds}{\Omega(s)}, \quad t \in J.
\]

This inequality implies that \(v(t) < \infty\). So, there is a constant \(K\) such that \(v(t) \leq K, \ t \in J\), and hence, \(\|x(t)\| \leq K, \ t \in J\), where \(K\) depends only on \(b\) and on the functions \(m\) and \(\Omega\).

Second, we must prove that the operator \(F : C \to C\) defined by

\[
(Fx)(t) = R(t, 0) [x_0 - g(0, x_0)] + g(t, x(t)) + \int_0^t R(t, s) A(s) g(s, x(s)) \, ds \\
+ \int_0^t R(t, s) \int_0^b B(s, \tau) g(\tau, x(\tau)) \, d\tau \, ds + \int_0^t R(t, \eta) G W^{-1} \\
\times \left[x_1 - R(b, 0)(x_0 - g(0, x_0)) - g(b, x(b)) - \int_0^b R(b, s) A(s) g(s, x(s)) \, ds \right] \Omega(\eta) \, d\eta \\
- \int_0^b R(b, s) \int_0^b B(s, \tau) g(\tau, x(\tau)) \, d\tau \, ds - \int_0^b R(b, s) f(s, x(s)) \, ds \\
+ \int_0^t R(t, s) f(s, x(s)) \, ds
\]

is a completely continuous operator.

Let \(B_k = \{x \in C, \ \|x\| \leq k\}\) for some \(k \geq 1\). We first show that \(F\) maps \(B_k\) into an equicontinuous family. Let \(x \in B_k\) and \(t_1, t_2 \in J\). Then, if \(0 < t_1 < t_2 \leq b\),

\[
\| (F x)(t_1) - (F x)(t_2) \| \\
\leq \| R(t_1, 0) - R(t_2, 0)\| \|x_0 - g(0, x_0)\| + \|g(t_1, x(t_1)) - g(t_2, x(t_2))\| \\
+ \int_0^{t_1} \|R(t_1, s) - R(t_2, s)\| A(s) g(s, x(s)) \, ds \| + \int_{t_1}^{t_2} R(t_2, s) A(s) g(s, x(s)) \, ds \|
\]
The right-hand side tends to zero as $t_2 - t_1 \to 0$, since $g$ is completely continuous and the compactness of $R(t, s)$ for $t, s > 0$ implies the continuity in the uniform operator topology.

Thus, $F$ maps $B_k$ into an equicontinuous family of functions. Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple.

It is easy to see that the family $FB_k$ is uniformly bounded. Next, we show $FB_k$ is compact. Since we have shown $FB_k$ is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that $F$ maps $B_k$ into a precompact set in $X$. Let $0 < t < s < b$ be fixed and $\epsilon$ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$(F_\epsilon x)(t) = R(t, 0) [x_0 - g(0, x_0)] + g(t, x(t)) + \int_0^{t-\epsilon} R(t, s)A(s)g(s, x(s)) \, ds$$
\[
\begin{align*}
&+ \int_0^{t-\epsilon} R(t,s) \int_0^s B(s,\tau) g(\tau, x(\tau)) \, d\tau \, ds + \int_0^{t-\epsilon} R(t,\eta) G \tilde{W}^{-1} \\
\times &\left[ x_1 - R(b,0) \left(x_0 - g(0,x_0)\right) - g(b,x(b)) - \int_0^b R(b,s) A(s) g(s,x(s)) \, ds \\
- &\int_0^b R(b,s) \int_0^s B(s,\tau) g(\tau, x(\tau)) \, d\tau \, ds - \int_0^b R(b,s) f(s,x(s)) \, ds \right] (\eta) \, d\eta \\
&+ \int_0^{t-\epsilon} R(t,s) f(s,x(s)) \, ds.
\end{align*}
\]

Since \( R(t,s) \) is a compact operator, the set \( Y_\epsilon(t) = \{(F,x)(t) : x \in B_k\} \) is precompact in \( X \) for every \( \epsilon, 0 < \epsilon < t \). Moreover for every \( x \in B_k \) we have

\[
\| (F,x)(t) - (F,x)(t) \|
\leq \int_{t-\epsilon}^{t} \| R(t,s) A(s) g(s,x(s)) \| \, ds + \int_{t-\epsilon}^{t} \| R(t,s) \int_0^s B(s,\tau) g(\tau, x(\tau)) \| \, d\tau \, ds
\]

\[
+ \int_{t-\epsilon}^{t} \| R(t,\eta) G \tilde{W}^{-1} \left[ x_1 - R(b,0) \left(x_0 - g(0,x_0)\right) - g(b,x(b)) \\
- &\int_0^b R(b,s) A(s) g(s,x(s)) \, ds - \int_0^b R(b,s) \int_0^s B(s,\tau) g(\tau, x(\tau)) \, d\tau \, ds \\
- &\int_0^b R(b,s) f(s,x(s)) \, ds \right] (\eta) \| \, d\eta
\]

\[
+ \int_{t-\epsilon}^{t} \| R(t,\eta) M_4 M_5 \left[ \| x_1 \| + M_1 (\| x_0 \| + L) + L + M_2 Lb \\
+ &M_1 M_3 Lb^2 + M_1 \int_0^b \alpha_k(s) \, ds \right] \| \| R(t,\eta) \| \, \alpha_k(s) \, ds \| d\eta
\]

Therefore, there are precompact sets arbitrarily close to the set \( \{(F,x)(t) : x \in B_k\} \). Hence, the set \( \{(F,x)(t) : x \in B_k\} \) is precompact in \( X \).

It remains to show that \( F : C \to C \) is continuous. Let \( \{x_n\}_{n=0}^\infty \subseteq C \) with \( x_n \to x \) in \( C \). Then, there is an integer \( r \), such that \( \|x_n(t)\| \leq r \), for all \( n \) and \( t \in J \), so \( x_n \in B_r \) and \( x \in B_r \). By (iv) \( f(t,x_n(t)) \to f(t,x(t)) \), for each \( t \in J \), and since \( \|f(t,x_n(t)) - f(t,x(t))\| \leq 2\alpha_r(t) \) and also \( g \) is completely continuous, we have by dominated convergence theorem

\[
\| Fx_n -Fx \|
\leq \sup_{t \in J} \| g(t,x_n(t)) - g(t,x(t)) \| + \int_0^t R(t,s) A(s) \| g(s,x_n(s)) - g(s,x(s)) \| \, ds
\]

\[
+ \int_0^s R(t,s) \int_0^s B(s,\tau) [g(\tau, x_n(\tau)) - g(\tau, x(\tau))] \, d\tau \, ds
\]

\[
+ \int_0^b R(b,\eta) G \tilde{W}^{-1} \left[ \int_0^b R(b,s) A(s) [g(s,x_n(s)) - g(s,x(s))] \, ds \\
+ \int_0^b R(b,s) \int_0^s B(s,\tau) [g(\tau, x_n(\tau)) - g(\tau, x(\tau))] \, d\tau \, ds \\
+ \int_0^b R(b,s) [f(s,x_n(s)) - f(s,x(s))] \, ds \right] (\eta) \, d\eta
\]
Thus, \( F \) is continuous. This completes the proof that \( F \) is completely continuous.

Finally, the set \( \zeta(F) = \{ x \in C : x = \lambda Fx, \lambda \in (0, 1) \} \) is bounded, as we proved in the first step. Consequently, by Schaefer’s theorem the operator \( F \) has a fixed point in \( C \). This means that any fixed point of \( F \) is a mild solution of (1) on \( J \) satisfying \( (Fx)(t) = x(t) \). Thus, system (1) is controllable on \( J \).

4. EXAMPLE

Consider the following parabolic partial integrodifferential equation of the form

\[
\frac{\partial}{\partial t} z(t, y) + \mu_1(t, z(t, y)) = a(t, y) \frac{\partial^2}{\partial y^2} z(t, y) + \int_0^t b(t, s) z(s, y) \, ds
\]

\[
+ \mu_2(t, y) + \mu_2(t, z(t, y)),
\]

with \( z(t, 0) = z(t, 1) = 0, t \geq 0, \)

\[
z(0, y) = z_0(y), 0 < y < 1, t \in J = [0, 1],
\]

where \( a(t, y) \) and \( b(t, s) \) are continuous functions such that \( \|b(t, s)\| \leq k \). Let \( g(t, w)(y) = \mu_1(t, w(y)) \) and \( f(t, w)(y) = \mu_2(t, w(y)) \).

Take \( X = L^2(J) \) and define \( A(t) : X \to X \) by \( (A(t)w)(y) = a(t, y)w'' \) with domain \( D(A) = \{ w \in X : w, w' \text{ are absolutely continuous}, w'' \in X, w(0) = w(1) = 0 \} \), generates an evolution system and \( R(t, s) \) can be extracted from the evolution system \([14,16]\) such that \( \|R(t, s)\| \leq n_1 \) and \( \|R(t, s)A(s)\| \leq n_2 \).

Let \( Gu : Y \to X \) be defined by

\[
(Gu)(t)(y) = \mu(t, y), \ y \in (0, 1).
\]

With the choice of \( A(t), B(t, s), g \) and \( f \), the equation (5) can be written in the abstract form of (1). Assume that the linear operator \( W \) is given by

\[
(Wu)(y) = \int_0^1 R(1, s) \mu(s, y) \, ds, \quad y \in (0, 1),
\]

has a bounded invertible operator \( W^{-1} \) in \( L^2(J, U) / \ker W \).

The functions \( \mu_1 \) and \( \mu_2 \) satisfy the following conditions.

(i) The function \( \mu_1 : J \times X \to X \) is completely continuous and there exists a constant \( k_1 > 0 \), such that

\[
\|\mu_1(t, w)\| \leq k_1.
\]
(ii) There exists an integrable function $q : J \rightarrow [0, \infty)$, such that

$$\|\mu_2(t, w)\| \leq q(t) \Omega_1 (\|w\|),$$

where $\Omega_1 : [0, \infty) \to (0, \infty)$ is continuous and nondecreasing. Also, we have

$$n_1 \int_0^1 q(s) \, ds < \int_0^\infty \frac{ds}{\Omega_1 (s)},$$

where $c = n_1 (\|z_0\| + k_1) + k_1 + n_2 k_1 + n_1 k_1 + n_1 N$. Here, $N$ depends on $\mu_1$ and $\mu_2$. Further, all the conditions stated in the above theorem are satisfied. Hence, system (5) is controllable on $J$.

REFERENCES