Topological entropy of maps on the real line

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Abstract

The aim of this paper is to introduce a definition of topological entropy for continuous maps such that, at least for continuous real maps, it keeps the following general philosophy: positive topological entropy implies that the map has a complicated dynamical behaviour. Besides, we pursue that our definition keeps some properties which are hold by the classic definition of topological entropy introduced for compact sets.

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1. Introduction

Let X be a Hausdorff topological space and let f : X → X be a continuous self-map on X. The pair (X, f) is called a dynamical system. If n ∈ N, then fn := f ∘ fn−1, f1 := f and f0 is the identity on X. For x ∈ X, the sequence {fn(x): n ∈ N} is called the orbit of x, denoted by Orb f(x). If f is a homeomorphism, the full orbit of x is the sequence {fn(x): n ∈ Z}, which will be denoted by FullOrb f(x). A subset K ⊆ X is said to be invariant (by f) if f(K) ⊆ K and it is said to be strictly invariant if f(K) = K.

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When $X$ is compact, the notion of topological entropy was introduced in [1] by using open covers of $X$. If the set $X$ is not compact but it is metric, a new definition of topological entropy was introduced in [5] in the setting of uniformly continuous maps as follows. Let $K \subset X$ be a compact set. Define the $(n, \varepsilon, K, f)$-separated set if for all $x, y \in E$, $x \neq y$, there is $i \in \{0, 1, \ldots, n - 1\}$ such that $d(f^i(x), f^i(y)) > \varepsilon$, where $d$ is the distance on $X$. We denote by $s_n(\varepsilon, K, f)$ the cardinality of a maximal $(n, \varepsilon, K, f)$-separated subset of $K$. The topological entropy is

$$h(f) := \sup \left\{ \lim_{n \to \infty} \limsup_{i \to 0} \frac{1}{n} \log s_n(\varepsilon, K, f) : K \subset X \text{ compact} \right\}.$$ 

When $X$ is a metric space, the definitions in [1,5] are equivalent. In this case the topological entropy can be used as tool to check if the dynamical behaviour of $f$ is complicated (see for instance [18,3] or [12]).

However, when $X$ is not compact, the definition of [5] does not respect this premise. For example, we consider the uniformly continuous map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$, $x \in \mathbb{R}$ and suppose $\mathbb{R}$ endowed with the Euclidean metric. It is clear that its dynamic is simple: the point 0 is fixed and if $x \neq 0$, then $\lim_{n \to \infty} |f^n(x)| = \infty$. On the other hand, by [5], it follows that $h(f) \geq \log 2$. Let us remark that several definitions of topological entropy for non-compact topological spaces have been introduced in the literature (see [11, 6,10] or [14]).

The aim of this paper is to introduce a definition of topological entropy for continuous maps such that, at least for continuous real maps, it keeps the following general philosophy: positive topological entropy implies that the map has a complicated dynamical behaviour. Besides, we pursue that our definition keeps some properties which are hold by the classic definition of topological entropy introduced for compact sets, and that we have sum up in the next result (see [19,3,1,5,7]).

**Theorem 1.1.** Let $X$ and $Y$ be two (metric) compact sets and let $f : X \to X$ and $g : Y \to Y$ be two continuous maps. Then the following properties are held:

(a) Let $\varphi : X \to Y$ be continuous such that $g \circ \varphi = \varphi \circ f$. Then:
   (a1) If the map $\varphi$ is injective, then $h(f) \leq h(g)$.
   (a2) If the map $\varphi$ is surjective, then $h(g) \leq h(f)$.
   (a3) If map $\varphi$ is bijective, then $h(f) = h(g)$.

(b) Suppose that $X = \bigcup_{i=1}^{n} X_i$, where $X_i$ are compacts and invariants by $f$. Then $h(f) = \max \{h(f|X_i) : i = 1, 2, \ldots, n\}$.

(c) For any integer $n \geq 0$ it is hold $h(f^n) = nh(f)$.

(d) Let $f \times g : X \times Y \to X \times Y$ be defined by $(f \times g)(x, y) = (f(x), g(y))$ for all $(x, y) \in X \times Y$. Then $h(f \times g) = h(f) + h(g)$.

(e) If $f$ is a homeomorphism, then $h(f) = h(f^{-1})$.

(f) Let $\varphi : X \to Y$ be a continuous surjective map such that $\varphi \circ f = g \circ \varphi$. Then $\max \{h(g), \sup \{h(f, \varphi^{-1}(y)) : y \in Y\}\} \leq h(f) \leq h(g) + \sup \{h(f, \varphi^{-1}(y)) : y \in Y\}$.

(g) If $f : X \to Y$ and $g : Y \to X$ are continuous, then $h(f \circ g) = h(g \circ f)$.

(h) Let $f : X \to Y$, $g : Y \to X$ be continuous and let $F : X \times Y \to X \times Y$ be defined by $F(x, y) = (g(y), f(x))$ for all $(x, y) \in X \times Y$. Then $h(F) = h(f \circ g) = h(g \circ f)$.
(i) If $X_\infty = \bigcap_{n \geq 0} f^n(X)$, then $h(f) = h(f|_{X_\infty}).$

(j) $h(f) = h(f|_{\Omega(f)})$ where $x \in \Omega(f)$ if for all neighborhood $U$ of $x$ there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$ ($\Omega(f)$ is called non-wandering set of $f$).

The paper is organized as follows. Below we introduce our definition of topological entropy for non-compact topological spaces and study its general properties. Second, we study a variational principle for this notion. Finally, we study the particular case of topological entropy for mappings defined on the real line.

2. Definition and general properties

Let $(X, d)$ be a metric space and let $f$ be a continuous self-map of $X$. Our definition of topological entropy for $f$ is the next:

$$\text{ent}(f) := \sup \{ h(f|_K): K \subseteq X, \text{ compact and invariant by } f \}.$$  

By Theorem 1.1(i), we have that

$$\text{ent}(f) = \sup \{ h(f|_K): K \in \mathcal{K}(X, f) \},$$

where $\mathcal{K}(X, f)$ is the family of all the compacts subsets of $X$ which are strictly invariant by $f$. Notice that this definition makes sense when $X$ is metric or simply a topological space.

Besides Bowen’s definition [5], there are another definitions of topological entropy on non-compact spaces. One of them is due to Hofer [10] when the space is not necessarily metric. He gives two definitions of topological entropy for a map $f: X \rightarrow X$. One of them is making $h(f) = h(f^*)$, where $f^*$ is the extension of $f$ to the Stone–Čech compactification of $X$. The other definition is based in that done in [1] but considering only finite open covers of $X$. Both definitions agree when $X$ is normal. Example 2 shows that this definition is not good for our purposes.

Another definitions are based on dimension theory (see [6] or [16]). However, these definitions do not hold that for real maps simplicity implies zero topological entropy, as the example in page 126 from [6] shows.

Our definition is in some sense inspired in the definition established by Gurevic in [9] for the case of shift homeomorphisms. In this case, Gurevic’s entropy agrees with the definition of classical topological entropy on the extension of the shift homeomorphism to the Alexandroff compactification. This fact is not guaranteed for our definition (see Section 3).

We start by studying some properties of the new definition of topological entropy, which we sum up in the next result.

**Theorem 2.1.** Let $X$ and $Y$ be two metric spaces, and let $f: X \rightarrow X$ and $g: Y \rightarrow Y$ be two continuous maps. Then the following properties are held:

(a) Let $\varphi: X \rightarrow Y$ be continuous such that $g \circ \varphi = \varphi \circ f$. Then:

(a1) If $\varphi: X \rightarrow Y$ is injective, then $\text{ent}(f) \leq \text{ent}(g)$. 


(a2) If \( \varphi : X \to Y \) is surjective and such that the set \( \varphi^{-1}(K) \) is compact for all compact \( K \subseteq X \), then \( \text{ent}(f) \geq \text{ent}(g) \).

(a3) If \( \varphi : X \to Y \) is an homeomorphism, then \( \text{ent}(f) = \text{ent}(g) \).

(b) The formula \( \text{ent}(f) = \max\{h(f|_{X_i}) : \ i = 1,2,\ldots,n\} \), where \( X = \bigcup^n_{i=1} X_i \), and \( X_i \) are invariant subsets by \( f \) for \( i = 1,2,\ldots,n \), is not true in general.

(c) For all \( n \in \mathbb{N} \), \( \text{ent}(f^n) = n \cdot \text{ent}(f) \).

(d) Let \( f \times g : X \times Y \to X \times Y \) be such that \( (f \times g)(x,y) = (f(x),g(y)) \) for all \( (x,y) \in X \times Y \). Then \( \text{ent}(f \times g) = \text{ent}(f) + \text{ent}(g) \).

(e) If \( f \) is an homeomorphism, then \( \text{ent}(f) = \text{ent}(f^{-1}) \).

(f) Let \( \varphi : X \to Y \) be continuous, surjective, verifying that \( \varphi \circ f = g \circ \varphi \) and such that if \( K \) is compact in \( Y \) then \( \varphi^{-1}(K) \) is compact in \( X \). Then

\[
\max\{\text{ent}(g), \sup\{h(f, \varphi^{-1}(y)) : y \in \varphi(K), \ K \in \mathcal{K}(X,f)\}\}
\leq \text{ent}(f) \leq \text{ent}(g) + \sup\{h(f, \varphi^{-1}(y)) : y \in Y\}.
\]

(g) Let \( f : X \to Y \) and \( g : Y \to X \) be continuous. Then \( \text{ent}(g \circ f) = \text{ent}(f \circ g) \).

(h) Let \( f : X \to Y \) and \( g : Y \to X \) be continuous, and \( F : X \times Y \to X \times Y \) is defined by \( F(x,y) = (g(y), f(x)) \), \( (x,y) \in X \times Y \). Then \( \text{ent}(F) = \text{ent}(f \circ g) = \text{ent}(g \circ f) \).

(i) Let \( X_\infty = \bigcap_{n \geq 0} F^n(X) \). Then \( \text{ent}(f) = \text{ent}(f|_{X_\infty}) \).

(j) Let \( \Omega(f) \) be the non-wandering set of \( f \). Then \( \text{ent}(f) = \text{ent}(f|_{\Omega(f)}) \).

Proof. (a1) Let \( K \in \mathcal{K}(X,f) \). Then \( \varphi(K) \in \mathcal{K}(Y,g) \). By Theorem 1.1(a1), \( (f|_K) \leq h(g|_{\varphi(K)}) \). Then

\[
\text{ent}(f) = \sup\{h(f|_K) : K \in \mathcal{K}(X,f)\} \leq \sup\{h(g|_{\varphi(K)}) : K \in \mathcal{K}(X,f)\}
\leq \sup\{h(g|_L) : L \in \mathcal{K}(Y,g)\} = \text{ent}(g).
\]

(a2) Let \( K \in \mathcal{K}(Y,g) \). Then \( \varphi^{-1}(K) \in \mathcal{K}(X,f) \). As \( \varphi \) is surjective, by Theorem 1.1(a2), \( h(f|_{\varphi^{-1}(K)}) \geq h(g|_K) \). Then

\[
\text{ent}(f) = \sup\{h(f|_L) : L \in \mathcal{K}(X,f)\} \geq \sup\{h(f|_{\varphi^{-1}(K)}) : K \in \mathcal{K}(Y,g)\}
\geq \sup\{h(g|_K) : K \in \mathcal{K}(X,f)\} = \text{ent}(g).
\]

(a3) It is a consequence of (a1) and (a2).

(b) Let \( f : X \to X \) be a minimal homeomorphism defined on a compact set with positive topological entropy (see [17]). Let \( x \in X \) and we define \( X_1 := \text{FullOrb}_f(x) = \{f^n(x) : n \in \mathbb{Z}\} \) and \( X_2 := X \setminus X_1 \). It is clear that both set are invariant by \( f \) and that they haven not compact invariant subsets (in case of that they have compact invariant subsets, the map \( f \) is not minimal). Therefore \( \text{ent}(f|_{X_i}) = 0, i = 1,2 \), while \( \text{ent}(f) = h(f) > 0 \).

(c) Let \( K \in \mathcal{K}(X,f) \). Then \( K \in \mathcal{K}(X,f^n) \). By Theorem 1.1(c), \( h(f^n|_K) = h((f|_K)^n) = n \cdot h(f|_K) \). Then

\[
\text{ent}(f^n) = \sup\{h(f^n|_L) : L \in \mathcal{K}(X,f^n)\} \geq \sup\{h(f^n|_K) : K \in \mathcal{K}(X,f)\}
= n \cdot \sup\{h(f|_K) : K \in \mathcal{K}(X,f)\} = n \cdot \text{ent}(f).
\]

Now we prove the converse inequality. Let \( K \in \mathcal{K}(X,f^n), n \in \mathbb{N} \). The set \( \hat{K} = \bigcup_{i=0}^{n-1} f^i(K) \) is compact in \( X \) and such that \( f(\hat{K}) = f(\bigcup_{i=0}^{n-1} f^i(K)) = f(K) \cup f^2(K) \cup \cdots \cup f^{n-1}(K) \cup K = \hat{K} \). Hence \( \hat{K} \in \mathcal{K}(X,f) \). Then, by Theorem 1.1(b) and (c),
By Theorem 1.1(d),

\[ n \cdot h(f|\hat{K}) = h\left((f|\hat{K})^n\right) = h\left(f^n|\hat{K}\right) = \sup \{ h(f^n|f^{i}(K)) : i = 0, 1, \ldots, n - 1 \} \geq h\left(f^n|K\right), \]

then

\[ n \cdot \text{ent}(f) \geq n \cdot \sup \{ h(f|\hat{K}) : K \in \mathcal{K}(X, f^n) \} = \sup \{ h(f^n|\hat{K}) : K \in \mathcal{K}(X, f^n) \} = \text{ent}(f^n). \]

(d) Let \( K \in \mathcal{K}(X \times Y, f \times g) \) denote by \( \Pi_i \) the map \( \Pi_1(x, y) = x \) and \( \Pi_2(x, y) = y \). By Theorem 1.1(d),

\[ h(f|K) \leq h(f \times g|K_1 \times K_2) = h(f|K_1) + h(g|K_2), \]

where \( K_i = \Pi_i(K), i = 1, 2 \) (notice that \( K \subseteq K_1 \times K_2 \)). Then:

\[ \text{ent}(f \times g) = \sup \{ h(f \times g|K) : K \in \mathcal{K}(X \times Y, f \times g) \} \leq \sup \{ h(f \times g|K_1 \times K_2) : K_1 \times K_2 \in \mathcal{K}(X \times Y, f \times g) \} \leq \sup \{ h(f|K_1) : K_1 \in \mathcal{K}(X, f) \} + \sup \{ h(g|K_2) : K_2 \in \mathcal{K}(Y, g) \} = \text{ent}(f) + \text{ent}(g). \]

We now prove the converse inequality. Let \( K_1 \in \mathcal{K}(X, f) \) and \( K_2 \in \mathcal{K}(Y, g) \). By Theorem 1.1(d),

\[ h(f \times g|K_1 \times K_2) = h(f|K_1) + h(g|K_2). \]

Then:

\[ \text{ent}(f \times g) = \sup \{ h(f \times g|K) : K \in \mathcal{K}(X \times Y, f \times g) \} \geq \sup \{ h(f \times g|K_1 \times K_2) : K_1 \in \mathcal{K}(X, f) \text{ and } K_2 \in \mathcal{K}(Y, g) \} = \sup \{ h(f|K_1) : K_1 \in \mathcal{K}(X, f) \} + \sup \{ h(g|K_2) : K_2 \in \mathcal{K}(Y, g) \} = \text{ent}(f) + \text{ent}(g). \]

(e) Let \( K \in \mathcal{K}(X, f) \). It is clear that \( K \in \mathcal{K}(X, f^{-1}) \). By Theorem 1.1(e),

\[ h(f|K) = h(f^{-1}|K). \]

Hence

\[ \text{ent}(f) = \sup \{ h(f|K) : K \in \mathcal{K}(X, f) \} = \sup \{ h(f^{-1}|K) : K \in \mathcal{K}(X, f) \} = \text{ent}(f^{-1}). \]

(f) Let \( K \in \mathcal{K}(X, f) \). Then \( \varphi(K) \in \mathcal{K}(Y, g) \). By Theorem 1.1(f),

\[ h(f|K) \leq h(g_{\varphi(K)}) + \sup \{ h(g, \varphi^{-1}(y)) : y \in \varphi(K) \} \]

and then

\[ \text{ent}(f) = \sup \{ h(f|K) : K \in \mathcal{K}(X, f) \} \leq \sup \{ h(g_{\varphi(K)}) + \sup \{ h(g, \varphi^{-1}(y)) : y \in \varphi(K) \} \} \leq \sup \{ h(g|L) : L \in \mathcal{K}(Y, g) \} + \sup \{ h(g, \varphi^{-1}(y)) : y \in Y \} = \text{ent}(g) + \sup \{ h(g, \varphi^{-1}(y)) : y \in Y \}. \]

As the other inequality is immediate, we finish the proof.

(g) Let \( K \in \mathcal{K}(X, g \circ f) \). Then \( f(K) \in \mathcal{K}(Y, f \circ g) \). By Theorem 1.1(g),

\[ h(g \circ f|K) = h(f \circ g|f(K)). \]

Then
ent(g \circ f) = \sup\{ h(g \circ f|_K): K \in \mathcal{K}(X,g \circ f) \}
= \sup\{ h(f \circ g|_f(K)): K \in \mathcal{K}(X,g \circ f) \}
\leq \sup\{ h(f \circ g|_L): L \in \mathcal{K}(Y,f \circ g) \} = \text{ent}(f \circ g).

By a symmetrical reasoning, we prove that \( \text{ent}(f \circ g) \leq \text{ent}(g \circ f) \).

(h) As \( F^2(x,t) = ((g \circ f)(x), (f \circ g)(y)) \), then
\[
2 \cdot \text{ent}(F) = \text{ent}(F^2) = \text{ent}\left( (g \circ f) \times (f \circ g) \right)
= \text{ent}(g \circ f) + \text{ent}(f \circ g) = 2 \cdot \text{ent}(g \circ f) = 2 \cdot \text{ent}(f \circ g).
\]

(i) This equality is immediate because any \( K \in \mathcal{K}(X,f) \) is contained in \( X_\infty \).

(j) Let \( i: \Omega(f) \to X \) be defined by \( i(x) = x \). Notice that \( (f \circ i)(x) = (i \circ f|_{\Omega(f)})(x) \) for any \( x \in \Omega(f) \). Some \( i \) is injective, by the paragraph (a1), \( \text{ent}(f) \geq \text{ent}(f|_{\Omega(f)}) \).

Let \( K \in \mathcal{K}(X,f) \) and let \( \bar{K} = K \cap \Omega(f) \in \mathcal{K}(X,f|_{\Omega(f)}) \). Moreover, \( (f|_K)|_{\Omega(f)} = f|_{\bar{K}} = (f|_{\Omega(f)})|_K \). By Theorem 1.1(j), \( h(f|_K) = h(f|_{K \cap \Omega(f)}) \). Then:
\[
\text{ent}(f) = \sup\{ h(f|_K): K \in \mathcal{K}(X,f) \}
= \sup\{ h(f|_{\Omega(f) \cap K}): K \in \mathcal{K}(X,f) \}
\leq \sup\{ h(f|_L): L \in \mathcal{K}(X,f|_{\Omega(f)}) \} = \text{ent}(f|_{\Omega(f)}). \tag*{\Box}
\]

Remark 1. The following example due to Henk Bruin shows that Theorem 2.1(a2) it is not true in general if \( \varphi \) does not hold the condition that \( \varphi^{-1}(K) \) is compact for all \( K \) compact. Let \( f: \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 2x \) for all \( x \in \mathbb{R} \) and let \( g: S^1 \to S^1 \) defined also by \( g(x) = 2x \) for all \( x \in S^1 \). Then \( g \) is a factor of \( f \) and \( \text{ent}(f) = 0 \) while \( \text{ent}(g) = \log 2 \) (see [1, Chapter 4]). Some other examples of this fact can be seen in [8,15].

Remark 2. When \( X \) is compact (non-metric) topological space and \( f: X \to X \) is continuous, there is a definition of topological entropy of \( f \) by using open covers of \( X \) (see [1]). If \( X \) is not compact, we can define again \( \text{ent}(f) = \sup\{ h(f|_K): K \in \mathcal{K}(X,f) \} \). Then it can be proved that properties (a), (c)–(e) and (g)–(i) of Theorem 2.1 are also true in this setting, that is, for non-metrizable topological spaces.

3. A restricted variational principle and compactifications

Let \( f: X \to X \) be continuous with \( X \) metric. Let \( \mathcal{M}(X,f) \) be the set of all the probability measures \( \mu \) defined on the Borel \( \sigma \)-algebra of \( X \), \( \beta \), such that it holds \( \mu(A) = \mu(f^{-1}(A)) \) for all \( A \in \beta \). Let \( \mathcal{E}(X,f) \) be the set of the measures \( \mu \in \mathcal{M}(X,f) \) for which the condition \( f^{-1}(A) = A \) implies that \( \mu(A) = 0 \) or \( \mu(A) = 1 \). The variational principle for topological entropy (see [19, Chapter 8]) states that if \( X \) is compact, then \( h(f) = \sup\{ h_\mu(f): \mu \in \mathcal{E}(X,f) \} \), where \( h_\mu(f) \) is the metric entropy of \( f \) (see the definition in [19, Chapter 4]). If \( X \) is not compact, we can define \( h_V(f) = \sup\{ h_\mu(f): \mu \in \mathcal{E}(X,f) \} \) (see [11]). Then it is immediate that \( \text{ent}(f) \leq h_V(f) \). The following example shows that the inequality can be strict.
**Example 1.** Consider the example of Theorem 2.1(b) and the map $f |_{X_2} : X_2 \to X_2$. Since $X_1$ is a non-periodic orbit, the set $\mathcal{M}(X_1, f) = \emptyset$, and therefore $\mathcal{E}(X_1, f) = \emptyset$. Then $\sup \{ h_\mu(f |_{X_2}) : \mu \in \mathcal{E}(X_2, f |_{X_2}) \} > 0$. On the other hand $\text{ent}(f |_{X_2}) = 0$ because $\mathcal{K}(X_2, f |_{X_2}) = \emptyset$.

We denote by $\mathcal{B}(X, f)$ the set of invariant ergodic measures $\mu$ of $f$ such that $\text{supp } \mu \subseteq K$, for some $K \in \mathcal{K}(X, f)$, where $\text{supp } \mu$ denotes the smallest compact subset of full measure. Then

$$\text{ent}(f) = \sup \{ h(f |_K) : K \in \mathcal{K}(X, f) \}$$

$$= \sup_{K \in \mathcal{K}(X, f)} \sup \{ h_\mu(f) : \mu \in \mathcal{M}(K, f |_K) \}$$

$$= \sup \{ h_\mu(f) : \mu \in \mathcal{B}(X, f) \},$$

which provides a variational principle for the topological entropy of non-compact sets.

The variational principle is connected with the following question. Let $f : X \to X$ be a continuous map where $X$ is not compact. Assume that $X^*$ is a compactification of $X$ such that there is a continuous extension of $f$ on $X^*$, denoted by $f^*$. What is the relationship between $\text{ent}(f)$ and $h(f^*)$? In general it is clear that $h(f^*) \geq \text{ent}(f)$. We study what conditions provide the equality. The next example of [10] shows that this is not possible in general.

**Example 2.** Let $\mathbb{Z}$ be the set of the integer numbers and let $f : \mathbb{Z} \to \mathbb{Z}$ be the map defined by $f(n) = n + 1$ for all $n \in \mathbb{Z}$. Let $\mathbb{Z}^*$ and $f^*$ be the Stone–Čech compactification of $\mathbb{Z}$. Then we can see in [10] that $h(f^*) = \infty$. Since $\mathcal{K}(\mathbb{Z}, f) = \emptyset$, we get that $\text{ent}(f) = 0$.

It is important to point out that the map of the above example has a very simple dynamics. In fact, the map $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x + 1$, has a simple dynamics and it is clear that $f = F |_{\mathbb{Z}}$.

**Theorem 3.1.** Let $X$ be metric and let $f : X \to X$ be continuous such that $\mathcal{B}(X, f) = \mathcal{E}(X, f)$. Assume that there is a compactification $X^*$ which is metric and such that $X^* \setminus X$ is countable. Assume there is a continuous extension $f^* : X^* \to X^*$. Then $\text{ent}(f) = h(f^*)$.

**Proof.** The argument for the proof is similar to one of [11, Lemma 1.5]. As any ergodic measure of $f$ belongs to $\mathcal{B}(X, f)$ and $X^* \setminus X$ is at most countable, then the set $\mathcal{E}(X^*, f^*) \setminus \mathcal{E}(X, f)$ contains at most measures associated to periodic points. Thus, $h_\mu(f^*) = 0$ for all $\mu \in \mathcal{E}(X^*, f^*) \setminus \mathcal{E}(X, f)$. Then $h(f^*) = \sup \{ h_\mu(f^*) : \mu \in \mathcal{E}(X^*, f^*) \} = \sup \{ h_\mu(f^*) : \mu \in \mathcal{E}(X^*, f^*) \setminus \mathcal{M}(X, f) \}$, and therefore $h(f^*) = \text{ent}(f)$. \[\square\]

The condition $\mathcal{B}(X, f) = \mathcal{E}(X, f)$ cannot be avoided as the following example points out.

**Example 3.** Consider the example defined in the proof of Theorem 2.1(b). In this example the compactification of the set $X_2$ is $X$ and the extension of $f_2$ is $f$. Then $h(f) > 0$ while $\text{ent}(f_2) = 0$. 
4. Topological entropy of maps on the real line

In this section we study the topological entropy of continuous maps defined on the real line. Given a compact interval \([a, b]\), let \(f_{[a,b]}: [a, b] \to [a, b]\) be the continuous map defined by

\[
f_{[a,b]}(x) := \begin{cases} 
  f(x) & \text{if } f(x) \in [a, b], \\
  b & \text{if } f(x) > b, \\
  a & \text{if } f(x) < a.
\end{cases}
\]

Recall that if \(x \in [a, b]\), the set \(\omega_{f_{[a,b]}}(x)\) is the set of limit points of the orbit of \(x\) under \(f_{[a,b]}\). Thus \(\omega(f_{[a,b]}) = \bigcup_{x \in [a,b]} \omega_{f_{[a,b]}}(x)\).

**Lemma 1.** Let \(f : \mathbb{R} \to \mathbb{R}\) be continuous and assume that \(\text{Orb}_f(a)\) and \(\text{Orb}_f(b)\) are in \([a, b]\). Then there is \(K \in \mathcal{K}(\mathbb{R}, f)\), \(K \subseteq [a, b]\), such that \(h(f_{[a,b]}) = h(f|_K)\).

**Proof.** If \([a, b]\) is invariant by \(f\) there is nothing to prove. Assume that \([a, b]\) is not invariant and let \(\mathcal{A} := \{x \in [a, b]; f(x) \notin [a, b]\}\) and \(\mathcal{A}_\infty := \bigcup_{n \geq 0} f^{-n}(\mathcal{A})\). It is clear that \(\text{Orb}_f(a)\) and \(\text{Orb}_f(b)\) are contained in \([a, b] \setminus \mathcal{A}_\infty\). We prove that for all \(x \in [a, b]\), \(\omega_{f_{[a,b]}}(x) \cap \mathcal{A}_\infty = \emptyset\). Let \(y \in \mathcal{A}_\infty\). Let \((c, d) \subset \mathcal{A}_\infty\) be such that \(y \in (c, d)\). Let \(x \in [a, b]\) be such that \(y \in \omega_{f_{[a,b]}}(x)\). Then there is \(n \geq 0\) such that \(f^n_{[a,b]}(x) \in (c, d)\) and then there is \(m \geq 0\) such that \(f^{n+m}_{[a,b]}(x) \in \mathcal{A}\). Therefore \(f^{n+m+1}_{[a,b]}(x) = a\) or \(f^{n+m+1}_{[a,b]}(x) = b\). Then \(\{f^k_{[a,b]}(x); k \geq n + m + 1\} \cap (c, d) = \emptyset\), and then \(y \notin \omega_{f_{[a,b]}}(x)\). As \(h(f_{[a,b]}) = h(f_{[a,b]}|_{\omega(f_{[a,b]})})\) and \(\omega(f_{[a,b]})\) is compact (see for example [4]), the proof finishes. 

**Theorem 4.1.** Let \(f : \mathbb{R} \to \mathbb{R}\) be continuous. Then

\[
\text{ent}(f) = \sup \{h(f_I); I \text{ compact interval containing } K \in \mathcal{K}(\mathbb{R}, f)\}.
\]

**Proof.** Let \(\alpha = \sup \{h(f_I); I \text{ compact interval containing } K \in \mathcal{K}(\mathbb{R}, f)\}\). Let \(K \in \mathcal{K}(\mathbb{R}, f)\) and let \(J\) be the smallest compact interval containing \(K\). Then it is clear that \(h(f|_K) \leq h(f|_J)\). Then \(\text{ent}(f) \leq \alpha\). By Lemma 1, we get the converse inequality. 

4.1. Piecewise monotone maps

A continuous map \(f : \mathbb{R} \to \mathbb{R}\) is said that to be piecewise monotone if there are \(-\infty = x_1 < x_2 < \cdots < x_n = +\infty\) such that \(f_{(x_i,x_{i+1})}\) is increasing or decreasing for \(i = 1, 2, \ldots, n\).

**Theorem 4.2.** Let \(f : \mathbb{R} \to \mathbb{R}\) be continuous and piecewise monotone. Then there is a compact interval \([a, b]\) such that \(\text{ent}(f) = h(f_{[a,b]})\).

**Proof.** By Theorem 2.1(i), we can consider that \(f\) is surjective working with \(f: \mathbb{R}_\infty \to \mathbb{R}_\infty\). If \(\mathbb{R}_\infty\) is bounded, there is nothing to prove. Therefore, assume that it is unbounded and of the way \(\mathbb{R}_\infty = [\alpha, +\infty), \alpha \in \mathbb{R}\) (the other cases are similar). Let \(L\) be the set given by the union of all the compact sets of \(\mathcal{K}(X_\infty, f)\). We consider two cases. If \(L\) is
bounded, let \( a := \min L \) and \( b := \max L \). If \( L \) is unbounded, let \( \mathcal{F} := L \setminus \text{Fix}(f) \), where \( \text{Fix}(f) \) denotes the set of fixed points of \( f \). Since \( \lim_{x \to +\infty} f(x) = +\infty \) and \( f \) is piecewise monotone, there is \( b \in \text{Fix}(f) \) such that \( y < b \) for all \( y \in \mathcal{F} \). Let \( a := \min L \). In both cases we consider the map \( f_{[a,b]} \). It is clear that \( \text{Orb}_f(a) \) and \( \text{Orb}_f(b) \) are contained in \( [a, b] \). By Lemma 1, there is \( K \in \mathcal{K}(\mathbb{R}_\infty, f) \) such that \( h(f_{[a,b]}) = h(f|_{K}) \). Since there are not elements of \( \mathcal{K}(\mathbb{R}_\infty, f) \) outside \( [a, b] \), except for the fixed points of \( f \), by Theorem 4.1, we conclude that \( \text{ent}(f) = h(f_{[a,b]}) \). \[ \square \]

**Remark 3.** The above result is not true when the map is not piecewise monotone. For example we consider the map \( f \) such that \( n \in \{3, \text{Chapter } 4 \} \) that \( f \) is piecewise monotone and define a continuous extension of \( f \). The above result is not true when the map is not piecewise monotone. For example we consider the map \( f \) such that \( n \in \{3, \text{Chapter } 4 \} \) that \( f \) is piecewise monotone and define a continuous extension of \( f \).

For a piecewise monotone map \( f \) denote by \( c_n(f) \) the number of monotonicity pieces of \( f^n, n \geq 1 \). In the case of continuous maps on a compact interval \( [a, b] \), where the entropy is bounded by \( \log k \), where \( k = \max \{|f^n(x)| : x \in [a, b]\} \) (see, e.g., [19, Chapter 7]).

For a piecewise monotone map \( f \) denote by \( c_n(f) \) the number of monotonicity pieces of \( f^n, n \geq 1 \). In the case of continuous maps on a compact interval, it is known that \( h(f) = \lim_{n \to \infty} \frac{1}{n} \log c_n(f) \) (see [3, Chapter 4]). Now we prove an analogous result.

**Theorem 4.3.** Assume \( f : \mathbb{R} \to \mathbb{R} \) is continuous and piecewise monotone. Then

\[
\text{ent}(f) = \lim_{n \to \infty} \frac{1}{n} \log c_n(f).
\]

**Proof.** Since \( \lim_{x \to +\infty} f(x) = \beta \) and \( \lim_{x \to -\infty} f(x) = \alpha \), we consider \([-\infty, +\infty]\), the two points compactification of the real line and define a continuous extension of \( f \) by \( f^*(x) = f(x) \) if \( x \in \mathbb{R} \), \( f^*(-\infty) = \alpha \) and \( f^*(+\infty) = \beta \). By Theorem 4.2, there is a compact interval \( [a, b] \) such that \( \text{ent}(f) = h(f_{[a,b]}) \). Moreover, following the proof of Theorem 4.2, we can see that \([a, b] \) can be chosen such that \( \mathbb{R} \setminus [a, b] \) does not contain compact invariant subsets except for fixed points or periodic points of period two (when \( f^*(-\infty) = +\infty \) and \( f^*(+\infty) = -\infty \)). Then \( B(\mathbb{R}, f) = E(\mathbb{R}, f) \) and by Theorem 3.1 \( h(f^*) = \text{ent}(f) \). On the other hand, since \( f^* \) is piecewise monotone, \( h(f^*) = \lim_{n \to \infty} \frac{1}{n} \log c_n(f^*) \). Since \( c_n(f^*) = c_n(f) \), the proof concludes. \[ \square \]

4.2. Entropy and horseshoes

We recall that a continuous map \( f : [a, b] \to [a, b] \) has a \( k \)-horseshoe, \( k \in \mathbb{N} \), if there are \( k \) compact subintervals \( J_i, 1 \leq i \leq k \) such that \( \bigcup_{i=1}^{k} J_i \subseteq f(J_i), 1 \leq i \leq k \). It can be seen in [3, Chapter 4] that if \( f \) has a \( k \)-horseshoe, then \( h(f) \geq \log k \). Moreover, if \( h(f) > 0 \), there are sequences of positive integers \( s_n \) and \( k_n \) such that \( f^{k_n} \) has an \( s_n \)-horseshoe and

\[
h(f) = \lim_{n \to \infty} \frac{1}{k_n} \log s_n.
\]

We prove a similar result for continuous maps on the real line.
Theorem 4.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Then

(a) If $f$ has an $s$-horseshoe, then $\text{ent}(f) > 0$.
(b) If $\text{ent}(f) > 0$, then there are sequences of positive integers $s_n$ and $k_n$ such that $f^{k_n}$ has an $s_n$-horseshoe and

$$\text{ent}(f) = \lim_{n \to \infty} \frac{1}{k_n} \log s_n.$$

Proof. (a) Let $J_1, \ldots, J_s$ be an $s$-horseshoe for $f$ and let $J$ be the smallest interval containing $J_1, \ldots, J_s$. Then it is clear that $h(f_J) > \log s$ and by Theorem 4.1, $\text{ent}(f) > \log s$.

(b) First, we assume that $\text{ent}(f) = \alpha < \infty$. Let $\epsilon_n$ be a sequence of positive real numbers such that $\lim_{n \to \infty} \epsilon_n = 0$. By Theorem 4.1, for any $m \in \mathbb{N}$ there is a compact interval $I_m$ and $\delta_m > 0$ such that $h(f_{I_m}) > 0$ and $\alpha - \epsilon_m \leq h(f_{I_m}) - \delta_m < h(f_{I_m}) < h(f_{I_m}) + \delta_m \leq \alpha + \epsilon_m$. Since $h(f_{I_m}) > 0$, there is $n_0(m) \in \mathbb{N}$ such that if $n \geq n_0(m)$, then

$$h(f_{I_m}) - \delta_m \leq \frac{1}{k_n(m)} \log s_{n(m)} \leq h(f_{I_m}) + \delta_m,$$

where $f_{I_m}^{k_n(m)}$ has a $s_{n(m)}$-horseshoe. This means that $f_{I_m}^{k_n(m)}$ has a $s_{n(m)}$-horseshoe (because if $I_m \subseteq f_{I_m}^{k_n(m)}(J)$ for some interval $J \subseteq I_m$, then $I_m \subseteq f_{I_m}^{k_n(m)}(J) \subseteq f_{I_m}^{k_n(m)}(J)$).

Now, we fix $n_m \geq n_0(m)$ such that

$$h(f_{I_m}) - \delta_m \leq \frac{1}{k_{n(m)}} \log s_{n(m)} \leq h(f_{I_m}) + \delta_m,$$

for any $m \in \mathbb{N}$. Then, there is a sequence of positive integers $n_m$ converging to infinite such that $\alpha - \epsilon_m \leq \frac{1}{k_{n(m)}} \log s_{n(m)} \leq \alpha + \epsilon_m$. Therefore

$$\lim_{m \to \infty} \frac{1}{k_{n(m)}} \log s_{n(m)} = \text{ent}(f).$$

Now we assume that $\text{ent}(f) = +\infty$. For any $m \in \mathbb{N}$, by Theorem 4.1, there is a compact subinterval $I_m$ such that $h(f_{I_m}) > m$. Then there is $n_0(m) \in \mathbb{N}$ such that

$$\frac{1}{k_{n(m)}} \log s_{n(m)} > m$$

for all $n \geq n_0(m)$ and such that $f_{I_m}^{k_{n(m)}}$ has an $s_{n(m)}$-horseshoe, which is again a horseshoe for $f_{I_m}^{k_{n(m)}}$. Fix $n_m \geq n_0(m)$ and then

$$\frac{1}{k_{n(m)}} \log s_{n(m)} > m$$

for all $m \in \mathbb{N}$. Then

$$\lim_{m \to \infty} \frac{1}{k_{n(m)}} \log s_{n(m)} = +\infty$$

and the proof concludes. $\square$
4.3. Entropy of topologically transitive maps

Transitive maps in compact spaces has been studied in the literature (see for example the survey [13]). In this survey, we can see that if \( f : [a, b] \to [a, b] \) is transitive, then \( h(f) > 0 \). We wonder if is true the same result for transitive maps of the real line.

Now we study the case of topologically transitive maps on the real line. A continuous map \( f : \mathbb{R} \to \mathbb{R} \) is said to be topologically transitive if for any non-empty open subsets \( V \) and \( W \) of \( \mathbb{R} \), there is \( n \in \mathbb{N} \) such that \( f^n(V) \cap W \neq \emptyset \). It is clear that if \( f \) is a transitive map and \( K \) is invariant, then \( \text{Int}(K) = \emptyset \). Transitive maps on the real line has been studied in [2] where it is proved that if \( \mathcal{C} \) is the critical points set of \( f \), then the transitivity of \( f \) implies that \( \mathcal{C} \) is unbounded (we said that \( x \in \mathbb{R} \) is a critical point of \( f \) if for any neighborhood \( V \) of \( x \) there are distinct \( y, z \in V \) and such that \( f(y) = f(z) \)).

**Theorem 4.5.** If \( f : \mathbb{R} \to \mathbb{R} \) is continuous and transitive, then \( \text{ent}(f) > 0 \). Moreover, for any positive real \( \alpha \) there is a transitive map \( f : \mathbb{R} \to \mathbb{R} \) such that \( \text{ent}(f) \geq \alpha \).

**Proof.** It is clear that \( \text{Fix}(f) \neq \emptyset \). We divide the proof in two cases: (1) there is \( a \in \text{Fix}(f) \) such that \( f \) is increasing on \([a, \epsilon)\) or decreasing on \((\epsilon, a]\) for some \( \epsilon > a \). For all \( a \in \text{Fix}(f) \) the map \( f \) is decreasing on \([a, \epsilon)\) and increasing on \((\epsilon, a]\) for some \( \epsilon > a \).

(1) Assume for instance that \( f \) is increasing on \([a, \epsilon)\) for some \( \epsilon > a \) (the other case is analogous). Firstly, notice that \( f(x) > x \) for all \( x \in (a, \epsilon) \), because otherwise the interval \([a, \epsilon)\) would be invariant by \( f \), and this contradicts that \( f \) is transitive. Let \( c \in \text{Fix}(f) \) be the smaller fixed point of \( f \) such that \( a < c \). Clearly the point \( c \) exists because otherwise \( f([a, +\infty)) \subset [a, +\infty) \) and then \( f \) would not be transitive. Then there is \( d > c \) such that \( f(d) < a \), because otherwise \([a, +\infty)\) would be again invariant and with non-empty interior, a contradiction. Let \( b \) be the smaller number in \((c, d)\) such that \( f(b) = a \). Then it must exist \( e \in (a, b) \) such that \( f(e) > b \), because otherwise \([a, b]\) would be invariant. Let \( J_1 = [a, e] \) and \( J_2 = [e, b] \). It is clear that \( f|_{[a, b]}(J_1) = f|_{[a, b]}(J_2) = [a, b] \), then \( f|_{[a, b]} \) has a 2-horseshoe and therefore \( h(f|_{[a, b]}) \geq \log 2 \). By Theorem 4.1 (see also Theorem 4.4), we conclude that \( \text{ent}(f) \geq \log 2 \).

(2) Let \( a \in \text{Fix}(f) \) and consider the map \( f^2 \). Now \( f^2 \) is increasing on \([a, \epsilon)\) for some \( \epsilon > a \). By [2], it is hold that \( f^2 \) is transitive on all \( \mathbb{R} \) or on any of the sets \([a, \infty)\) and \((\infty, a)\). Reasoning as in the previous case (assuming \( f^2|_{[a, \infty)} \) is transitive if it is not transitive on all the real line) there are \( b > e > a \) such that \( f^2(b) = a \), \( f^2(e) > b \) and such that \( h(f^2|_{[a, b]}) \geq \log 2 \). Then there is \( K \in \mathcal{K}(\mathbb{R}, f^2) \) such that \( h(f^2|_K) = h(f^2|_{[a, b]}) \geq \log 2 \) and, by Theorem 2.1(c), \( \text{ent}(f) > 0 \).

Now we prove the second part of the result. Given \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{Z}^+ \) be such that \( \log m \geq \max\{\alpha, 4\} \). Now we build a continuous map \( f : \mathbb{R} \to \mathbb{R} \) as follows. Consider the sequence of pairs \((n, a_n), n \geq 0\) such that \( a_n = -n/m \) if \( n \) is even and \( a_n = (n + 1)m/2 \) if \( n \) is odd. Then define \( f \) on \([n, n + 1]\) to be linear and continuous. Finally, define for \( x < 0 \) let \( f(x) = -f(-x) \). Clearly, for \( n = 0, \ldots, m - 1 \) it is clear that \([0, m]\) \subset f\((n, n + 1)\) and then \( f|_{[0, m]} \) has an \( m \)-horseshoe. Then \( h(f|_{[0, m]}) \geq \log m \) and therefore \( \text{ent}(f) \geq \log m \geq \alpha \).

To conclude the proof we need to show that \( f \) is transitive. To this end, notice that \( \lim_{n \to \infty} f^n(n, n + 1) = \mathbb{R} \) for all \( n \in \mathbb{Z} \). This means that for any open subset \( U \) of the real line and any integer \( n \) there is \( k \in \mathbb{N} \) such that \( f^k(n, n + 1) \cap U \neq \emptyset \). Now let \((a, b) \subset \mathbb{R} \).
Since the slope of each linear piece of \( f \) has slope with modulus greater than 4, it is immediate to check that there is \( k \in \mathbb{N} \) such that \( f^k(a, b) \) contains an interval \((n, n+1)\) for some \( n \in \mathbb{N} \). This concludes the proof. □

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