The jump number problem on interval orders: 
A 3/2 approximation algorithm

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Abstract

In this paper, we consider the jump number problem on interval orders and use arc-diagram representations of posets to provide an approximation algorithm for the problem in this case. First, a complete characterization of arc-diagrams of interval orders is presented. Then, based on the properties of such representations, it is shown that semi-strongly greedy linear extensions (introduced by the author in 1987), in the case of interval orders are at most 50% worse than optimal linear extensions. This shows also that the pseudo-polynomial time backtracking algorithm for solving the jump number problem on arbitrary posets (Sysło, 1988) is a linear-time 3/2-approximation algorithm when the problem is restricted to interval orders. Moreover, in several cases the proposed algorithm proves the optimality of the solutions generated.

1. Preliminaries

1.1. Posets and their representations

Let \( (P, \leq) \) be a partially ordered set, simply denoted by \( P \) and called a poset. Let \( |P| = n \). For two elements \( p, q \in P \), we say that \( q \) covers \( p \) if \( p < q \) in \( P \) and \( p \leq r < q \) implies \( p = r \). Let us denote
\[
N_p^- = \{ q \in P : p \text{ is covered by } q \text{ in } P \},
\]
\[
N_p^+ = \{ q \in P : q < p \text{ in } P \},
\]
\[
N_p^0 = \{ q \in P : p < q \text{ in } P \}.
\]
Moreover, let
\[
\mathcal{R} = \{ N_p^- : p \in P \} \cup \{ P \} \quad \text{and} \quad \mathcal{R} = \{ N_p^+ : p \in P \} \cup \{ P \}.
\]
A poset (i.e., \( P \) as well as \( \leq \)) can be represented by vertices and arcs of digraphs (i.e., directed graphs), see [8] for details, and we shall make use here of the so-called vertex and arc diagrams.
A digraph may contain loops and multiple arcs, so it is defined as $D = (V, A, t, h)$, where $V$ is the vertex set, $A$ is the arc set, and $t, h$ (for tail and for head) are two incidence mappings $t, h: A \to V$. An arc $a \in A$ is of the form $a = (t(a), h(a))$. A sequence of arcs $\pi = (a_1, a_2, \ldots , a_l)$ is a path of length $l$ if $h(a_i) = t(a_{i+1})$ for $i = 1, 2, \ldots , l - 1$. The path $\pi$ begins with the arc $a_1$ and with the vertex $t(a_1)$, and terminates with $a_l$ and with $h(a_l)$. Therefore, we may write $t(\pi) = t(a_1)$ and $h(\pi) = h(a_l)$. A digraph is acyclic if it contains no path of length greater than 1 and such that $h(a_i) = t(a_i)$. The transitive closure of $D$ is denoted by $tc \ D = (V, tc \ A, t^*, h^*)$, where $(a_1, a_2, \ldots , a_l), l \geq 1$, is a path in $D$ if and only if $tc \ A$ contains arc $b$ such that $t^*(a_i) = t^*(b)$ and $h^*(a_i) = h^*(b)$. Let us denote $A^* = tc(A) \cup \{(v, v): v \in V\}$.

The vertex diagram, known better as the Hasse diagram of $P$ is a digraph $D^H(P) = (P, A)$, in which $(p, q) \in A$ if and only if $q$ covers $p$ in $P$.

The main result of this paper is obtained by using arc diagrams, known also as PERT networks, in which the poset elements are assigned to arcs and the relation is preserved along the paths of the digraph. Formally, an arc diagram of a poset $P$ is an acyclic digraph $D^A(P) = (V, R, t, h)$ without loops (but possibly with parallel arcs) and a mapping $\phi: P \to R$ such that for every $p, q \in P$, $p \neq q$ we have

$$p < q \text{ in } P \iff (h^*(\phi(p)), t^*(\phi(q))) \in R^*,$$

where $t^*, h^*$ are the incidence mappings of $tc \ D^A(P)$ and $R^* = tc(R) \cup \{(v, v): v \in V\}$. An arc diagram will be simply denoted by a digraph $D^A(P)$ in which some arcs (drawn in solid lines) are labelled by the poset elements and the poset relation between them is preserved along the paths in $D^A(P)$. Let us denote $S = R - \phi(P)$. An arc $a \in \phi(P)$ is a poset arc and otherwise $a$ is called a dummy arc (and drawn in dotted lines). A path $\pi$ in $D^A(P)$ is a poset path if it consists entirely of poset arcs.

Fig. 1 shows vertex and arc diagrams of some sample posets.

1.2. Interval orders

An interval order is a poset $(P, \leq)$ whose elements can be put in a one-to-one correspondence with intervals in the real line $P \leftrightarrow \{I_p\}_{p \in P}$ such that $p < q$ if and only if $x \in I_p$ and $y \in I_q$ imply $x < y$.

Interval orders can be characterized as follows (see [4] for proofs).

![Fig. 1. Vertex and arc diagrams of some posets.](image-url)
Theorem 1. For a poset \((P, \leq)\), the following conditions are equivalent:

1. \((P, \leq)\) is an interval order.
2. \((P, \leq)\) contains no subposet consisting of two disjoint chains of length 2 each.
3. \((\mathcal{M}, \subset)\) is a linear order.
4. \((\mathcal{N}, \subset)\) is a linear order.

Corollary 1. We have \(|\mathcal{M}| = |\mathcal{N}|\).

Proof. It is sufficient to show that there exists a bijection between \(\mathcal{M}\) and \(\mathcal{N}\). Let us define a mapping \(\varphi: \mathcal{M} \to \mathcal{N}\) such that

\[ \varphi(M_i) = N_j, \quad \text{where} \quad N_j = \bigcap_{q \in M_i} N_q^+ \quad \text{if} \quad M_i \neq \emptyset \quad \text{and otherwise} \quad \varphi(M_i = \emptyset) = P. \]

It is easy to check that \(\varphi\) is well defined. We show that \(\varphi\) is bijective. To this end, if \(M_1, M_2 \in \mathcal{M}\) and \(M_1 \subset M_2\) then there exist \(p \in (M_2 - M_1)\) and \(q \in (\varphi(M_1) - \varphi(M_2))\). On the other hand, for every \(N_i\) there exists \(M_i\) such that \(\varphi(M_i) = N_j\), where \(M_i = \{r : r < s \quad \text{for every} \quad s \in N_j\}\). Both these implications follow by Theorem 1(3) and (4). \(\square\)

Let us denote \(k + 1 = |\mathcal{M}|\) and assume \(\mathcal{M} = (M_0, M_1, \ldots, M_{k-1}, M_k)\), where \(\emptyset = M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = P\) and \(\mathcal{N} = (N_0, N_1, N_2, \ldots, N_k)\), where \(P = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_k = \emptyset\). Integer \(k = k(P)\) is called in [6] the length of an interval order \(P\). It is clear that \(\varphi(M_i) = N_i\). Now, for every \(p \in P\) we define \(l(p)\) such that \(N_p^- = N_{l(p)}\) and \(r(p)\) such that \(N_p^+ = M_{r(p)}\).

It is easy to show that, see [6].

Corollary 2. The assignment \(P \Rightarrow \{I_p = [l(p), r(p)] : p \in P\}\) is an interval representation of the interval order \(P\).

We call \(\{I_p : p \in P\}\) defined above a canonical representation of \(P\).

2. Arc-diagram of interval orders

We now extend a canonical interval representation \(\{I_p : p \in P\}\) of an interval poset \((P, \leq)\) to its arc-diagram \(D^A(P) = (V, R)\) with vertex set \(V\) and arc set \(R\), where

1. \(V = \{0, 1, 2, \ldots, k\}\), with \(k = k(P)\) defined above,

2. \(a_p = (l(p), r(p)) \in R\) for every \(p \in P\)
and dummy arcs are

\[(d3) \quad (i, i+1) \in R \quad \text{if} \quad (i, i+1) \text{ is not a poset arc, for } i = 1, 2, \ldots, k - 2.\]

We have the following complete characterization of arc diagrams of interval orders:

**Theorem 2.** Let \((P, \leq)\) be a poset and \(D = (V, R, t, h)\) be an arc diagram of \(P\) together with a mapping \(\phi\). Then, \(P\) is an interval order if and only if \(D\) contains a path \(\pi = (v_0, v_1, \ldots, v_n)\) consisting of all vertices of \(V\) and all arcs of \(R - \phi(P)\) belong to \(\pi\) as its non-terminal arcs.

**Proof.** First, the construction preceding the theorem results in an arc diagram of an interval poset \(P\), since \(p < q \in P\) if and only if \(r(p) \leq l(q)\) and there is a path in \(D^\Delta(P)\) (may be of length 0) from \(r(p)\) to \(l(q)\) consisting of poset and dummy arcs. Therefore the relation between the elements of \(P\) is preserved along the paths between the corresponding arcs in \(D^\Delta(P)\). In this case, \(\pi = (0, 1, \ldots, k)\) is a path in \(D^\Delta(P)\) and contains all dummy arcs. Moreover, \((0, 1)\) is not a dummy arc in \(D^\Delta(P)\), since otherwise we would have \(M_0 = M_1 = 0\) — a contradiction. By a similar argument, \((k - 1, k)\) is also a poset arc in \(D^\Delta(P)\).

On the other hand, let us assign the sequence \((v_0, v_1, \ldots, v_n)\) of all vertices of \(D\) to points \((0, 1, \ldots, n)\) in the real line. Now, every arc of \(D\) is an interval and we associate with \(D\) the following order:

for every two arcs \(a, b \in \phi(P)\), we have \(a < b\) if there exists in \(D\) a path (may be of length 0) from the head of \(a\) to the tail of \(b\).

It is clear that \(P\) is an interval order. \(\square\)

An arc diagram defined for an interval order \(P\) by rules \((d1)-(d3)\) is called a **canonical arc diagram** of \(P\). Such diagrams have the following properties.

**Property 1.** A canonical arc diagram \(D^\Delta(P)\) of an interval order \(P\), defined by \((d1)-(d3)\), has the following properties:

(a1) Every vertex \(i\), for \(0 < i \leq k\), is a head of a poset arc and, for \(0 \leq i < k\), is a tail of a poset arc.

(a2) \(D^\Delta(P)\) has at most \(k - 2\) dummy arcs.

(a3) For every dummy arc \(a\), \(h(a)\) is a head of at least two arcs, the arc \(a\) and a poset arc.

(a4) For a poset arc \(p \in P\) such that \(l(p) \leq r(p) - 2\), \(h(p)\) is also a head of another arc different from \(p\).

**Proof.** Property (a1) follows by \((d1)\) and the construction of the vertex set of \(D^\Delta(P)\). Property (a2) follows by \((d3)\) — no dummy arc can be incident with the source or with
the sink of $D^\lambda(P)$ and property (a3) follows by (a1) since $h(a)$ is also a head of a poset arc. To show (a4), it is sufficient to observe that $h(p)$ is a head of $p$ and of the arc $(h(p) - 1, h(p))$, which may be a poset arc or a dummy arc. 

3. The jump number problem on interval orders

A linear extension of a poset $(P, \leq)$ is a total order $L = p_1, p_2, \ldots, p_n$ of $P$ such that if $p_i < p_j$ in $P$ then $i < j$. A pair $(p_i, p_{i+1})$ is a jump of $L$ if $p_i$ is not smaller than $p_{i+1}$ in $P$. The jumps partition $L$ into chains $C_i$ of $P$, so we can write $L = C_0 + C_1 + \cdots + C_s$. The jump number $s(P)$ of $P$ is equal to minimum $s$ over all linear extensions $L$ of $P$ and the jump number problem consists in evaluating $s(P)$ and constructing an optimal linear extension (i.e., the one with $s(P)$ jumps). Although the problem is NP-complete (see [1]), there exist polynomial-time algorithms for some special classes of posets (e.g., for N-free posets, see below) and pseudo-polynomial time algorithms for arbitrary posets (see [10]). The complexity status of the jump number problem for interval orders has been settled by Juta Mitas [6] — the problem remains NP-complete.

3.1. Greedy chains and greedy linear extensions

A chain $C$ in a poset $P$ is greedy if there are no elements $p \in (P - C)$ and $q \in C$ such that $p < q$ and moreover for no $r$ which covers $\sup C$ in $P$, the chain $C \cup \{r\}$ has this property. A linear extension $L = C_0 + C_1 + \cdots + C_s$ of $P$ is greedy if $C_i$ is a greedy chain in the subposet $P - \bigcup_{j<i} C_j$.

Every poset has an optimal linear extension which is greedy (although an arbitrary greedy linear extension even for interval orders may be a very bad approximation of an optimal one). This, rather simple observation was strengthened in [9] (see also [11]), where we proved that every poset has an optimal semi-strongly greedy linear extension. The main result of this paper will follow from general properties of such extensions for interval orders represented by canonical arc diagrams. Therefore, in what follows, we restrict our attention to interval orders represented by their canonical arc diagrams (for the corresponding results on arbitrary posets, see [9] and also [11]).

A greedy chain $C$ in $P$ induces a greedy path $\pi$ in $D^\lambda(P)$, which contains poset arcs corresponding to the elements of $C$. To indicate this relation we shall denote $C$ by $C_\pi$ and $\pi$ by $\pi(C)$. We can easily show

Lemma 1. For an interval order $P$, a greedy path $\pi = (x_1, x_2, \ldots, x_f)$ in a canonical arc diagram $D^\lambda(P)$ of $P$ has the following properties:

(g1) no $h(x_i)$ for $i = 1, 2, \ldots, f - 1$ is a head of an arc different from $x_i$ and either $P$ has no element which is not in $\pi$ or there exists an arc $y$ in $D^\lambda(P)$ (possibly a dummy arc) such that $y \neq x_f$ and $h(y) = h(x_f)$, in the latter case we say that the arc $y$ stops $\pi$ to go further.
(g2) every $x_i (i = 1, 2, \ldots, f)$ is a poset arc,
(g3) $t(x_1) = 0$, $h(x_i) = t(x_{i+1}) = i$ for $i = 1, 2, \ldots, f - 1$, and $h(x_f) \geq f$, i.e., all but possibly the last arc (interval) form a path of unit intervals in the canonical representation of $P$.

**Proof.** Property (g1) follows by the construction of a greedy path (from a greedy chain). To show (g2), let us observe that $x_f$ must be a poset arc and, by properties (a3) and (g1), no arc $x_i$, for $1 \leq i < f$, can be a dummy one. For property (g3), note that if an arc $x_i (i < f)$ is not a unit interval, that is if $t(x_i) + 2 \leq h(x_i)$, then by (a4), $h(x_i)$ is also a head of $(h(x_i) - 1, h(x_i))$, and therefore $(x_1, x_2, \ldots, x_i)$ could not be extended to a greedy path $\pi$. 

Informally, a path $\pi$ in $DA(P)$ is greedy if it is a sequence of unit intervals (arcs) terminated possibly by a non-unit interval and no intermediate vertices if $\pi$ are heads of other arcs of $DA(P)$. For instance, the poset in Fig. 2 has three greedy paths: (b), (a, c) and (a, d).

A poset $P$ is **N-free** if $DH(P)$ contains no induced subgraph isomorphic to $DH(N)$, or equivalently, if $P$ has an arc diagram with no dummy arcs. (The poset $N$ is shown in Fig. 1.) If $P$ is N-free then every greedy linear extension solves the jump number problem for $P$. In the other words, an optimal linear extension of an N-free poset $P$ may begin with an arbitrary greedy chain of $P$ (or equivalently, a greedy path of $DA(P)$). In [9] we have introduced **strongly greedy paths** which share this property in arbitrary posets. When $P$ is an interval order, then a greedy path $\pi$ is **strongly greedy** if in addition to properties (g1)-(g3) of greedy paths, it satisfies

(g4) either (a) $h(\pi) = k$, i.e., $h(\pi)$ is the sink $DA(P)$, or
(b) $h(\pi)$ is the head of a poset arc $b$ ($b \neq x_f$) such that every path terminating with $b$ contains no vertex which is incident with a dummy arc, or equivalently, no vertex of $\{0, 1, 2, \ldots, t(b)\}$ is incident with a dummy arc.

It was proved in [9] that

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Fig. 2. An interval order and its canonical arc diagram [3].
Theorem 3 (Syslo [9]). If \( \pi \) is a strongly greedy path in \( D^A(P) \) of \( P \), then every greedy linear extension \( L \) of \( P \) can be transformed to a greedy linear extension \( L^* \) of \( P \) which begins with the chain \( C_\pi \) and \( s(P, L^*) \leq s(P, L) \).

An arc diagram \( D^A(P) \) of a poset \( P \) provides a simple lower bound to the jump number:

Corollary 3 (Syslo [10]). If \( D^A(P) \) is an arc diagram of a poset \( P \) then

\[
s_1 = \sum_{v \in V} \max\{0, \text{indeg}_P(v) - 1\} \leq s(P),
\]

(1)

where \( \text{indeg}_P(v) \) is the number of poset arcs coming into \( v \).

Proof. Note that \( \text{indeg}_P(v) \) which is greater than 1 cannot be decreased by any greedy path which does not terminate at \( v \). Hence, only a greedy path terminating at \( v \) may reduce \( \text{indeg}_P(v) \), and each such path contributes one to the jump number of \( P \). This proves the bound. \( \square \)

An arc diagram \( D^A(P) \) of a poset \( P \), in general, and of an interval order in particular may not contain a strongly greedy path (see Fig. 3). In such a case, a poset contains a semi-strongly greedy path, which is a greedy path \( \pi \) that additionally to \((g1)-(g3)\) satisfies also the following condition:

\((sg4)\) \( \pi \) goes through a vertex which is a tail of a dummy arc but not a head of a dummy arc.

The poset shown in Fig. 2 contains two semi-strongly greedy paths \((a, c)\) and \((a, d)\), and in Fig. 3 — also two \((a, c)\) and \((a, d)\).

The following fact holds for arbitrary posets (see [9]) and we prove it for interval orders.

Lemma 2. If an arc diagram \( D^A(P) \) of an interval order \( P \) contains no strongly greedy path then it contains a semi-strongly greedy path.

Proof. If \( D^A(P) \) contains no strongly greedy paths then it contains a dummy arc. Let us consider the path \( \rho = (x_1, x_2, \ldots, x_{g-1}, x_g) \) in \( D^A(P) \) such that \( t(x_1) = 0 \), \( h(x_i) = t(x_{i+1}) = i \) for \( i = 1, 2, \ldots, g - 1 \), \( h(x_g) = g \) and \( g \) is the tail of the first dummy arc in \( D^A(P) \). If any vertex \( i \in \{1, 2, \ldots, g\} \) is the head of a poset arc, then \( (x_1, x_2, \ldots, x_i) \)

Fig. 3. A poset with no strongly greedy path.
is a strongly greedy path. Otherwise, \( \rho \) is an initial part of a greedy path which can be extended to a semi-strongly greedy one, since \( g \) is the tail of a dummy arc. □

Note that condition (sg4) for a greedy path \( \pi \) to be semi-strongly greedy in \( D^\wedge(P) \), for an interval order \( P \), is equivalent to

\[(sg4') t(x_f) \text{ is a tail of a dummy arc and } t(x_f) + 2 \leq h(x_f).\]

It has been proved in [9] for an arbitrary poset \( P \), in general, and for interval posets in particular that

**Theorem 4** (Syslo [9]). *If an arc diagram \( D^\wedge(P) \) of \( P \) contains no strongly greedy paths then \( P \) has an optimal linear extension which begins with a semi-strongly greedy path.*

### 3.2. Algorithm

Theorems 3 and 4 guarantee that every poset has an optimal linear extension \( L = C_0 + C_1 + \cdots + C_s \), hereafter called *semi-strongly greedy* such that chain \( C_i \) is strongly greedy in \( P_i = P - \bigcup_{j < i} C_j \) or semi-strongly greedy in \( P_i \) if \( P_i \) contains no strongly greedy chain. Formally, a semi-strongly greedy linear extension can be generated by the following algorithm:

**Algorithm: Semi-strongly greedy (SSG)**

1. Let \( D = D^\wedge(P) \) for an interval order \( P \);
   \[ Q = P; \ L = \emptyset; \]
   \[ \text{while } D \neq \emptyset \text{ do begin} \]
   2. *if* \( D \) contains a strongly greedy path \( \rho \) *then* \( \pi = \rho \)
   3. *else* \( \pi = \) a semi-strongly greedy path of \( D \);
   4. \( L = L + C_\pi; \)
   5. Remove \( \pi \) from \( D \) and reduce the resulting digraph to a canonical arc diagram of \( Q - C_\pi; \)
   6. \( Q = Q - C_\pi \)
   \[ \text{end.} \]

Algorithm SSG consists of one forward step of a backtracking, pseudo-polynomial time algorithm presented in [10] for solving the jump number problem on arbitrary posets. In the other words, instead of looking for an optimal solution over all semi-strongly greedy linear extensions, it finds one such extension. We shall show that for every interval order \( P \), each of its semi-strongly greedy linear extensions contains at most 50% more jumps than an optimal one. This will be based on the fact that a strongly or semi-strongly greedy path in \( D^\wedge(P) \), when removed from \( D^\wedge(P) \), reduces the number of dummies by at most 3.
Let us now describe in detail step 5, of algorithm SSG. Since the removal of a greedy chain from an interval order results in the poset which is also interval, without loss of generality we may assume that $D$ is a canonical arc diagram of an interval order $P$ and $\pi = (x_1, x_2, \ldots, x_f)$ is a greedy (in particular, strongly or semi-strongly) path in $D$. The removal of $\pi$ from $D$ should result in a canonical arc diagram of the poset $P - C_\pi$. By properties (g1)-(g3), $D^\pi(P - C_\pi)$ can be obtained from $D$ by the following algorithm:

Algorithm: REDUCTION

1. Remove arcs $x_1, x_2, \ldots, x_{f-1}, x_f$ and replace all vertices of indegree 0 by one vertex.
2. If $h(x_f) \geq t(x_f) + 2$ then:
   a. If there is a dummy arc $a_1 = (t(x_f), t(x_f) + 1)$ in $D$ then remove also $a_1$;
   b. If no poset arc other than $x_f$ enters $h(x_f)$ then remove dummy arc $a_2 = (h(x_f) - 1, h(x_f))$.
   c. Moreover, if $D$ contains a poset arc $x = (h(x_f) - 1, h(x_f) + 1)$ and dummy arc $a_3 = (h(x_f), h(x_f) + 1)$, then remove also $a_3$.
   d. Replace $h(x_f) - 1$ and $h(x_f)$ by one vertex.
3. Renumber the vertices of the resulting digraph starting from 0.

The correctness of algorithm REDUCTION can be proved by considering all possible patterns of a greedy path $\pi$ in $D$. Regardless of the situation, we remove all arcs of $\pi$ (step c1). By (g3) in the definition of a greedy path, either $h(x_f) = t(x_f) + 1$ or $h(x_f) \geq t(x_f) + 2$. In the former case we do nothing more since a dummy arc can be incident only with $h(x_f)$ but there exists a poset arc $y, y \neq x_f$ such that $h(y) = h(x_f)$. In the latter case we may have three dummy arcs adjacent to $x_f$, which can be removed together with $x_f$, namely $a_1, a_2$ and $a_3$, as defined above. If $D$ contains $a_1$, then there exists a poset arc $p = (j, t(x_f) + 1)$, where $0 \leq j < t(x_f)$. In this case, $N_p^-$ becomes $\emptyset$ in $P - C_\pi$ and $N_p^+$ is the largest set in $\mathcal{H}(P - C_\pi)$. Hence we shall have $t(p) = 0$ and $h(p) = 1$ in $D^\pi(P - C_\pi)$, and therefore $a_1$ is removed from $D$. If $x_f$ is not the only poset arc entering $h(x_f)$, then no other dummy arc can be removed. Otherwise, for poset arcs $q$ and $r$ such that $t(q) = h(x_f) - 1$ and $t(r) = h(x_f)$ we have $N_q^- = N_r^-$ in $P - C_\pi$. Therefore, $a_1$ can be removed and its end-vertices — contracted. If in this case, $D$ contains dummy arc $a_3$ and poset arc $x$, then after the contraction, $a_3$ and $x$ become parallel, so $a_3$ can be removed.

We have the following very important observation:

**Lemma 3.** Every semi-strongly greedy linear extension of an interval order contains the same number, $s_1$, of greedy paths (strongly or semi-strongly) which are stopped by poset arcs.

**Proof.** It is enough to observe that for an interval order $P$, algorithms SSG and REDUCTION reduce $s_1$ by 1 if and only if a greedy path $\pi$ which is currently removed is stopped by a poset arc and remains the same otherwise. □
When we restrict the removal of paths from arc diagrams only to strongly and semi-strongly greedy ones then we obtain:

**Theorem 5.** The removal of a strongly greedy path \( \pi \) from \( D^\wedge(P) \) reduces the number of dummy arcs by 0 or 1.

**Proof.** If \( \pi \) is a strongly greedy path, then if \( h(x_f) = t(x_f) + 1 \) then no dummy arc can be removed. However, if \( h(x_f) > t(x_f) + 2 \) then there exists a poset arc \( y, y \neq x_f \), with the same head as \( x_f \), and therefore only \( a_1 \) can be removed. \( \square \)

**Theorem 6.** The removal of a semi-strongly greedy path \( \pi \) from \( D^\wedge(P) \) reduces the number of dummy arcs by at most 3, and by 2 or 3 when \( \pi \) is stopped by a dummy arc.

**Proof.** If \( \pi \) is a semi-strongly greedy path, then we can remove at least \( a_1 \), and possibly also \( a_2 \) and \( a_3 \). If \( \pi \) is stopped by a dummy arc then also \( a_2 \) is removed, and possibly also \( a_3 \). \( \square \)

Let \( s_{ssg}(P, L) \) denote the number of jumps in a semi-strongly greedy linear extension \( L \) generated by algorithm SSG and let \( d \) denote the number of dummy arcs in \( D^\wedge(P) \), a canonical arc diagram of \( P \).

Now to improve bound (1) we take into considerations the number of dummy arcs in \( D^\wedge(P) \). In the process of generating a semi-strongly greedy linear extension, all dummy arcs are removed by successive greedy paths. By Lemma 3 and Theorems 5 and 6 we have

\[
s_1 + (d - s_1)/3 \leq s(P),
\]  

since every path counted in \( s_1 \) may be a strongly greedy one which in the best case (Theorem 5) removes one dummy arc and the remaining dummies have to be removed by semi-strongly greedy paths, 3 at a time in the best case.

On the other hand, each semi-strongly greedy path which is not counted in \( s_1 \) removes in the worst case two dummies at a time (Theorem 6). Therefore,

\[
s_{ssg}(P, L) \leq s_1 + d/2.
\]  

From (2) we have that

\[
s_1 + d/2 \leq 3s(P)/2
\]

and combining with (3) we obtain

\[
s_{ssg}(P, L) \leq 3s(P)/2.
\]

Thus we have reached the main result of this paper:
Theorem 7. Algorithm SSG applied to an interval order represented by its canonical arc diagram generates a semi-strongly greedy linear extension which is at most 50% worse than an optimal one.

Theorems 5 and 6, and the analysis of the performance above suggest the following modification of step 3 of algorithm SSG:

3’ else \( \pi = \) a semi-strongly greedy path which is stopped by a poset arc or which removes three dummy arcs (case c221 in algorithm REDUCTION);

3.3. Complexity of the algorithm

The time complexity of algorithm SSG has been partly established in [10], where we showed that, once an arc diagram \( D^A(P) \) of a poset \( P \) is given, a semi-strongly greedy linear extension of \( P \) can be generated in time linear in the number of vertices and arcs in \( D^A(P) \). We show that the remaining steps take no longer. To this end, it remains to show that a canonical representation \( D^A(P) \) of an interval order \( P \) can be also constructed in linear time (step 1) and step 5 (performed by using algorithm REDUCTION) takes also linear time. Step 1 can be implemented using an approach of [5], where a linear time algorithm is given for recognizing an interval digraph (i.e., a digraph whose transitive closure is an interval order). The latter is also obvious, since one reduction step of \( D \) takes time proportional to the number of removed vertices and arcs.

4. Conclusions

We have presented a 3/2-approximation algorithm for solving the jump number problem on interval orders. The algorithm makes a significant use of arc-diagram representations of such posets. These representations of interval orders are also completely characterized in this paper.

Other 3/2-approximation algorithms for the same problem have been proposed by Felsner [3] and Mitas [6]. The algorithm presented here is superior to those of [3] and [6] on at least two accounts.

Firstly, algorithm SSG is in fact a special version of a general-purpose algorithm of [10], and the result of this paper says that if one forward step of the pseudo-polynomial time algorithm of [10] is applied to an interval order it produces a 3/2-approximation solution.

Secondly, it is easy to show that for every interval order, algorithm SSG produces a solution which is at least as good as that generated by the algorithm of [3]. Moreover, what is more important, the notions of strongly and semi-strongly greedy chains and their properties (proved in general in [9] and [11]) allow us to draw a conclusion in several cases that a linear extension generated by SSG is not only
3/2-approximate but optimal one. For instance, when every chain \( C_i \) is strongly greedy in \( P \), or \( C_i \) is either the only semi-strongly greedy chain \( P \), or the removal of \( C_i \) results in the removal of three dummy arcs from the arc diagram.

References