JOURNAL OF COMBINATORIAL THEORY, Series B **38**, 23–30 (1985)

Stability in CAN-Free Graphs

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Communicated by the Managing Editors

Received June 20, 1983

A class $\mathscr{F}$ of graphs characterized by three forbidden subgraphs C, A, N is considered; C is the claw (the unique graph with degree sequence (1, 1, 1, 3)), A is the antenna (a graph with degree sequence (1, 2, 2, 3, 3, 3) which does not contain C), and N is the net (the unique graph with degree sequence (1, 1, 1, 3, 3, 3)). These graphs are called CAN-free. A construction is described which associates with every CAN-free graph G another CAN-free graph G' with strictly fewer nodes than G and with stbility number $\alpha(G') = \alpha(G) - 1$. This gives a good algorithm for determining the stability number of CAN-free graphs. © 1985 Academic Press, Inc.

1. INTRODUCTION

The determination of the stability number $\alpha(G)$ in a graph G (i.e., the maximum number of pairwise nonadjacent nodes) is in general a difficult problem.

Efficient algorithms have been devised for some special classes of graphs: for perfect graphs it is known that the stability number can be computed in polynomial time, and very efficient methods have been described for several classes of perfect graphs [2]. Also for claw-free graphs (i.e., graphs containing no induced $K_{1,3}$) polynomial algorithms have been proposed (see [3, 4]).

A different approach has been described in [1]: it consists in a construction which associates with a graph G another graph G' the stability number

* This research was carried out while this author was visiting the Department of Combinatorics and Optimization at the University of Waterloo. The support of NSERC is gratefully acknowledged.

0095-8956/85 $3.00

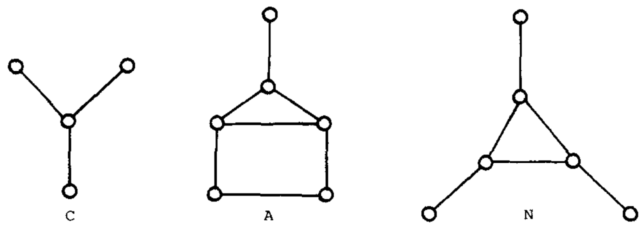

FIG. 1. C = the claw; A = the antenna; and N = the net.

of which is exactly one less than that of G. In general G' may have more nodes than G. Our purpose is to describe here a construction which transforms any graph G of some class to be described into another graph G' of the same class with stability number exactly one less than that of G. Repeated applications of this transformation will lead to a polynomial time algorithm for the stability number of any graph in the class.

We shall consider a proper subclass $\mathscr{F}$ of claw-free graphs; it will be characterized by 3 forbidden subgraphs C, A, N defined as follows (see Fig. 1): C is the claw (i.e., the unique graph with degree sequence (1, 1, 1, 3)); A is the antenna (i.e., a graph with degree sequence (1, 2, 2, 3, 3, 3)); and N will be the net (i.e., the unique graph with degree sequence (1, 1, 1, 3, 3, 3)). Graphs in $\mathscr{F}$ will be called *CAN-free.*

We will show that the class $\mathscr{F}$ is closed for the transformation $G \to G'$ and furthermore that G' has fewer nodes than G. This will give a simple and efficient algorithm for obtaining the stability number of a graph in $\mathscr{F}$.

Although polynomial algorithms are known for finding the stability number in the more general class of claw-free graphs, the construction given here is interesting because of its simplicity and because it may suggest a quite different approach.

In the next section the transformation and some of its properties will be described; Section 3 will contain the proof that the class $\mathscr{F}$ is closed for the transformation. Finally in Section 4, we shall show that the stability number is reduced by 1 and that a polynomial algorithm can be derived for obtaining the stability number of CAN-free graphs.

2. THE TRANSFORMATION $G \to G'$

It will be convenient to call a pair of nonadjacent nodes i, j with $i < j$ a *nonedge*; it will be denoted by $\overline{[i, j]}$.

Construction of G'

(a) Let 0 be any node of $G = (X, U)$ and $N(0) = \{1, ..., p\}$ the set of its neighbours.

(b) With each nonedge $\overline{[i, j]}$ in $N(0)$ we associate a set of nodes $N\overline{[i, j]} = (N(i) \cup N(j)) - N(0)$ consisting of all neighbours of i or of j which are not neighbours of 0.

(c) We consider the partial order defined on the set of nonedges in $N(0)$ by

$$\overline{[i, j]} \succ \overline{[k, l]} \Leftrightarrow N\overline{[i, j]} \supseteq N\overline{[k, l]}.$$

Let I^* be the collection of minimal nonedges in this partial order; if several nonedges $\overline{[i, j]}$ in I^* have the same $N\overline{[i, j]}$, we keep only one of them in I^*.

(d) Introduce into G' the subgraph R of G induced by $X - (N(0) \cup \{0\})$. For each nonedge $\overline{[i, j]}$ in I^* introduce a new node (i, j). Introduce a link between every pair of new nodes and link a new node (i, j) with a node r in R if either i or j was linked with r in G.

This construction is illustrated in Fig. 2: In G the nonedges to be considered are $\overline{[1, 3]}$, $\overline{[1, 4]}$, $\overline{[2, 3]}$, $\overline{[2, 4]}$; since $N\overline{[1, 3]} = N\overline{[1, 4]} = \{5, 6\}$, $N\overline{[2, 3]} = \{5\}$, $N\overline{[2, 4]} = \emptyset$ we have $I^* = \{\overline{[2, 4]}\}$ and we get the graph G' of Fig. 2b.

The following property will be useful for deriving an algorithm for determining $\alpha(G)$.

PROPOSITION 2.1. *Let G be a CAN-free graph, and* 0 *an arbitrary node of G; if $\overline{[i, j]}$ and $\overline{[k, l]}$ are two nonedges in $N(0)$ with $\{i, j\} \cap \{k, l\} \neq \emptyset$, then we have $\overline{[i, j]} \succ \overline{[k, l]}$ or $\overline{[k, l]} \succ \overline{[i, j]}$.*

Proof. Without loss of generality, let us assume that $k = i$; in G node 0 is linked to i, j, l; since G is claw-free, j and l must be linked (otherwise there would be a claw $(0; i, j, l)$.

Assume that $\overline{[i, l]}$ and $\overline{[i, j]}$ are not comparable in the partial order.

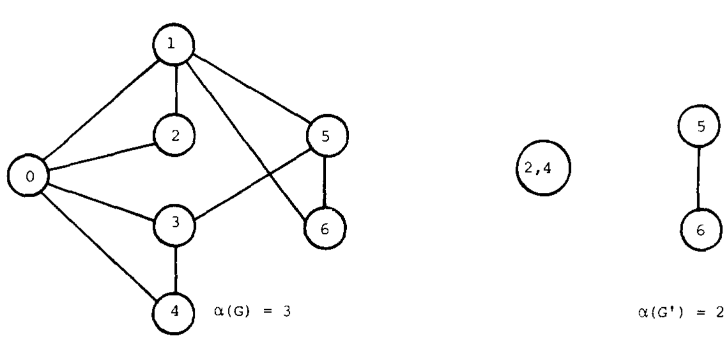

FIG. 2. (a) The graph G; (b) the transformed graph G'.

This means that j is linked to some node r which is not linked to 0, i, or l; also l is linked to a node s which is not linked to 0, i, or j. But then we have an induced antenna if r and s are linked or an induced net if they are not linked.

Hence $\overline{[i, j]}$ and $\overline{[i, l]}$ are comparable. Q.E.D.

As a consequence of Proposition 2.1 all new nodes (i, j) in G' will correspond to disjoint subsets $\{i, j\} \subseteq N(0)$. Furthermore since nodes in $\{0\} \cup N(0)$ have been removed from G and at most $\lfloor |N(0)|/2 \rfloor$ new nodes have been introduced when constructing G', we notice that G' has fewer nodes than G.

3. Closedness of CAN-free Graphs

In this section we shall prove that the class of CAN-free graphs is closed under the transformation described in Section 2. For simplifying the notation $[i, j]$ (resp. $\overline{[i, j]}$) will mean that nodes i and j are linked (resp. not linked).

Proposition 3.1. *If G is a CAN-free graph containing the configuration H^* given in Fig. 3, then* $\overline{[i, k]}$.

Proof. If $[i, k]$, then $\overline{[i, j]}$ (otherwise $(i; j, k, l)$ is a claw). But then $[m, j]$ and $[m, l]$ (otherwise H^* induces an A or an N). Now $(m; j, k, l)$ is a claw. Hence $\overline{[i, k]}$ must hold. ∎

Theorem 3.1. *If G' is the transform of a CAN-free graph G, then G' does not contain A or N.*

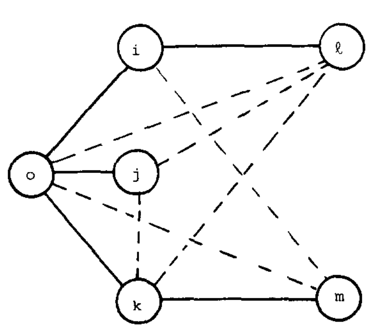

Fig. 3. Configuration $H^*(j, 0, i, k, l, m)$ (dark lines indicate edges and dotted lines non-edges).

Proof. We have to show that G' does not contain the configuration H given in Fig. 4. Since G does not contain A or N, G does not contain H; let us assume that G' contains H. Then clearly at least one node of H is a new node of G'. From the construction we know that all new nodes form a clique in G'. We have several cases to examine.

Case 1. b, c, d are new nodes. Let $\overline{[i, j]}$, $\overline{[k, l]}$, and $\overline{[m, n]}$ be the nonedges in the neighbourhood of node 0 in G which correspond to new nodes b, c, d, respectively. From Proposition 2.1 we know that nodes i, j, k, l, m, n are all distinct. Since all new nodes form a clique in G', a, e, and f must be old nodes and hence are also nodes of G. Then we have in G

(1) $[a, i]$ or/and $[a, j]$ and $\overline{[a, q]}$ for $q = k, l, m, n$

(2) $[e, k]$ or/and $[e, l]$ and $\overline{[e, q]}$ for $q = i, j, m, n$

(3) $[f, m]$ or/and $[f, n]$ and $\overline{[f, q]}$ for $q = i, j, k, l$.

Assume without loss of generality that $[a, j]$, $[e, l]$, $[f, n]$. Then by Proposition 3.1, $\overline{[j, l]}$, $\overline{[j, n]}$, $\overline{[l, n]}$, and $(0; j, l, n)$ is a claw. So this case is not possible.

Case 2. b, c are new nodes, d is old. Let $b = (i, j)$ and $c = (k, l)$ with i, j, k, l all distinct. Since $[c, d]$, $[b, d]$ in G', then ($[i, d]$ or/and $[j, d]$) and ($[k, d]$ or/and $[l, d]$) in G. Assume without loss of generality that $[j, d]$ and $[k, d]$. Then $\overline{[a, j]}$, otherwise $(j; 0, a, d)$ is a claw; furthermore $\overline{[e, k]}$ (otherwise $(k; 0, d, e)$ is a claw and $[j, k]$ (otherwise $(d; j, k, f)$ is a claw). Since $[a, b]$ and $[c, e]$ in G', $[a, i]$ and $[e, l]$ follow in G. Now by Proposition 3.1 applied to $H^*(k, 0, i, l, a, e)$ we have $\overline{[i, l]}$; then $[k, i]$ and $[l, j]$ and G contains $H(l, j, k, d, i, f)$ so this case is not possible.

Note. A similar reasoning shows that the cases (b, d new nodes; c old) and (c, d new nodes; b old) are also impossible. Hence at most one of the nodes b, c, d may be a new node.

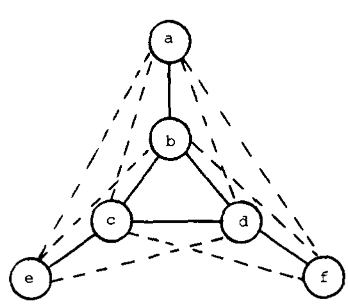

FIG. 4. Configuration $H(a, b, c, d, e, f)$ (dark lines indicate edges and dotted lines nonedges).

Case 3. b is a new node; c, d are old nodes. Let $b=(i, j)$; then $[i, c]$ or/and $[j, c]$ in G. Assume without loss of generality that $[i, c]$. It follows that $[i, d]$, otherwise $[c; i, d, e)$ is a claw. But then G contains $H(0, i, c, d, e, f)$. Hence this case is not possible.

Note. A similar reasoning shows that neither c nor d can be new nodes.

Case 4. The only new node is a. Let $a=(i, j)$; then $[b, i]$ or/and $[b, j]$; assume without loss of generality that $[b, i]$. Then G contains $H(i, b, c, d, e, f)$. This case is again impossible.

Note. A similar argument shows that neither e alone nor f alone can be new nodes.

Case 5. The only new nodes are e and f. Let $e=(i, j)$ and $f=(k, l)$. Assume without loss of generality that $[c, j]$, $[d, k]$. Then G contains $H(a, b, c, d, j, k)$.

We have now examined all cases where H contains at least one new node. Hence G' cannot contain H (so it does not contain A or N as induced subgraphs).

It remains to show that the transform G' of a CAN-free graph contains no claw; we will then have shown that the class of CAN-free graphs is closed under the transformation.

THEOREM 3.2. *Let G' be the transform of a CAN-free graph G, then G' is claw-free.*

Proof. Suppose there is a claw $(a; b, c, d)$ in G'; we will show that this implies the existence of a claw in G. Since G is claw-free, at least one node among a, b, c, d must be a new node. We have several cases to examine.

Case 1. The only new node is a. Let $a=(i, j)$; assume without loss of generality that $[b, i]$; then $\overline{[c, i]}$ (otherwise $(i; 0, b, c)$ is a claw); also $\overline{[d, i]}$ (otherwise $(i; 0, b, d)$ is a claw). Hence $[c, j]$, $[d, j]$, and $(j; 0, c, d)$ is a claw. So this case is not possible.

Case 2. One of b, c, d is a new node and a is an old node. Assume $b=(i, j)$ is a new node; since $[a, b]$, we may assume without loss of generality $[a, i]$; but then $(a; i, c, d)$ is a claw. This case is not possible.

Case 3. a and one node among b, c, d are new nodes. Let $a=(i, j)$, $b=(k, l)$ with i, j, k, l all distinct according to Proposition 2.1. Assume without loss of generality that $[c, i]$; then $\overline{[d, i]}$ (otherwise $(i; 0, c, d)$ is a claw). Hence $[d, j]$ and we have $\overline{[c, j]}$ (otherwise $(j; 0, c, d)$ is a claw). Furthermore $[i, k]$ or $[i, l]$ but not both (otherwise if $\overline{[i, k]}$, $\overline{[i, l]}$ $(0; i, k, l)$ is a claw and if $[i, k]$, $[i, l]$ then $(i; c, k, l)$ is a claw because $\overline{[c, k]}$, $\overline{[c, l]}$). Assume without loss of generality $[i, k]$ and $\overline{[i, l]}$. Then we have $[j, l]$ (otherwise $(0; i, j, l)$ is a claw); also $\overline{[j, k]}$ (otherwise $(j; k, l, d)$ is a claw since $\overline{[k, d]}$, $\overline{[l, d]}$). Then $\overline{[i, j]}$ and $\overline{[j, k]}$ are comparable according to Proposition 2.1 and we have $[c, i]$, $\overline{[c, j]}$, $\overline{[c, k]}$, so $c \in N\overline{[i, j]} - N\overline{[j, k]}$. Hence $\overline{[i, j]} \succ \overline{[j, k]}$ and $N\overline{[i, j]} \neq N\overline{[j, k]}$. So by the construction of G' (i, j) is not a node of G'. Hence a does not exist.

So all cases are impossible; we conclude that G' does not contain any claw if G is CAN-free. ∎

4. Reduction of the Stability Number

We now have to show that the stability number of G' is exactly one less than that of G. This will follow immediately from the two following properties:

Proposition 4.1. *Let S be a nonempty stable set in G; then there exists a stable set S' in G' with $|S'| = |S| - 1$.*

Proof. Let $N'(0) = \{0\} \cup N(0)$; for any stable set S in G, we have $|S \cap N'(0)| \leqslant 2$. If $S \cap N'(0) = \emptyset$, then take $S' = S - x$ for an arbitrary node x of S. If $S \cap N'(0) = \{x\}$, then take $S' = S - x$. If $S \cap N'(0) = \{i, j\}$, then clearly $i, j \neq 0$ and there is a nonedge $\overline{[i, j]}$ in $N(0)$; if (i, j) is a new node of G', then we may take $S' = (S - \{i, j\}) \cup (i, j)$. If (i, j) is not a new node of G', there is, according to step (c) of the construction of G', a new node (k, l) such that $\overline{[i, j]} \succ \overline{[k, l]}$. Notice that $\{k, l\} \not\subseteq S$ since $|S \cap N'(0)| \leqslant 2$. So in the subgraph of G induced by $X - N'(0)$, every neighbour of k or of l is a neighbour of i or of j; such neighbours are not in S since i and j are in S. Hence we may take $S' = (S - \{i, j\}) \cup (k, l)$. Q.E.D.

Proposition 4.2. *Let S' be a stable set in G'; then there exists a stable set S in G with $|S| = |S'| + 1$.*

Proof. Let K' be the clique formed by all new nodes in G'. If $S' \cap K' = \emptyset$, then we may take $S = S' \cup \{0\}$. If $S' \cap K' = \{(i, j)\}$, then we may take $S = (S' - (i, j)) \cup \{i, j\})$ because i and j are not linked in G and since (i, j) was in S', no neighbour of i or of j is in $S' - (i, j)$. Q.E.D.

Corollary 4.1. $\alpha(G') = \alpha(G) - 1$.

It now follows from this result that the stability number of a CAN-free graph can be computed by repeatedly applying the transformation of Section 2. According to the results in Section 3, we will have CAN-free graphs at each step, the stability number will decrease by one at each step and the number of nodes will also decrease.

Since the transformation will be applied at most n times, when n is the number of nodes in G and since the construction of G' has complexity $O(n^2)$, we have a complexity $O(n^3)$.

The general construction described in [1] can be adapted for dealing with the weighted case (without giving a polynomial time algorithm). There is apparently no straightforward extension of the construction described here for weighted graphs.

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