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Finite dimensionality and regularity of attractors for a 2-D semilinear wave equation with nonlinear dissipation

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Abstract

We consider a semilinear wave equation, defined on a two-dimensional bounded domain Ω, with a nonlinear dissipation. Our main result is that the flow generated by the model is attracted by a finite dimensional global attractor. In addition, this attractor has additional regularity properties that depend on regularity properties of nonlinear functions in the equation. To our knowledge this is a first result of this type in the context of higher dimensional wave equations.

Keywords: Semilinear wave equation; Nonlinear dissipation; Global attractors; Finite dimensional attractors; Regularity of attractors

1. Introduction

We consider the following semilinear wave equation defined on a bounded, sufficiently smooth domain $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} &w_{tt} + g(w_t) - \Delta w + f(w) = 0, \quad \text{in } \Omega \times (0,\infty), \\ &w|_{\partial\Omega\times(0,\infty)} = 0, \end{aligned} \tag{1}$$

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with the initial conditions $w(0) = w_0$, $w_t(0) = w_1$, in Ω.

The following standing assumptions are imposed on the nonlinear functions f, g:

Assumption 1. *g is strictly increasing, $g(0) = 0$ and there exist positive constants m, M, N, p, q such that*

$$\begin{aligned}
&g'(s) \geqslant m > 0 \quad \textit{for } |s| \geqslant 1,\\
&|g'(s)| < M\big[s^{q-1} + 1\big] \quad \textit{for all } s,\ 1 \leqslant q < \infty; \qquad (2)\\
&f \in \mathcal{C}^1(\mathbb{R}), \qquad \lim_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1,\\
&|f'(s)| \leqslant N\big[1 + |s|^{p-1}\big] \quad \textit{for } s \in R,\ 1 \leqslant p < \infty, \qquad (3)
\end{aligned}$$

where λ_1 denotes the first eigenvalue of Δ with zero Dirichlet data.

Since Eq. (1) is a locally Lipschitz perturbation of a monotone second-order equation defined in $\mathcal{H} \equiv H_0^1(\Omega) \times L_2(\Omega)$, the existence of unique local (in time) solutions follows from standard nonlinear semigroup theory [1,2]. The ultimate dissipativity condition imposed on f guarantees an existence of *a priori* bounds in finite energy space $\mathcal{H}$. Thus local solutions become global and we are in a position to define a continuous semiflow $T(t)$ on $\mathcal{H}$ by the formula

$$\begin{aligned}
&T(t)(w_0, w_1) = (w(t), w_t(t)), \quad t \geqslant 0,\\
&(w_0, w_1) \in \mathcal{H} \equiv H_0^1(\Omega) \times L_2(\Omega),
\end{aligned}$$

where $w(t)$ satisfies Eq. (1).

The main goal of this paper is to study long-time behaviour of $T(t)$ and, in particular, questions related to the existence, regularity, and dimensionality of global attractors (we use here the standard definition of global attractor as in [3–5]).

If the nonlinear term f is dissipative (i.e., $f(s)s \geqslant 0$), then the flow $T(t)$ is uniformly stable, and $T(t)$ converges to 0 in the operator norm with the rates which depend on the behaviour of a function $g(s)$ at the origin. In fact, these rates can be calculated exactly (see [6]) by solving an appropriate nonlinear ODE. In such a case, asymptotic behaviour of the flow $T(t)$ is very simple and the attractor collapses to a single equilibrium. If, instead, the nonlinear function f is subject to a more general condition as in (3), asymptotic behaviour of the flow is more complex and is confined (as we shall see) to an appropriate global attractor. Our main aim is to show that the asymptotic behaviour of the flow is *finite dimensional*. Here is our main result:

Theorem 1. *With reference to the flow $T(t)$ associated with* (1), *subject to Assumption* 1:

(1) (*Compactness*). *There exist a global, compact attractor* $\mathcal{A} \in \mathcal{H}$ *whose size is independent of the dissipation parameter* $m \geqslant m_0 > 0$ (*see Theorem* 4).
(2) (*Regularity*). *Under the additional growth conditions*

$$g'(s) \geqslant m > 0 \quad \text{for } |s| \leqslant 1, \quad \text{and}$$
$$g(s)s \geqslant m|s|^l \quad \text{for } |s| > 1 \text{ with } l > q - 1$$

the attractor $\mathcal{A}$ *has the additional regularity*: $\exists R > 0$, *such that* $\mathcal{A} \in B_{H^2(\Omega)\times H^1(\Omega)}(0, R)$, *where* $B_X(0, R)$ *denotes a ball in* X *with a radius* R (*see part* (1) *in Theorem* 5).
If, in addition $q = 1$ *and* $0 < m \leqslant g'(s) \leqslant M < \infty$, *then* R *does not depend on the dissipation parameter* m , *but it may depend on* M/m (*see part* (2) *of Theorem* 5).
(3) (*Finite dimensionality*). *In addition to previous hypotheses, we assume that* $|g''(s)| \leqslant C$, *for* $s \in R$. *Then the Hausdorf dimension of the attractor* $\mathcal{A}$ *is finite* (*see Theorem* 8).

Remark 1.1. The assumption that the domain is two-dimensional is critical only for the proof of finite dimensionality of the attractor. Compactness and also regularity of the attractor can be proved in the n-dimensional case subject to suitable growth conditions imposed on g, f—including the cases of critical Sobolev's exponents for the function f.

An interesting question is that of C^∞ regularity of attractors. The regularity of attractors, besides being an important property in the theory of dynamical systems, is also a very fundamental issue in the context of numerical approximations and construction of inertial manifolds. This type of additional regularity is typical for dynamics with an inherent smoothing effect, e.g., parabolic like systems. In the hyperbolic case, the C^∞ regularity of attractors is known [7] for equations with *linear dissipation* only. In fact, the linearity of dissipation is used critically in the proofs [7]. In what follows we shall show that if the parameter representing dissipation is sufficiently large, the C^∞ regularity of attractors is also enjoyed by hyperbolic flows with nonlinear dissipation. In order to state our result we introduce parameter $l_g \equiv \sup_{s\in R}[|g''(s)| + |g'(s) - g'(0)|]$.

Theorem 2 (C^∞ regularity). *In addition to assumptions in part* (2) *of Theorem* 1, *we assume that* $f, g \in C^\infty(R)$ *and the following condition holds*: $l_g/m \ll 1$. *Then the flow on the attractor is infinitely many times differentiable. More precisely,* $\mathcal{A} \in B_{H^n(\Omega)\times H^{n-1}(\Omega)}(0, R_m)$, $n \geqslant 3$, *where* R_m *may depend on the parameter of dissipation* m. (*See Theorem* 7.)

Note that the additional assumption imposed by Theorem 2 is always satisfied if we replace $g(s)$ by $g(s) + Cs$ for a sufficiently large value of C. The above observation leads to the following corollary.

Corollary 1.2. *There exists a constant* $C_0 > 0$ *such that the original dynamics with the damping* $g(s) + Cs$*, where* $g(s)$ *satisfies the assumptions of Theorem* 1 *and* $C \geqslant C_0$ *generates* C^∞ *flow on the attractor.*

The above corollary leads to the following control-theoretic interpretation: one may control the smoothness of attractors by adding a linear velocity feedback to the original nonlinear system.

Proofs of these two theorems follow from six theorems: 3, 4, 5, 6, 7, and 8 presented in the main body of the paper.

We would like to say a few words about the literature related to this problem. There is a large literature devoted to the stability and existence of global attractors for semilinear wave equations [3,5,7–12]. The majority of the results in the literature deal with the case of *linear* dissipation $g(s)$. In such case, the existence of compact attractors is known—see [3,5,8] and references therein. A more delicate problem, in the hyperbolic case, is that of finite dimensionality and regularity of attractors. While these properties are typical of parabolic-like flows with an inherent smoothing mechanism [13], in the hyperbolic flows regularity and finite dimensionality are much less expected (see [14]). This is due to the lack of smoothing effect propagated by the original dynamics and related spectral distribution where infinitely many eigenvalues of the linearization lie on a vertical line in the complex plane [14]. The very first results establishing finite dimensionality for a wave equation with *linear dissipation* are in [7,15], and later in [14]. Linearity of the dissipation is used critically in all the arguments pertaining to regularity and finite dimensionality of the attractor.

If the dissipation is nonlinear, the situation becomes, as noted in the literature [3,10,16,35,36], much more subtle. In order to recognize the difficulty, it suffices to realize that in hyperbolic problems dissipation, to be effective (i.e., to change the essential spectrum of linearization), cannot be relatively compact [17,18]. Thus, the dissipative term in the equation belongs to the main part of the operator. If this term is, in addition, nonlinear, it becomes very sensitive with respect to any perturbation type of argument (used typically in the study of attractors and their properties).

In the case of nonlinear dissipation, the results available in the literature [3, 8,19–21] provide an existence of *global attractors* under hypothesis $0 < m \leqslant g'(s) \leqslant M,\ s \in R$. If there is no upper bound on the derivative of g, but there is a structural hypothesis relating the growth of g with respect to the growth of f, the existence of a compact global attractor is proved in [11,22]. However, there are virtually no results in the literature dealing with *regularity and finite dimensionality* of attractors in the context of higher (than one) dimensional wave equation with *nonlinear* dissipation (some restricted regularity of attractors for semilinear wave equations with nonlinear boundary damping is proved in [38]). The reason for this is simple: since the flow is *not differentiable* with respect to finite energy topology, the standard methods for estimating the dimension of the attractor are not applicable [3,23,24].

For this reason alone the problem of *finite dimensionality* and regularity of attractors for wave equations with nonlinear dissipation has been an open problem in the literature. The only result existing (to the best of our knowledge) is for a one-dimensional wave equation [16], where strong Sobolev's embeddings $H^1(\Omega) \subset C(\Omega)$ available in the 1-D case are critically used. Needless to say the above argument does not apply to higher dimensions. Thus, what we consider as the main contributions of this paper, which concentrates on *nonlinear dissipation*, are (i) finite dimensionality of the attractor in the two-dimensional case (Theorem 8), and (ii) regularity of attractors subject to various regularity assumptions imposed on nonlinear terms f, g (Theorems 5, 6, 7). Our arguments rely critically on the "sharp regularity" of multipliers in Besov's spaces [25].

Let us conclude with a few remarks, pointing out some open problems and future directions worth pursuing. Note that while the size of the attractor, when measured in finite energy space $\mathcal{H}$, does not depend on the dissipation parameter $m > 0$, its regularity measured in higher Sobolev's norms does. More precisely, the size of the attractor when measured in $H^3(\Omega) \times H^2(\Omega)$ (or higher norms) depends on m and it may increase when the dissipation m becomes larger. On the other hand, large values of the damping parameter m are responsible for "fast" decay rates to the attractor (see Lemma 3.2). This raises an interesting question on how to "optimize" the damping parameter in order to achieve "good" attractiveness properties of the attractor along with reasonable regularity.

The related issue is that of finite dimensionality versus damping. Optimization of the damping parameter m, in order to obtain the lowest estimate for the dimension of attractor, is interesting and important for applications problems. It is clear that "more damping" does not mean stronger decay properties. In fact, a large value of m leads to the so-called "overdamping"—a phenomenon well known among engineers. We hope that the results of this paper, including specific estimates relating the damping parameter to the properties of the attractor may be a first step toward this type of quantitative analysis.

Notation. In what follows we shall use the following notation:

$(u, v)_\Omega \equiv \int_\Omega u(x)v(x)\,dx$, $|u|_\Omega \equiv |u|_{L_2(\Omega)}$.

$H^s(\Omega)$ are the usual Sobolev's spaces [26]. We recall that $H^{-s}(\Omega) = (H_0^s(\Omega))'$, $s > 0$.

$|u|_{s,\Omega} \equiv |u|_{H^s(\Omega)}$.

$A \equiv -\Delta$, $D(A) = H^2(\Omega) \times H_0^1(\Omega)$, and A^r, $0 \leqslant r \leqslant 1$, denote fractional powers of A.

Constants C_i, c_i are generic constants, different in different occurrences. $C(s)$ denotes a function that is bounded for bounded values of the argument.

The remainder of the paper is devoted to the proofs of Theorems 1 and 2. This will be accomplished by proving supporting results in Theorems 3–8.

2. Existence of an absorbing set

At the outset we mention that the main results established in Sections 2 and 3, which deal with absorption and compactness property, can be proved in a more direct (simpler) way than is done in the paper. However, our more extensive treatment provides more information about the flow (such as dependence on the parameter of dissipation and rates of convergence)—which is not strictly necessary in order to conclude the existence of a global attractor. These additional properties will be critically used for the proof of the remaining statements in Theorems 1 and 2.

Theorem 3. *Under Assumption* 1

(i) *There exists an absorbing set* $\mathcal{B}$ *in* $\mathcal{H}$, *for the problem* (1)–(3), *i.e., for all* $R_0 > 0$ *and initial data* $(w_0, w_1) \in \mathcal{H}$ *with the property* $|(w_0, w_1)|_{\mathcal{H}} \leqslant R_0$, *and there exists a* $t_0 = t(R_0)$ *such that*

$$(w(t), w_t(t)) \in \mathcal{B} \quad \text{for } t \geqslant t_0. \tag{4}$$

(ii) *Moreover, the size of the absorbing set does not depend on* m, M *as long as* $m > m_0 > 0$. *This is to say that* $\mathcal{B} \in B_{\mathcal{H}}(0, R)$, *where* R *does* not *depend on* m, M, R_0. *However, the time* t_0 *may depend on* m, M, R_0.

Remark 2.1. The second part of the theorem, which provides control of the size of the absorbing set with respect to the dissipation parameter m, will be critically used in the study of regularity of the attractor.

Proof. Define the following linear and nonlinear energies for the problem:

$$E_w(t) = \frac{1}{2}(w_t(t), w_t(t))_\Omega + \frac{1}{2}(\nabla w, \nabla w)_\Omega, \tag{5}$$

$$\mathcal{E}_w(t) \equiv E_w(t) + (F(w), 1)_\Omega, \quad \text{where } F(x) = \int_0^x f(y)\,dy. \tag{6}$$

Since regular initial data produce smooth solutions, we can freely perform differential calculus on smooth solutions. Final inequalities applicable to finite energy initial data are obtained by the usual density argument.

By multiplying (1) by w_t and integrating by parts, we obtain the dissipativity relation

$$\mathcal{E}_w(0) - \mathcal{E}_w(t) = \int_0^t (g(w_s), w_s)_\Omega \, ds. \tag{7}$$

The following relations between the two energies follow from the assumptions imposed on function f and Sobolev's embeddings

$$c_0 E_w(t) - C_0 \leqslant \mathcal{E}_w(t) \leqslant C(E_w(t)) \tag{8}$$

for suitable positive constants c, C_0 and the function $C(s)$.

We introduce the Lyapunov function

$$V(t) \equiv \mathcal{E}_w(t) + \varepsilon(w(t), w_t(t))_\Omega. \tag{9}$$

Since

$$-\varepsilon c_1 E_w(t) \leqslant \varepsilon(w(t), w_t(t))_\Omega \leqslant \varepsilon C_1 E_w(t),$$

where c_1, C_1 are universal constants, from definition of $V(t)$ and (8) we obtain

$$(c_0 - \varepsilon c_1)E_w(t) - C_0 \leqslant V(t) \leqslant C(E_w(t)) + \varepsilon C_1 E_w(t), \quad t \in R, \tag{10}$$

and taking ε small we conclude that

$$cE_w(t) - C_0 \leqslant V(t) \leqslant C(E_w(t)), \quad t \in R, \tag{11}$$

where c, C_0 are generic constants.

Differentiating (9) with respect to t and substituting (1) yields

$$V_t = (\mathcal{E}_w)_t + \varepsilon|w_t|^2_\Omega + \varepsilon(w, \Delta w - g(w_t) - f(w))_\Omega.$$

After integrating by parts and substituting (7) we obtain

$$\begin{aligned} V_t = {} & -(g(w_t), w_t)_\Omega + \varepsilon|w_t|^2_\Omega - \varepsilon|\nabla w|^2_\Omega \\ & - \varepsilon(w, g(w_t))_\Omega - \varepsilon(w, f(w))_\Omega. \end{aligned} \tag{12}$$

We will define the sets $\Omega_{A(t)}, \Omega_{B(t)} \subset \Omega$, such that $\Omega_{A(t)} = \{x \in \Omega\colon |w_t(t,x)| < 1\}$ and $\Omega_{B(t)} = \{x \in \Omega\colon |w_t(t,x)| \geqslant 1\}$. Since by (2), $|w_t| \leqslant g(w_t)/m$ on $\Omega_{B(t)}$ we obtain that

$$\varepsilon|w_t(t)|^2_\Omega \leqslant \frac{\varepsilon}{m}\big(g(w_t(t)), w_t(t)\big)_\Omega + \varepsilon \int_{\Omega_{A(t)}} |w_t(t)|^2 \, dx. \tag{13}$$

By (3), for all s large enough $-f(s)s < \lambda_1 s^2$. Therefore we can split the term $(w, f(w))_\Omega$ into the term corresponding to the small w and the term corresponding to large w and, using Poincare's inequality and (2) we obtain the estimates

$$-(w, f(w))_\Omega < \lambda_1 |w|^2_\Omega + K_{\Omega,f} \leqslant |\nabla w|^2_\Omega + K_{\Omega,f}, \tag{14}$$

$$\left|(w, g(w_t))_\Omega\right| \leqslant \left|(w, g(w_t))_{\Omega_{A(t)}}\right| + \left|(w, g(w_t))_{\Omega_{B(t)}}\right|, \tag{15}$$

$$\left|(w, g(w_t))_{\Omega_{A(t)}}\right| \leqslant \delta \int\limits_\Omega w^2(t)\,dx + C_\delta M \int\limits_{\Omega_{A(t)}} g(w_t) w_t\,dx. \tag{16}$$

We estimate $|(w, g(w_t))_{\Omega_{B(t)}}|$ using Hölder's inequality, Sobolev's embedding $H^1(\Omega) \subset L_r(\Omega)$ for $r \geqslant 1$ and (2):

$$\begin{aligned}\left|(w, g(w_t))_{\Omega_{B(t)}}\right| &\leqslant \left(\int\limits_{\Omega_{B(t)}} |w|^{\bar r}\right)^{1/\bar r} \left(\int\limits_{\Omega_{B(t)}} |g(w_t)|^r\right)^{1/r} \\ &\leqslant C|w|_{1,\Omega} \left(\int\limits_{\Omega_{B(t)}} |g(w_t) g(w_t)|^{r-1}\right)^{1/r} \\ &\leqslant C|w|_{1,\Omega} \left(\int\limits_{\Omega_{B(t)}} |g(w_t)| M^{r-1} |w_t|^{(r-1)q}\right)^{1/r}. \end{aligned} \tag{17}$$

In (17) we choose $r = 1 + 1/q$. Then using $1/r < 1$ and $g(w_t)w_t > C > 0$ on $\Omega_{B(t)}$, we obtain

$$\begin{aligned}\left|\big(w(t), g(w_t(t))\big)_{\Omega_{B(t)}}\right| &\leqslant C M^{1/(q+1)} |w(t)|_{1,\Omega} \left(\int\limits_{\Omega_{B(t)}} g(w_t(t)) w_t(t)\right)^{1/r} \\ &\leqslant C M^{1/(q+1)} E_w^{1/2}(t) \left(\int\limits_{\Omega_{B(t)}} g(w_t(t)) w_t(t)\right). \end{aligned} \tag{18}$$

By (8), $E_w(t) \leqslant (1/c_0)(\mathcal{E}_w(0) + C_0) \leqslant (1/c_0) C(E_w(0))$. Therefore,

$$\left|(w(t), g\big(w_t(t)\big)\big)_{\Omega_{B(t)}}\right| \leqslant C M^{1/(q+1)} C(E_w(0)) \left(\int\limits_{\Omega_{B(t)}} g(w_t) w_t\right). \tag{19}$$

Thus, from (16) and (19) we derive for $t \geqslant 0$

$$\begin{aligned}(w, g(w_t))_\Omega &\leqslant \delta |w|^2_{0,\Omega} + C_\delta M \int\limits_{\Omega_{A(t)}} g(w_t) w_t \\ &\quad + M^{1/(q+1)} C(E_w(0)) \int\limits_{\Omega_{B(t)}} g(w_t) w_t \\ &\leqslant \delta |w|^2_{0,\Omega} + \big(C_\delta M + M^{1/(q+1)} C(E_w(0))\big)(g(w_t), w_t)_\Omega. \end{aligned} \tag{20}$$

Substituting the estimates (13), (14), and (20) into (12), we obtain for $t \geqslant 0$

$$V_t \leqslant -\left(1 - \frac{\varepsilon}{m} - \varepsilon C_\delta M - \varepsilon M^{1/(q+1)} C(E_w(0))\right)(w_t, g(w_t))_\Omega - \varepsilon|\nabla w|_\Omega^2 + \varepsilon\delta|w|_{0,\Omega}^2 + \varepsilon \int_{\Omega_{A(t)}} |w_t|^2 + \varepsilon K_{\Omega,f}. \tag{21}$$

Using Poincare's inequality, we find a small constant δ such that

$$-\varepsilon|\nabla w|_\Omega^2 + \varepsilon\delta|w|_{0,\Omega}^2 \leqslant -\frac{\varepsilon}{2}|\nabla w|_\Omega^2.$$

Now we choose a small $\varepsilon = \varepsilon(m, M, E_w(0))$ such that

$$1 - \frac{\varepsilon}{m} - \varepsilon C_\delta M - \varepsilon M^{1/(q+1)} C(E_w(0)) \geqslant \frac{1}{2}. \tag{22}$$

Adding and subtracting the term $\epsilon \int_{|w_t|\geqslant 1} |w_t|^2\,dx$ to the right side of (21) yields for $t \geqslant 0$

$$\begin{aligned} V_t(t) &\leqslant -\frac{m}{2} \int_{\Omega_{B(t)}} |w_t|^2\,dx - \varepsilon \int_{\Omega_{A(t)}} |w_t|^2\,dx - \frac{\varepsilon}{2}|\nabla w|_\Omega^2 + \varepsilon K_{\Omega,f} \\ &\leqslant -\frac{\varepsilon}{2} E_w(t) + \varepsilon K_{\Omega,f}, \end{aligned} \tag{23}$$

where $K_{\Omega,f}$ is a generic constant different in various occurrences, and we assume that $m \geqslant m_0 > 0$, so that $\varepsilon \leqslant m_0$. Rescaling the constants we infer that

$$V_t(t) \leqslant -\varepsilon E_w(t) + \varepsilon K_{\Omega,f}, \quad t \geqslant 0, \tag{24}$$

where, we recall, $\varepsilon = \varepsilon(m, m_0, M, E_w(0))$.

From (24), we will derive a bound (independent of the initial conditions and independent of parameters m, M) on $E_w(t)$ for all $t > t_0$, where t_0 is suitably chosen and may depend on the initial energy and also m, M.

Suppose initially that $E_w(t_0) \leqslant 2K_{\Omega,f}$, for some $t_0 > 0$. Then, by (7) and (8) we have that $E_w(t) \leqslant C(K_{\Omega,f})$, $t \geqslant t_0$.

If, instead, $E_w(t) < 2K_{\Omega,f}$, $\forall t \geqslant 0$, than (24) implies that

$$V_t(t) + \varepsilon K_{\Omega,f} \leqslant 0, \quad t \geqslant 0. \tag{25}$$

Hence

$$V(t) + \varepsilon K_{\Omega,f}\, t \leqslant V(0), \quad t \geqslant 0. \tag{26}$$

Letting $t \to \infty$ leads to a contradiction, in view of lower bound on V (11) and finiteness of $V(0)$. Thus, we must have

$$E_w(t) \leqslant C(K_{\Omega,f}), \quad t \geqslant t_0.$$

Theorem is proved. □

3. Compactness property

Theorem 4. *Under Assumption* 1 *there exists a global and compact attractor in* $\mathcal{H}$.

Proof. The proof of Theorem 4 is based on a decomposition of the flow $T(t)$ into two parts: uniformly stable and compact. This is to say that $T(t)(w_0, w_1) = S(t)(w_0, w_1) + K(t)(w_0, w_1)$, where $S(t)$ is a uniformly stable semigroup on $\mathcal{H}$ and the operator $K(t): \mathcal{H} \to \mathcal{H}$ is compact for all $t > 0$. Once this is accomplished the assertion of Theorem 4 follows from Theorem 3.6 in [3]. To this end we apply the decomposition $w = z + u$, where

$$z_{tt} - \Delta z + g(z_t) = f(w), \tag{27}$$

$$z(0) = 0, \quad z_t(0) = 0, \quad z|_{\partial\Omega} = 0 \tag{28}$$

and

$$u_{tt} - \Delta u + g(u_t + z_t) - g(z_t) = 0, \tag{29}$$

$$u(0) = w_0, \quad u_t(0) = w_1, \quad u|_{\partial\Omega} = 0. \tag{30}$$

For a given solution $(w(t), w_t(t)) = T(t)(w_0, w_1)$, Eq. (27) is a standard monotone problem with a forcing term $f(w) \in L_1(0, T; L_2(\Omega))$. Thus, the monotone operator theory [1] yields a unique solvability of (27) with $z \in C([0, T]; H^1(\Omega)) \cap C^1([0, T]; L_2(\Omega))$. Having obtained solution z, we solve Eq. (29), which is, again, a maximal monotone problem driven by the initial conditions $(w_0, w_1) \in \mathcal{H}$. Thus the monotone operator theory provides us with a unique solution $(u, u_t) \in C([0, \infty); \mathcal{H})$. In what follows we need more information on solutions z and u. This is given in the two lemmas stated below.

Lemma 3.1. *With reference to Eq.* (27) *the map* $K(T): (w_0, w_1) \to (z(T), z_T(T))$ *is compact on* $\mathcal{H}$ *for each* $T > 0$.

The second lemma deals with decay rates for the semigroup $S(t)$ defined by

$$S(t)(w_0, w_1) \equiv (u(t), u_t(t)), \tag{31}$$

where $u(t)$ satisfies (29). In order to state this result we need to introduce some notation:

$$g_a(s) \equiv g(a + s) - g(a) \quad \text{for } s, a \in R,$$

$$\hat{g}(s) \equiv \inf_{a \in R} s g_a(s).$$

Since $\hat{g}(s)$ is monotone increasing and zero at the origin, by the construction in [6] (see (1.3) in [6]), there exist a function $h(s)$ that is continuous, concave, and monotone increasing (see [38]), $h(0) = 0$ and such that

$$s^2 + \hat{g}^2(s) \leqslant h(s\hat{g}(s)) \quad \text{for } |s| \leqslant 1.$$

By using function h defined above we can construct, as in [6], an ODE equation describing decay rates for $S(t)$.

To accomplish this we define $h^*(s) \equiv h(s) + s^{2q/(1+2q)} + s^{2/(q+1)}$, $p(x) \equiv [I + h^*]^{-1}(Kx)$, where K is a suitable positive constant.

With $q \equiv I - [I + p]^{-1}$ we define $\mathcal{S}(t, r)$ as a (unique) solution to the ODE

$$\mathcal{S}_t(t, r) + q(\mathcal{S}(t, r)) = 0, \qquad \mathcal{S}(0, r) = E(0), \quad E(0) \leqslant r. \tag{32}$$

We note that due to monotonicity of h, q is strictly monotone increasing and $\mathcal{S}(t, r) \to 0$, $t \to \infty$, as desired.

Now we are ready to state our second lemma, which provides uniform decay rates for solutions to (29) originating in the absorbing set $\mathcal{B}$.

Lemma 3.2. *With reference to Eq.* (29) *we have* $\forall \epsilon > 0$, $\exists C_\epsilon > 0$

$$E_u(t) \leqslant S(t, E_w(0)), \tag{33}$$

where $\mathcal{S}(t, r) \to 0$ *for* $t \to \infty$ *and* $r \leqslant R_0 \leqslant C(\mathcal{B})$ *and it is given above in* (32) *with the constant* K *depending on the size of the absorbing set* $\mathcal{B}$.

If, in addition, the function g *satisfies for some* $\varepsilon > 0$

$$g(s)s \geqslant m|s|^{q-1+\varepsilon}, \quad |s| \geqslant 1, \tag{34}$$

then one can take $h^*(s) = h(s)$. *In such case,* (33) *provides decay rates for solution* u, *which, in turn, depend on the growth of nonlinearity at the origin which is characterized by function* h. *Thus, in particular, if* $g(s)s \geqslant ms^2$, $s \in R$, *the decay rates obtained from* (33) *are exponential.*

Lemma 3.1 and Lemma 3.2 together with Theorem 3.6 in [3] yield the assertion stated in the theorem. Thus it remains to prove both lemmas.

Proof of Lemma 3.1. The proof of the lemma is based on analysis of the following string of maps

$$(w_0, w_1) \Rightarrow w(\cdot) \Rightarrow f(w(\cdot)) \Rightarrow (z(T), z_t(T)), \tag{35}$$

acting between the spaces $\mathcal{H} \Rightarrow C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L_2(\Omega)) \Rightarrow C([0, T]; L_2(\Omega)) \Rightarrow \mathcal{H}$.

Our goal is to show that the superposition of these maps is compact $\mathcal{H} \to \mathcal{H}$.

First, by well-posedness of the original flow $T(t)$ the map $(w_0, w_1) \to w(\cdot)$ is bounded and continuous: $\mathcal{H} \to C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L_2(\Omega))$. On the other hand, by using compactness criterion due to Aubin and Simon [27] together with Sobolev's embeddings we infer that the injection $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L_2(\Omega)) \subset C([0, T]; H_0^{1-\epsilon}(\Omega))$ is compact $\forall \epsilon > 0$. Hence the injection $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L_2(\Omega)) \subset C([0, T]; L_{2/\epsilon}(\Omega))$ is also compact $\forall \epsilon > 0$.

Applying the above result with $\epsilon = 1/p$ implies compactness of the map

$$w \to f(w), \tag{36}$$

$$C\big[(0,T), H_0^1(\Omega)\big] \cap C^1[(0,T), L_2(\Omega)] \to C[(0,T), L_2(\Omega)]. \tag{37}$$

To see this it suffices to show that the map $w \to f(w)$ acting between the spaces $L_{2p}(\Omega) \to L_2(\Omega)$ is bounded and continuous.

However, this follows from differentiability of f together with the growth condition and Sobolev's embeddings. Indeed, growth condition imposed on f' yields the boundedness

$$|f(w)|^2_{L_2(\Omega)} \leqslant C \int_\Omega \big[|w|^{2p} + 1\big]\, dx \leqslant C\big[|w|^{2p}_{L_{2p}(\Omega)} + 1\big].$$

By applying the integral form of the Mean Value Theorem [28] we infer continuity

$$\begin{aligned} &|f(w_n) - f(w)|^2_{L_2(\Omega)} \\ &\quad \leqslant C \int_\Omega |w_n - w|^2\big[|w|^{2p-2} + 1\big]\, dx \\ &\quad \leqslant C|w_n - w|^2_{L_{2p}} |w|^{(p-1)/2}_{L_{2p}} \to 0 \quad \text{as } w_n \to w \text{ in } L_{2p}(\Omega). \end{aligned}$$

Thus the map given in (36) is compact.

In order to assert compactness of the superposition of maps in (35) it suffices to show that the map $f \to (z(T), z_t(T))$ acting on the spaces $L_2((0,T), L_2(\Omega)) \to \mathcal{H}$ is bounded and continuous. Here, we recall z satisfies

$$z_{tt} - \Delta z + g(z_t) = f, \tag{38}$$

$$z(0) = 0, \quad z_t(0) = 0, \quad z|_{\partial\Omega} = 0. \tag{39}$$

However, this follows from the fact that Eq. (38) is a maximal monotone problem driven by a forcing term f. Thus, standard maximal monotone operator theory yields the desired result [1,29]. The proof of Lemma 3.1 is thus complete. □

Proof of Lemma 3.2. This proof follows the method used in [6], which leads to explicit decay rates obtained for the solutions. Although this information is not necessary at the level of asserting compactness of the attractor (uniform convergence of solutions to 0 would suffice), more precise information on decay rates will be needed later in the process of proving finite dimensionality of the attractor.

To proceed with the proof, our first step is, as usual, energy identity:

$$E_u(t) + \int_{t_0}^{t} \int_\Omega \big(g(u_t + z_t) - g(z_t)\big) u_t \, dx\, ds = E_u(t_0). \tag{40}$$

In what follows we introduce the notation $Q \equiv \Omega \times (0,T)$, $\Sigma \equiv \partial\Omega \times (0,T)$ with some (fixed) $T > 0$. Multiplying (29) by u and integrating by parts yields

$$\int_0^T E_u(t)\,dt \leqslant C[E_u(0) + E_u(T)] + C\int_Q |u_t|^2\,dQ + \int_Q (g(u_t + z_t) - g(z_t))u\,dQ. \tag{41}$$

Combining the above inequality with (40) yields

$$\sup_{t\in[0,T]} E_u(t) + \int_0^T E_u(t)\,dt \leqslant C\int_Q (g(u_t + z_t) - g(z_t))u_t\,dQ + C\int_Q |u_t|^2\,dQ + \int_Q (g(u_t + z_t) - g(z_t))u\,dQ. \tag{42}$$

Our task now is to estimate the last two terms in (42). To accomplish this we shall use the following properties of function g_a, obtained directly from the corresponding properties of $g(s)$ (see [38]),

$$sg_a(s) \geqslant m|s|^2, \quad |s| \geqslant 1, \quad \text{and} \quad h(sg_a(s)) \geqslant |s|^2, \quad |s| \leqslant 1, \tag{43}$$

where, we recall, h is concave, increasing, and zero at the origin.

Using the function h along with the properties in (43) we estimate the second term on the RHS of (42)

$$\int_Q |u_t|^2\,dQ \leqslant \int_{Q;\,|u_t|\leqslant 1} |u_t|^2\,dQ + \int_{Q;\,|u_t|\geqslant 1} |u_t|^2\,dQ \leqslant \frac{1}{m}\int_Q (g(u_t + z_t) - g(z_t))u_t\,dQ + \int_Q h(g(u_t + z_t) - g(z_t)u_t)\,dQ. \tag{44}$$

Using Jensen's inequality (associated with concavity of h) and introducing the notation $\mathcal{F} \equiv \int_Q (g(u_t + z_t) - g(z_t))u_t\,dQ$, we obtain from (44)

$$\int_Q |u_t|^2\,dQ \leqslant C[I + h]\mathcal{F}. \tag{45}$$

As for the third term on the RHS of (42) we claim the following inequality:

Proposition 3.3.

$$\int_Q (g(u_t + z_t) - g(z_t))u\,dQ \leqslant 1/2 \sup_{t\in[0,T]} E_u(t) + C(T,\mathcal{B})\big[\mathcal{F}^{2q/(2q+1)} + \mathcal{F}^{2/(q+1)}\big]. \tag{46}$$

Under the additional assumption (34) *we obtain a stronger estimate*

$$\int_Q (g(u_t + z_t) - g(z_t))u\,dQ \leqslant 1/2 \sup_{t\in[0,T]} E_u(t) + C(T,\mathcal{B})\mathcal{F}. \tag{47}$$

Proof of Proposition 3.3. The proof is based on splitting the region Q into three subsets (see [37]): $Q = Q_1 + Q_2 + Q_3$, where

$$Q_1 \equiv \big\{(t,x) \in Q\colon\ |u_t(t,x)| \geqslant 1\big\}, \tag{48}$$

$$Q_2 \equiv \big\{(t,x) \in Q\colon\ |u_t(t,x)| \leqslant 1,\ |z_t(t,x)| \geqslant R,\ |w_t(t,x)| \geqslant R-1\big\}, \tag{49}$$

$$Q_3 \equiv \big\{(t,x) \in Q\colon\ |u_t(t,x)| \leqslant 1,\ |z_t(t,x)| \leqslant R,\ |w_t(t,x)| \leqslant R+1\big\}, \tag{50}$$

where the constant R will be determined later. In order to estimate to the integral term in Proposition 3.3 it suffices to estimate the contribution on each subset Q_i. We begin with standard Holder's inequality where $1/r + 1/\bar{r} = 1,\ r > 1$

$$\int_{Q_i} (g(u_t + z_t) - g(z_t))u\,dQ \leqslant C_\epsilon \left(\int_{Q_i} [g(u_t + z_t) - g(z_t)]^r\,dQ\right)^{2/r} + \epsilon|u|^2_{L_{\bar{r}}(Q_i)}. \tag{51}$$

By Sobolev's embeddings $|u|^2_{L_{\bar{r}}(Q_i)} \leqslant CT^{2/\bar{r}}|u|^2_{C([0,T],H^1(\Omega)|}$, so by rescaling suitably ϵ we obtain

$$\int_{Q_i} (g(u_t + z_t) - g(z_t))u\,dQ \leqslant C_T \left(\int_{Q_i} [g(u_t + z_t) - g(z_t)]^r\,dQ\right)^{2/r} + \frac{1}{2} \sup_{t\in[0,T]} E_u(t). \tag{52}$$

In the arguments below we shall use different values of a constant $r > 1$ for different regions Q_i. Note that due to the absorbing property we obtain

$$|W(t)|_{\mathcal{H}} + |U(t)|_{\mathcal{H}} + |Z(t)|_{\mathcal{H}} \leqslant C(\mathcal{B}).$$

Hence the following *a priori* regularity holds:

$$\int_0^\infty \int_Q g(w_t)w_t(t)\,dt \leqslant C(\mathcal{B}) \quad \text{and} \quad \int_{mT}^{(m+1)T} \int_Q g(z_t)z_t(t)\,dt \leqslant C(\mathcal{B})T,$$
$$m = 0, 1, \ldots . \tag{53}$$

We are ready to estimate contribution of integration over each set Q_i

$$\begin{aligned}
&\left(\int_{Q_1} [g(u_t + z_t) - g(z_t)]^r \, dQ \right)^{2/r} \\
&\quad \leqslant \left(\int_{Q_1} \sqrt{[g(u_t + z_t) - g(z_t)]}[g(u_t + z_t) - g(z_t)]^{r-1/2} \, dQ \right)^{2/r} \\
&\quad \leqslant C\left(\int_{Q_1} [g(u_t + z_t) - g(z_t)]u_t \, dQ \right)^{1/r} \\
&\qquad \times \left(\int_{Q_1} \big[|g(w_t)| + |g(z_t)|\big]^{2r-1} \, dQ \right)^{1/r} \\
&\quad \leqslant C\mathcal{F}^{1/r}\left(\int_{Q_1} \big[|g(w_t)||w_t|^{2q(r-1)} + |g(z_t)||z_t|^{2q(r-1)}\big] \, dQ \right)^{1/r} \\
&\quad \leqslant C\mathcal{F}^{1/r}\left(\int_{Q_1} \big[|g(w_t)w_t| + |g(z_t)z_t| + C\big] \, dQ \right)^{1/r} \\
&\quad \leqslant C(\mathcal{B})T\mathcal{F}^{2q/(2q+1)},
\end{aligned} \tag{54}$$

where after setting $2q(r-1) = 1$ we have applied growth conditions on g and a priori bound in (53). For the second set Q_2 we have (note that r in this case is different than that selected for the Q_1 region)

$$\begin{aligned}
&\left(\int_{Q_2} [g(u_t + z_t) - g(z_t)]^r \, dQ \right)^{2/r} \\
&\quad \leqslant C\left(\int_{Q_2} \big[|g(w_t)|^r + |g(z_t)|^r\big] \, dQ \right)^{2/r}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\Bigg(\int_{Q_2}\big[|g(w_t)||w_t|^{(r-1)q}+|g(z_t)|z_t|^{(r-1)q}\big]\,dQ\Bigg)^{2/r}\\
&\leqslant C\Bigg(\int_{Q_2}\big[|g(w_t)||w_t|^{1-\delta}+|g(z_t)||z_t|^{1-\delta}\big]\,dQ\Bigg)^{2/r}\\
&\leqslant CR^{-2\delta/r}\Bigg(\int_{Q_2}\big[|g(w_t)||w_t|+|g(z_t)||z_t|\big]\,dQ\Bigg)^{2/r}\\
&\leqslant C(\mathcal{B},T)R^{-2\delta q/(q+1-\delta)}, \qquad (55)
\end{aligned}
$$

where we have selected $(r-1)q=1-\delta,\ 0\leqslant\delta\ <1$ and applied again growth conditions imposed on g and *a priori* regularity in (53).

For the last region Q_3 we apply the Mean Value Theorem and the growth condition imposed on g',

$$
\begin{aligned}
&\Bigg(\int_{Q_3}[g(u_t+z_t)-g(z_t)]^r\,dQ\Bigg)^{2/r}\\
&\quad\leqslant\Bigg(\int_{Q_3}[g(u_t+z_t)-g(z_t)]^{r/2}\Bigg(\int_0^1 g'(su_t+z_t)\,dsu_t\Bigg)^{r/2}dQ\Bigg)^{2/r}\\
&\quad\leqslant C\Bigg(\int_{Q_3}\big[|g(u_t+z_t)-g(z_t)|^{r/2}|u_t|^{r/2}\big]\\
&\qquad\times\big[|w_t|^{r(q-1)/2}+|z_t|^{r(q-1)/2}+1\big]\,dQ\Bigg)^{2/r}\\
&\quad\leqslant CR^{(q-1)}\Bigg(\int_{Q_3}\big[(g(u_t+z_t)-g(z_t))u_t\big]^{r/2}\,dQ\Bigg)^{2/r}\\
&\quad\leqslant CR^{(q-1)}\mathcal{F}, \qquad (56)
\end{aligned}
$$

where in the last step we have taken $r=2$.

Now we select (large) R so that

$$R^{q-1}\mathcal{F}=R^{-2\delta q/(q+1-\delta)}.$$

This leads to $R=\mathcal{F}^{1/t}$, where $t=(1+q)(1-q-\delta)/(1+q-\delta)<0$. The above choice leads to

$$R^{q-1}\mathcal{F}=R^{-2\delta q/(q+1-\delta)}=\mathcal{F}^{-2\delta q/(1+q)(1-q-\delta)}.$$

Collecting (54)–(56) with the above choice of parameters and taking δ close to 1 we obtain

$$\left(\int_Q [g(u_t+z_t)-g(z_t)]^r\,dQ\right)^{2/r} \leqslant C(\mathcal{B},T)\big[\mathcal{F}^{2q/(2q+1)}+\mathcal{F}^{2/(1+q)}\big]. \tag{57}$$

Combining (57) with (52) leads to the first statement in Proposition 3.3.

As for the second statement, the argument is much simpler. Indeed, if the additional growth condition (34) is satisfied, we use the integral version of the Mean Value Theorem, which gives $g(u_t+z_t)-g(z_t)=\int_0^1 g'(su_t+z_t)\,ds u_t$. Hence

$$\begin{aligned}
&\int_Q (g(u_t+z_t)-g(z_t))u\,dQ \\
&\quad=\int_Q \sqrt{\int_0^1 g'(su_t+z_t)\,ds\,u_t}\sqrt{\int_0^1 g'(su_t+z_t)\,ds\,u\,dQ} \\
&\quad\leqslant C_\epsilon\int_Q\int_0^1 g'(su_t+z_t)\,ds\,u_t^2\,dQ+\epsilon\int_Q\int_0^1 g'(su_t+z_t)\,ds\,u^2\,dQ \\
&\quad\leqslant C_\epsilon\mathcal{F}+\epsilon\left(\int_Q\big[1+|w_t|^{q-1}+|z_t|^{q-1}\big]^r\,dQ\right)^{1/r}|u|^2_{L_{\bar r}(Q)} \\
&\quad\leqslant C_\epsilon\mathcal{F}+\epsilon C\left(\int_Q\big[1+g(w_t)w_t(t)+g(z_t)z_t(t)\big]^r\,dQ\right)^{1/r}|u|^2_{L_{\bar r}(Q)} \\
&\quad\leqslant C_\epsilon\mathcal{F}+\epsilon C(\mathcal{B})T|u|^2_{C([0,T],H^{1-\delta}(\Omega))}.
\end{aligned} \tag{58}$$

Rescaling ϵ gives the second statement in the proposition. □

Applying the inequality in Proposition 3.3 along with (45) to (42) gives

$$E_u(T)\leqslant C(\mathcal{B},T)[I+h^*]\mathcal{F}. \tag{59}$$

By evoking once more (40) we obtain

$$E_u(T)\leqslant C_{T,\mathcal{B}}[I+h^*]\mathcal{F}\leqslant C_{\mathcal{B},T}[I+h^*](E_u(0)-E_u(T)), \tag{60}$$

$$[I+h^*]^{-1}C^{-1}_{T,\mathcal{B}}E_u(T)+E_u(T)\leqslant E_u(0). \tag{61}$$

Defining $p \equiv [I + h^*]^{-1} C_{T,\mathcal{B}}^{-1}$ we obtain

$$E_u(T) + p(E_u(T)) \leqslant E_u(0).$$

Reiterating the same argument on an arbitrary interval $[mT, (m+1)T]$ yields (note that $p(s)$ is independent of m)

$$E_u((m+1)T) + p\big(E_u((m+1)T)\big) \leqslant E_u(mT) \quad \Rightarrow$$
$$E_u(mT) \leqslant [I + p]^{-m} E_u(0).$$

Now, the conclusion in the lemma follows from comparison with Lemma 3.1 in [6]. □

Lemma 3.1 and Lemma 3.2 imply the statement in Theorem 4. □

4. Regularity of the attractor

In order to prove that the attractor is finite dimensional, the following additional regularity of the attractor plays a critical role.

Theorem 5.

(1) *In addition to Assumption* 1 *we assume that* $g'(s) \geqslant m > 0$, *for* $s \in R$, *and* $g(s)s \geqslant m|s|^l$, $l > q - 1$, *for* $|s| \geqslant 1$. *Then there exists a constant* $R > 0$ (*possibly depending on* m, M) *such that* $\mathcal{A} \in B_{H^2 \times H^1}(0, R)$.
(2) *If* $0 < m_0 \leqslant m \leqslant g'(s) \leqslant M$ *for all* $s \in \mathbb{R}$, *then* R *does not depend on* m, *but it may depend on* M/m.

Proof of Theorem 5. *Proof of part* (1). We set $W = (w, w_t)$, where $(w(t), w_t(t))$ denotes the original trajectory. Since $\mathcal{A} = T(t)\mathcal{A}$ for all $t > 0$, for any point in the attractor $W_0 \in \mathcal{A}$, there is a trajectory $W(t)$ passing trough this point and such that $W(t) \in \mathcal{A}$ for $t \to -\infty$. Therefore, such trajectory is bounded in $\mathcal{H}$ for all $t \in R$ and we can assume that $W(t_0) = W_0$ for some $t_0 > 0$.

We define the difference quotient

$$D_h W(t) \equiv \frac{W(t+h) - W(t)}{h} \quad \text{for } h > 0.$$

Then $D_h w(t)$ satisfies

$$(D_h w)_{tt} - \Delta D_h w + f_1(w(t), h) D_h w + g_1(w_t(t), h)(D_h w)_t = 0$$
$$\text{in } \Omega \times (t_0, \infty),$$
$$D_h w = 0 \text{ on } \partial\Omega \times (t_0, \infty) \quad \text{for } t_0 \in \mathbf{R}, \tag{62}$$

where by the Mean Value Theorem $f_1(w(t),h) \equiv \int_0^1 f'(sw(t+h)+(1-s)\times w(t))\,ds$ and $g_1(w_t,h) \equiv \int_0^1 g'(sw_t(t+h)+(1-s)w_t(t))\,ds$.

Let $T_h(t,s)$ be an evolution on $\mathcal{H}$ generated by the equation

$$\begin{aligned}
&u_{tt} - \Delta u + g_1(w_t(t),h)u_t = 0 \quad \text{in } \Omega \times (t_0,\infty),\\
&u = 0 \quad \text{in } \partial\Omega \times (t_0,\infty),\\
&(u(t_0),u_t(t_0)) = U(t_0) \in \mathcal{H} \quad \text{for some } t_0 \in \mathbf{R},
\end{aligned}$$

and we set $T_h(t,t_0)U(t_0) = U(t) = (u(t),u_t(t))$.

Note that the above equation is linear but time dependent through $g_1(w_t(t),h)$. We shall prove that the evolution $T_h(t,s)$ is exponentially stable with parameters that do not depend on initial condition $U(0)$, the original trajectory W, and the parameter h.

Lemma 4.1. *Under the assumptions stated in Theorem* 5, *there exist constants* $C>0,\ \omega>0$ *depending on the size of the attractor* $\mathcal{A}$ *but independent of* h, W *and such that*

$$|T_h(t,s)|_{\mathcal{L}(\mathcal{H})} \leqslant Ce^{-\omega(t-s)}.$$

Proof of Lemma 4.1. The proof of the lemma parallels the arguments given in Lemma 3.2. We will not repeat all the details, but we will provide the main steps with particular emphasis on points where the arguments of Lemma 3.2 need to be modified.

We begin, as always, with energy identity

$$E_u(t) + \int_{t_0}^{t}\int_{\Omega} g_1(w_t,h)u_t^2\,dx\,ds = E_u(t_0), \tag{63}$$

and we denote $\mathcal{F} \equiv \int_0^T\int_\Omega g_1(w_t,h)u_t^2\,dx\,dt$.

As we shall see, the critical property responsible for exponential decays is the fact that $m \leqslant g_1(w_t,h)$, where the constant m is independent of the solution w_t and on the parameter h.

As in (42) we obtain

$$\sup_{t\in[0,T]} E_u(t) + \int_0^T E_u(t)\,dt \leqslant C\int_Q g_1(w_t,h)u_t^2\,dQ + C\int_Q |u_t|^2\,dQ + \int_Q g_1(w_t,h)u_t u\,dQ. \tag{64}$$

Since $g' \geqslant m$, the inequality in (64) implies that

$$\sup_{t\in[0,T]} E_u(t) + \int_0^T E_u(t)\,dt \leqslant C\Big[\mathcal{F} + \int_Q g_1(w_t,h)u_t u\,dQ\Big]. \tag{65}$$

Thus we need to estimate the last term in (65). This is done as

$$(g_1(w_t,h)u_t,u)_\Omega \leqslant \epsilon\big(g_1(w_t,h),u^2\big)_\Omega + C_\epsilon\big(g_1(w_t,h),u_t^2\big)_\Omega. \tag{66}$$

By using growth conditions on g' and Sobolev's embeddings we obtain

$$\begin{aligned}\big(g_1(w_t,h),u^2\big)_\Omega &\leqslant C\int_\Omega \big(1+|w_t|^{q-1}\big)u^2\,dx \\ &\leqslant C\Big[\int_\Omega \big(1+|w_t|^{r(q-1)}\big)\,dx\Big]^{1/r} |u|^2_{1,\Omega},\end{aligned}$$

where $r^{-1}+\bar r^{-1}=1,\ r>1$. By selecting suitable r, so that $r(q-1)=l>q-1$

$$\big(g_1(w_t,h),u^2\big)_\Omega \leqslant C_{q,m,M}\Big[1+\int_\Omega g(w_t)w_t\,dx\Big]^{1/r} |u(t)|^2_{1,\Omega}, \tag{67}$$

and by Jensen's inequality and Sobolev's embedding

$$\begin{aligned}&\int_0^T \big(g_1(w_t,h),u^2\big)_\Omega\,dt \\ &\quad\leqslant C_{T,q,m,M}\Big[1+\int_0^T\int_\Omega g(w_t)w_t\,dx\,dt\Big]^{1/r} \sup_{t\in[0,T]} E_u(t).\end{aligned} \tag{68}$$

Using the dissipativity relation for the original problem together with the fact that initial data for W originate in $\mathcal{A}$ we infer that $\int_0^T\int_\Omega g(w_t)w_t\,dx\,dt \leqslant C(\mathcal{A})$, which combined with (68) gives

$$\int_0^T \big(g_1(w_t,h),u^2\big)_\Omega\,dt \leqslant C_{T,\mathcal{A},q,m,M} \sup_{t\in[0,T]} E_u(t). \tag{69}$$

Combining (66) and (69) yields

$$\int_0^T (g_1(w_t,h)u_t,u)_\Omega\,dt \leqslant \epsilon C(T,\mathcal{A},q)\sup_{t\in[0,T]} E_u(t) + C_\epsilon \int_0^T \big(g_1(w_t,h),u_t^2\big)_\Omega\,dt \tag{70}$$

and rescaling ϵ

$$\int_0^T (g_1(w_t,h)u_t,u)_\Omega\,dt \leqslant 1/2 \sup_{t\in[0,T]} E_u(t) + C_{\mathcal{A},T,q,m,M}\mathcal{F}. \tag{71}$$

The above estimate when inserted into (65) gives

$$\sup_{t\in[0,T]} E_u(t) + \int_0^T E_u(t)\,dt \leqslant C_{\mathcal{A},T,m,M,q}\mathcal{F}. \tag{72}$$

The same arguments as those given in the proof of Lemma 3.2 imply the final conclusion with function $p(s)$, which is linear. This, in turn, yields

$$|T_h(t,s)U|_{\mathcal{H}} \leqslant Ce^{-\omega(t-s)} \quad \text{for some } \omega>0 \text{ and } t\geqslant s, \tag{73}$$

and the constants C and ω do not depend on w_t and h. However, C and ω do depend on m and M, q and the size of $\mathcal{A}$, i.e., $C(\mathcal{A})$. □

In what follows we shall need the following result which provides the regularity of the function $f_1(w(t),h)$.

Lemma 4.2. *Under Assumption* 1 *imposed on* f *for any* $\delta>0$, *there exist positive constants* $C(\delta,\mathcal{A})$, *independent of* $t\in R$, $h\in[0,1]$ *such that*

$$\big|f_1(w(t),h)D_h w(t)\big|_{0,\Omega} \leqslant \delta|D_h w(t)|_{1,\Omega} + C(\delta,\mathcal{A}). \tag{74}$$

Proof. By applying the assumption imposed on function f followed by Hölder's inequality we obtain

$$\begin{aligned} &\big|f_1(w(t),h)D_h w(t)\big|^2_{L_2(\Omega)} \\ &\quad\leqslant C\int_\Omega \big(1+|w(t+h)|+|w(t)|\big)^{2(p-1)}|D_h w(t)|^2 \\ &\quad\leqslant C\left[\int_\Omega \big(1+|w(t+h)|+|w(t)|\big)^{2\bar{q}(p-1)}\right]^{1/\bar{q}}\left[\int_\Omega |D_h w(t)|^{2q}\right]^{1/q}, \end{aligned} \tag{75}$$

where $1/\bar{q} + 1/q = 1$. Hence

$$\begin{aligned} \left|f_1(w(t),h)D_h w(t)\right|_{L_2(\Omega)} &\leqslant C\big(|w(t)|_{L_{2\bar{q}(p-1)}(\Omega)}\big)|D_h w(t)|_{L_{2q}(\Omega)} \\ &\leqslant C\big(|w(t)|_{H^1(\Omega)}\big)|D_h w(t)|_{H^\delta(\Omega)} \end{aligned} \tag{76}$$

for some small $\delta > 0$, where we have used Sobolev's embeddings $H^\delta(\Omega) \subseteq L_{2q}(\Omega),\ \delta = 1 - 1/q$.

By using the moment inequality

$$|D_h w(t)|_{H^\delta(\Omega)} \leqslant |D_h w(t)|^{\delta}_{H^1(\Omega)} |D_h w(t)|^{1-\delta}_{L_2(\Omega)}$$

followed by Young's inequality we obtain

$$|D_h w(t)|_{H^\delta(\Omega)} \leqslant \delta |D_h w(t)|_{H^1(\Omega)} + C(\delta)|D_h w(t)|_{L_2(\Omega)}.$$

Inserting the above inequality into (76) gives

$$\begin{aligned} &\left|f_1(w(t),h)D_h w(t)\right|_{L_2(\Omega)} \\ &\quad\leqslant C(|w|_{H^1(\Omega)})\big[\delta|D_h w(t)|_{H^1(\Omega)} + C(\delta)|D_h w(t)|_{L_2(\Omega)}\big] \\ &\quad\leqslant C(|w|_{H^1(\Omega)})\big[\delta|D_h w(t)|_{H^1(\Omega)} + C(\delta)|w_t(t)|_{L_2(\Omega)}\big], \end{aligned} \tag{77}$$

which gives the desired inequality in Lemma 4.2 after noting that $|w(t)|_{1,\Omega} \leqslant C$, $\forall t \in R$. $\square$

To proceed with the proof of Theorem 5 we use a variation of the parameters formula applied to Eq. (62). Indeed, the solution of (62) coincides with a solution to the following integral equation (a variation of the constant formula)

$$\begin{aligned} D_h W(t) &= T_h(t,t_0)D_h W(t_0) \\ &\quad + \int_{t_0}^{t} T_h(t,s)\begin{pmatrix} 0 \\ f_1(w(s),h)D_h w(s)\end{pmatrix} ds. \end{aligned} \tag{78}$$

For each fixed h, the function $D_h W(t_0)$ is bounded in $\mathcal{H}$, uniformly for all $t_0 \in \mathbf{R}$. Using (73) we can pass to the limit $t_0 \to -\infty$ (with a fixed h) and obtain the formula

$$D_h W(t) = \int_{-\infty}^{t} T_h(t,s)\begin{pmatrix} 0 \\ f_1(w(s),h)D_h w(s)\end{pmatrix} ds. \tag{79}$$

By applying Lemma 4.1 and Lemma 4.2 to formula (79) we obtain

$$\begin{aligned} \|D_h W(t)\|_{\mathcal{H}} &\leqslant \int_{-\infty}^{t} e^{-\omega(t-s)}\big(\delta\|D_h W(s)\|_{\mathcal{H}} + C_{m,M}(\delta,\mathcal{A})\big)\,ds \\ &\leqslant \delta \sup_{-\infty<s<t}\|D_h W(s)\|_{\mathcal{H}} + C(\delta,\mathcal{A}). \end{aligned} \tag{80}$$

Taking $\delta < 1$ we obtain the bound

$$\|D_h W(t)\|_{\mathcal{H}} \leqslant C_{m,M}(\mathcal{A}) \quad \text{for } t \in \mathbf{R} \text{ and } h \in [0,1].$$

From here, the standard convergence argument gives $\|W_t(t)\|_{\mathcal{H}} \leqslant C_{m,M}(\mathcal{A})$ for all $t \in R$, which, in turn, implies that

$$\|w_{tt}(t)\|_{L_2(\Omega)} + \|w_t(t)\|_{H^1(\Omega)} \leqslant C_{m,M}(\mathcal{A}) \quad \text{for all } t \in \mathbf{R}.$$

In order to obtain H^2 regularity of $w(t)$ we go back to the original equation written as

$$\begin{aligned} &\Delta w(t) = w_{tt}(t) - f(w(t)) - g(w_t(t)), \quad \text{in } \Omega, \\ &w(t) = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Since $|g(w_t)(t)|_{0,\Omega)} \leqslant C(|w_t(t)|_{1,\Omega})$ and $|f(w(t))|_{0,\Omega} \leqslant C(|w_t(t)|_{1,\Omega})$, the term on the RHS of the elliptic equation is in $L_2(\Omega)$ uniformly in $t \in R$, with the constant depending only on $C(\mathcal{A})$. Thus, by elliptic theory we infer that

$$\|w(t)\|_{H^2(\Omega)} \leqslant C_{m,M}(\mathcal{A}) \quad \text{for all } t \in \mathbf{R}. \tag{81}$$

Since any point in the attractor can be identified with some $W(t_1)$, where $W(t)$ is a full trajectory on the attractor to which the argument provided above applies, the proof of the first part of the theorem is completed.

Proof of part (2). In order to establish independence of R with respect to m (for large m) we need to apply a different argument which requires the additional assumption in part (2). With $u = D_h w$ (62) becomes

$$u_{tt} - \Delta u + g_1(w_t, h)u_t + f_1(w,h)u = 0, \quad u|_{\partial\Omega} = 0.$$

Let $V(t) \equiv E_u(t) + \varepsilon(u(t), u_t(t))_\Omega$. Then

$$V_t = (E_u)_t + \varepsilon|u_t|^2_\Omega + \varepsilon(u, \Delta u - g_1(w_t,h)u_t - f_1(w,h)u)_\Omega, \tag{82}$$

$$(E_u)_t = -(g_1(w_t,h)u_t, u_t)_\Omega - (f_1(w,h)u, u_t)_\Omega. \tag{83}$$

Our task is to show that with suitably small ϵ

$$V_t + \varepsilon V \leqslant \varepsilon K_{f_1,\Omega,\mathcal{A},M/m}. \tag{84}$$

We will estimate each term in (82) separately. Using (76) we get

$$\begin{aligned} &\big|(f_1(w,h)u, u_t)_\Omega\big| \\ &\quad \leqslant \mathcal{C}_f(\mathcal{A})|u|_{H^\delta}|u_t|_{0,\Omega} \leqslant \mathcal{C}_f(\mathcal{A})[\varepsilon_0|u|_{1,\Omega} + C_{\varepsilon_0}|u|_{0,\Omega}]|u_t|_{0,\Omega} \\ &\quad \leqslant \frac{\sqrt{\varepsilon}\varepsilon_0}{\sqrt{\varepsilon}}\mathcal{C}_f(\mathcal{A})|u|_{1,\Omega}|u_t|_{0,\Omega} + C_{\varepsilon_0}\mathcal{C}_f(\mathcal{A})\mathcal{C}(\mathcal{A})|u_t|_{0,\Omega} \\ &\quad \leqslant \frac{\varepsilon}{2}|\nabla u|^2_{0,\Omega} + \frac{\varepsilon_0^2\mathcal{C}_f(\mathcal{A})^2}{2\varepsilon}|u_t|^2_{0,\Omega} + \delta(C_{\varepsilon_0}\mathcal{C}_f(\mathcal{A})) + \frac{1}{\delta}|u_t|^2_{0,\Omega}, \end{aligned} \tag{85}$$

$$\big|\varepsilon(g_1(w_t,h)u_t, u)_\Omega\big| \leqslant \frac{\varepsilon}{2}|g_1(w_t,h)u_t|^2_{0,\Omega} + \frac{\varepsilon}{2}|u|^2_{0,\Omega}. \tag{86}$$

From (82) and (83) we obtain

$$V_t + \varepsilon V = -((g_1(w_t,h)-\varepsilon)u_t,u_t)_\Omega - (f_1(w,h)u,u_t)_\Omega - \varepsilon|\nabla u|^2_\Omega - \varepsilon(g_1(w_t,h)u_t,u)_\Omega - \varepsilon(f_1(w,h)u,u)_\Omega + \frac{\varepsilon}{2}|u_t|^2_\Omega + \frac{\varepsilon}{2}|\nabla u|^2_\Omega + \varepsilon^2(u,u_t)_\Omega. \tag{87}$$

Substituting the estimates (85) and (86) into (82), we derive

$$\begin{aligned} V_t + \varepsilon V \leqslant{} & -\big((g_1(w_t,h)-\varepsilon)u_t,u_t\big)_\Omega + \left(\frac{\varepsilon_0^2\mathcal{C}_f(\mathcal{A})^2}{2\varepsilon} + \frac{1}{\delta}\right)|u_t|^2_{0,\Omega} \\ & + \frac{\varepsilon}{2}|\nabla u|^2_{0,\Omega} + \delta(C_{\varepsilon_0}\mathcal{C}_f(\mathcal{A})) - \varepsilon|\nabla u|^2_\Omega + \frac{\varepsilon}{2}|u_t|^2_\Omega \\ & + \frac{\varepsilon}{2}|g_1(w_t,h)u_t|^2_{0,\Omega} - \varepsilon(f_1(w,h)u,u)_\Omega + \frac{\varepsilon}{2}|u|^2_{0,\Omega} \\ & + \frac{\varepsilon}{2}|\nabla u|^2_\Omega + \frac{\varepsilon^2}{2}|u_t|^2_{0,\Omega} + \frac{\varepsilon^2}{2}|u|^2_{0,\Omega} \\ \leqslant{} & -\Bigg(\bigg(g_1(w_t,h) - \frac{\varepsilon}{2}g_1(w_t,h)^2 - \varepsilon \\ & - \frac{\varepsilon^2}{2} - \frac{\varepsilon_0^2\mathcal{C}_f(\mathcal{A})^2}{2\varepsilon} - \frac{1}{\delta}\bigg)u_t,u_t\Bigg)_\Omega \\ & + \varepsilon|u|^2_{0,\Omega} - \varepsilon(f_1(w,h)u,u)_\Omega + \frac{\varepsilon^2}{2}|u|^2_{0,\Omega} \\ & + \delta(C_{\varepsilon_0}\mathcal{C}_f(\mathcal{A})). \end{aligned} \tag{88}$$

Without loss of generality we may assume that m is large. We take $\varepsilon = 1/M$, $\varepsilon_0 = \sqrt{m/M}/4\mathcal{C}_f(\mathcal{A})$, $\delta = 4\varepsilon M/m$. Then

$$-\Bigg(\bigg(g_1(w_t,h) - \frac{\varepsilon}{2}g_1(w_t,h)^2 - \varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon_0^2\mathcal{C}_f(\mathcal{A})^2}{2\varepsilon} - \frac{1}{\delta}\bigg)u_t,u_t\Bigg)_\Omega < 0.$$

Since $|u|^2_{0,\Omega} \leqslant \mathcal{C}(\mathcal{A})$, the remaining terms in (88) can be bounded by a constant

$$\begin{aligned} & \varepsilon|u|^2_{0,\Omega} - \varepsilon(f_1(w,h)u,u)_\Omega + \frac{\varepsilon^2}{2}|u|^2_{0,\Omega} + \delta(C_{\varepsilon_0}\mathcal{C}_f(\mathcal{A})\mathcal{C}(\mathcal{A}))^2 \\ & \quad < \varepsilon K_{\Omega,f,\mathcal{A},M/m}. \end{aligned}$$

Thus (84) is proved. Integrating (84) we obtain

$$V(t) \leqslant e^{-\varepsilon(t-t_0)}V(t_0) + \varepsilon\int_{t_0}^{t} e^{-\varepsilon(t-s)}K_{\Omega,f,\mathcal{A},M/m} \tag{89}$$

$$\begin{aligned} & \leqslant e^{-\varepsilon(t-t_0)}V(t_0) + \frac{\varepsilon}{\varepsilon}\big[1 - e^{-\varepsilon(t-t_0)}\big]K_{\Omega,f,\mathcal{A},M/m} \\ & \leqslant e^{-\varepsilon(t-t_0)}V(t_0) + K_{\Omega,f,\mathcal{A},M/m}. \end{aligned} \tag{90}$$

Taking $t_0 \to -\infty$ we obtain $V(t) \leqslant K_{\Omega,f,\mathcal{C}(\mathcal{A}),M/m}$; hence $E_u(t) \leqslant K_{\Omega,f,\mathcal{C}(\mathcal{A}),M/m}$. The rest of the proof follows, as in part (1), by applying elliptic theory to the static part of the equation. □

Remark 4.3. In the case of *linear* dissipation $g(s)$, additional regularity of the attractor was proved in [7]. In fact, in [7] it was shown that the attractor is C^k, for any $k > 0$ provided sufficient regularity is imposed on a nonlinear function f. In our case instead, due to the nonlinearity of g the additional regularity of the attractor is restricted to "one derivative," and this is regardless of smoothness of f and g. The technical reason for this is due to the fact that the analysis of regularity of the attractor involves linearized *evolution* operators rather than *semigroups*, as in [7]. Due to the hyperbolicity of the problem, the time dependence of the coefficients in the evolution is *rough*, regardless of the smoothness of the nonlinear function g. This prevents further propagation of smoothness into the attractor, a fact that is the main obstacle in dealing with hyperbolic problems and nonlinear dissipation. In order to obtain higher regularity of the attractor one needs to impose further restrictions on the nonlinear function $g(s)$. In the theorem below we shall show that by assuming that the parameter of dissipation m is suitably large, one indeed obtains a higher regularity of $\mathcal{A}$.

Theorem 6. *In addition to assumptions in part* (2) *of Theorem* 5 *we assume that* $f, g \in C^2(R)$ *and moreover* $(1/m)\sup_{s\in R}[|g''(s)| + |g'(s) - g'(0)|] \ll 1$. *Then* $\mathcal{A} \subset H^3(\Omega) \times H^2(\Omega)$. *More precisely,* $\mathcal{A} \in B_{H^3\times H^2}(0, R_m)$, *where* R_m *may depend on the parameter* m.

Remark 4.4. We note that the hypothesis of Theorem 5 is satisfied trivially when function g is linear. In this case this result was proved in [7]. Thus, our result in Theorem 6 generalizes that of [7] to problems with nonlinear dissipation.

Proof. From the assumption imposed by Theorem 6 we infer that $M/m \leqslant C$. Thus part (2) in Theorem 5 implies that for all $W(0) \in \mathcal{A}$ we have that

$$|w_{tt}(t)|_{0,\Omega} + |w_t(t)|_{1,\Omega} \leqslant \mathcal{C}(\mathcal{A}) \quad \text{for all } t \in \mathbb{R}, \tag{91}$$

and the constant $\mathcal{C}$ is independent of m, M. Denote $u \equiv w_{tt}$, $U \equiv (u, w_t)$. Then $u \in \mathcal{C}(R, L_2(\Omega))$ satisfies

$$\begin{aligned} &u_{tt} - \Delta u + g'(w_t)u_t + g''(w_t)u^2 + f'(w)u + f''(w)w_t^2 = 0,\\ &U(t_0) \in L_2(\Omega) \times H_0^{-1}(\Omega). \end{aligned} \tag{92}$$

We find it convenient to rewrite the above equation in the form

$$u_{tt} - \Delta u + g'(0)u_t + \gamma u = R(t), \tag{93}$$

where

$$R(t) \equiv g''(w_t)u^2 + f'(w)w_{tt} + f''(w)w_t^2 + \gamma w_{tt} + [g'(0) - g'(w_t)]u_t \\ \in C\bigl(R; H^{-1}(\Omega)\bigr),$$

where we have used regularity in (91). The positive constant γ will be selected later.

Let $B : D(B) \subset H_0^1(\Omega) \times L_2(\Omega) \to H_0^1(\Omega) \times L_2(\Omega)$ denote the generator of the damped wave equation in (93). This is to say $B(u,z) \equiv [z, \Delta u - \gamma u + g'(0)z]$, $D(B) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

The following stability property is well known

$$\left|e^{Bt}\right|_{\mathcal{L}(\mathcal{H})} \leqslant Ce^{-\omega_{\gamma,m} t}, \tag{94}$$

where the constant $\omega_{\gamma,m}$ is positive. Moreover, an elementary spectral argument shows that by selecting γ proportional to m^2 we can achieve $\omega_{m,\gamma(m)} = m/2$. This is to say, that by calibrating static damping in the wave equation we can obtain decay rates for the semigroup which are of the same magnitude as that of the dynamic damping measured by the constant m.

Since e^{Bt} is a semigroup (B is time independent), the invariance of fractional powers of operator B with respect to the dynamics e^{Bt} implies that

$$\left|e^{Bt}\right|_{\mathcal{L}(\mathcal{H}_1)} \leqslant Ce^{-\omega_{\gamma,m} t}, \tag{95}$$

where $\mathcal{H}_1 \equiv L_2(\Omega) \times H^{-1}(\Omega)$. With the above notation we can write the solution to (93) via a variation of parameters formula, first on the space $\mathcal{H}_1$,

$$U(t) = e^{B(t-t_0)}U(t_0) + \int_{t_0}^{t} e^{B(t-s)} \begin{bmatrix} 0 \\ R(s) \end{bmatrix} ds. \tag{96}$$

Letting $t_0 \to -\infty$ and exploiting the exponential stability in (95) together with *a priori* regularity of $R(t)$ stated above yields the formula

$$U(t) = \int_{-\infty}^{t} e^{B(t-s)} \begin{bmatrix} 0 \\ R(s) \end{bmatrix} ds, \tag{97}$$

which gives a representation of solutions on the attractor with respect to weaker topology in $\mathcal{H}_1$. Our main task is to show that the above formula defines, subject to the assumptions stated, elements with higher regularity, i.e., in $\mathcal{H}$. To see this we shall estimate $\mathcal{H}$ norms of expressions in (97). By appealing to (94) we obtain the *a priori* estimate

$$|U(t)|_{\mathcal{H}} \leqslant C \int_{-\infty}^{t} e^{-\omega_{m,\gamma(m)}(t-s)} |R(s)|_{L_2(\Omega)}\, ds \\ \leqslant C \frac{1}{\omega_{m,\gamma(m)}} \sup_s |R(s)|_{0,\Omega}. \tag{98}$$

We shall next estimate the term $R(t)$.

Proposition 4.5.

$$\begin{aligned}
&|f'(w)w_{tt}|_{0,\Omega} + \left|f''(w)w_t^2\right|_{0,\Omega} + \gamma|w_{tt}|_{0,\Omega} \\
&\quad \leqslant \mathcal{C}(\mathcal{A}) + \mathcal{C}_\gamma(A) = C_m(\mathcal{A}), \\
&\left|[g'(0) - g'(w_t)]u_t\right|_{0,\Omega} \leqslant l_g|u_t|_{0,\Omega}, \\
&\left|g''(w_t)u^2\right|_{0,\Omega} \leqslant |g''|_{L^\infty}|u|_{1,\Omega}^{1/2}|w_{tt}|_{1,\Omega}^{1/2} \leqslant \mathcal{C}(\mathcal{A})|g''|_{L^\infty}|u|_{1,\Omega}.
\end{aligned} \tag{99}$$

Proof. The first two estimates follow directly from properties of function f, g and the improved regularity stated in part (2) of Theorem 5 (see also (91)). For the last estimate

$$\begin{aligned}
\left|g''(w_t)u^2\right|_{0,\Omega} &\leqslant C|g''|_{L^\infty}|u|_{L_4(\Omega)}^2 \leqslant C|g''|_{L^\infty}|u|_{1/2,\Omega}^2 \\
&\leqslant C|g''|_{L^\infty}|u|_{1,\Omega}|u|_{0,\Omega} \leqslant \mathcal{C}(\mathcal{A})|g''|_{L^\infty}|u|_{1,\Omega}|w_{tt}|_{0,\Omega} \\
&\leqslant \mathcal{C}(\mathcal{A})|g''|_{L^\infty}|u|_{1,\Omega},
\end{aligned} \tag{100}$$

where we have used Gagliardo–Nirenberg inequality, the moment inequality [30], and (91). □

Applying the result of Proposition 4.5 and denoting $l_g \equiv |g''|_{L^\infty} + \sup_s |g'(s) - g'(0)|$, we obtain

$$|R(t)|_{0,\Omega} \leqslant C(\mathcal{A})l_g\left[|u(t)|_{1,\Omega} + |u_t(t)|_{0,\Omega}\right] + C_m(\mathcal{A}).$$

Combining the above inequality with (98) yields

$$\begin{aligned}
|U(t)|_{\mathcal{H}} &\leqslant \mathcal{C}(\mathcal{A})\frac{1}{\omega_{m,\gamma(m)}}l_g \sup_t\left[|u(t)|_{1,\Omega} + |u_t(t)|_{0,\Omega}\right] + C_m(\mathcal{A}) \\
&\leqslant \mathcal{C}(\mathcal{A})\frac{1}{\omega_{m,\gamma(m)}}l_g \sup_t |U(t)|_{\mathcal{H}} + C_m(\mathcal{A}).
\end{aligned} \tag{101}$$

Since $l_g/m \ll 1$ and $\omega_{m,\gamma(m)} \geqslant m/2$, we obtain that $l_g/\omega_{m,\gamma(m)} \ll 1$. Therefore taking the supremum over t in (101) we obtain

$$\sup_{t\in R}\left[|u(t)|_{1,\Omega} + |u_t(t)|_{0,\Omega}\right] \leqslant \mathcal{C}_m(\mathcal{A}). \tag{102}$$

Now, going back to the elliptic problem, $\Delta w = w_{tt} - g(w_t) - f(w)$, in Ω, $w|_{\partial\Omega} = 0$, and using the improved regularity of w_{tt} and w_t from Theorem 5, we obtain that $\Delta w \in C(R; H_0^1(\Omega))$. From elliptic theory it follows that $w \subset C(R; H^3(\Omega))$. Thus $|w(t)|_{3,\Omega} \leqslant \mathcal{C}_m(\mathcal{A})$, as desired. □

Now we will show that under conditions of Theorem 6 with $f, g \in C^\infty(R)$, we obtain the $\mathcal{C}^\infty$ regularity of the attractor.

Theorem 7. *In addition to assumptions in Theorem* 6 *we assume that* $f, g \in C^n(R)$. *Then* $\mathcal{A} \subset H^n(\Omega) \times H^{n-1}(\Omega)$ *for all* $n > 1$. *More precisely,* $\mathcal{A} \in B_{H^n \times H^{n-1}}(0, R_m)$, *where* R_m *may depend on the dissipation parameter* m.

Proof. The statement of the theorem follows via the boot-strap argument. We shall show that one obtains $|w_t(t)|_{3,\Omega} + |w(t)|_{4,\Omega} \leqslant C_m$, $t \in R$. The above regularity will be "boot strapped" to a higher level.

After third differentiation (in distributional sense) of Eq. (1), we obtain an equation which allows for the same proof as in Theorem 6. Denoting $v \equiv w_{ttt}$, $u = w_{tt}$, $V = (v, v_t)$ we obtain

$$V(t) = \int_{-\infty}^{t} e^{B(t-s)} \begin{pmatrix} 0 \\ R(s) + [g'(0) - g'(w_t(s))]v_t(s) \end{pmatrix} ds, \tag{103}$$

where $R(s) \equiv f'''(w)w_s^3 + 3f''(w)w_s u(s) + f'(w)v - g'''(w_s)u(s)^3 - 3g''(w_t)uv + \gamma_m w_{ttt}$.

After taking $\mathcal{H}$ norms of both sides and using (73) we obtain

$$|V(t)|_{\mathcal{H}} \leqslant \int_{-\infty}^{t} e^{-\omega_{m,\gamma}(t-s)} \big[|R(s)|_{0,\Omega} + l_g |v_t(s)|_{0,\Omega} \big] ds. \tag{104}$$

Using regularity properties of Theorem 6, the following estimates are straightforward, $\forall t \in R$,

$$\begin{aligned} |R(t)|_{0,\Omega} &\leqslant \big| \big[f'''(w)w_s^3 + 3f''(w)w_s u + g'''(w_s)u^3 + f'(w)v \\ &\qquad + g''(w_t)uv + \gamma_m v \big](t) \big|_{0,\Omega} \\ &\leqslant C_{m,\mathcal{A}}. \end{aligned} \tag{105}$$

After substituting the estimates (105) into (104) we derive

$$\left(1 - \frac{l_g}{m}\right) |V(t)|_{\mathcal{H}} \leqslant C_{m,\mathcal{A}}, \quad t \in R,$$

and thus we obtain that

$$|w_{ttt}(t)|_{1,\Omega} + |w_{tttt}(t)|_{0,\Omega} \leqslant C_{m,\mathcal{A}}, \quad t \in R. \tag{106}$$

The above implies, by using the structure of the wave equation, that $|\Delta w_t(t)|_{1,\Omega} \leqslant C_m$, hence by elliptic regularity $|w_t(t)|_{3,\Omega} \leqslant C_m$. Interpolating the above regularity with (106) yields $|w_{tt}(t)|_{2,\Omega} \leqslant C_m$. Applying Δ to both sides of the wave equation and reading off the regularity of the nonlinear terms gives $|\Delta^2 w(t)|_{0,\Omega} \leqslant C_m$, $w = \Delta w = 0$ on $\partial\Omega$. By elliptic regularity applied to this biharmonic problem we conclude that $|w(t)|_{4,\Omega} \leqslant C_m$, $t \in R$, as desired.

This argument can be reiterated to yield a higher order regularity. □

5. Finite dimensionality of the attractor

Theorem 8. *Under the assumptions of the first part of Theorem* 5, *and, moreover,* $|g'(s)| + |g''(s)| \leqslant M$, *the attractor* $\mathcal{A}$ *has a finite Hausdorff dimension.*

The proof of the theorem relies on application of the theorem due to Ladyzhenskaya, which we recall below.

Theorem 9 ([31, Theorem 1.1]). *Let* M *be a compact set of a Hilbert space* H *and assume that on it there is the defined transformation* $V : M \to V(M) \subset H$ *such that* $M \subseteq V(M)$. *Moreover, assume that for some* $n > 0$ *and for any points* v *and* $\tilde{v}$ *of the set* M *one has*

$$\begin{aligned} &|V(v) - V(\tilde{v})|_H \leqslant l|v - \tilde{v}|_H, \\ &|Q_n V(v) - Q_n V(\tilde{v})|_H \leqslant \delta|v - \tilde{v}|_H, \quad \delta < 1, \end{aligned} \tag{107}$$

where Q_n *is the projection onto a subspace of codimension* n. *Then the set* M *has a zero* α *measure for* $\alpha > \alpha_0$ (*see* [31] *for the definition of* α *measure*) *and for any*

$$\alpha > \alpha_0 \equiv \frac{n \ln[2c^2 l^2/(1-\delta^2)]}{\ln[2/(1+\delta^2)]},$$

its Hausdorff dimension does not exceed α (*the constant* c *is an absolute constant*).

Proof of Theorem 8. We shall apply Theorem 9 with $M \equiv \mathcal{A}$, $V(v) \equiv T(t_0)W$, where $T(t_0)W$ denotes the nonlinear flow at the time t_0 associated with the original equation corresponding to the initial datum $W \in \mathcal{A}$. In view of the already-established compactness of the attractor $\mathcal{A}$, the result of Theorem 8 will follow from Theorem 9 as soon as we demonstrate validity of the following estimates holding for any two points W and $\widehat{W}$ in the attractor $\mathcal{A}$

$$\left|T(t_0)W - T(t_0)\widehat{W}\right|_{\mathcal{H}} \leqslant l\left|W - \widehat{W}\right|_{\mathcal{H}}, \tag{108}$$

$$\left|Q_n\left[T(t_0)W - T(t_0)\widehat{W}\right]\right|_{\mathcal{H}} \leqslant \delta\left|W - \widehat{W}\right|_{\mathcal{H}} \tag{109}$$

for some $n > 0$, $t_0 > 0$, $0 < \delta < 1$, and $l < \infty$. Q_n is an orthogonal projection of $\mathcal{H}$ on a suitably selected subspace of codimension n.

To accomplish this, we denote $W(t) \equiv T(t)W$, $\widehat{W}(t) \equiv T(t)\widehat{W}$ and use the decomposition

$$T(t)W - T(t)\widehat{W} = U(t) + Z(t), \tag{110}$$

where $u(t)$ and $z(t)$ satisfy

$$\begin{cases} u_{tt} - \Delta u + g(u_t + \hat{w}_t) - g(\hat{w}_t) = 0 \quad \text{in } Q, \\ u = 0 \quad \text{on } \Sigma, \\ (u(0), u_t(0)) = W(0) - \widehat{W}(0) \quad \text{in } \Omega \end{cases} \tag{111}$$

and

$$\begin{cases} z_{tt} - \Delta z + g(w_t) - g(w_t - z_t) = f(w) - f(\hat{w}) \quad \text{in } Q, \\ z(0) = z_t(0) = 0 \quad \text{in } \Omega, \\ z = 0 \quad \text{on } \Sigma. \end{cases} \tag{112}$$

Verification of condition (108) is straightforward. It follows from the monotonicity of the dissipation and local Lipschitz property of f together with the *a priori* bound of solutions. The crux of the proof is to establish the validity of (109). To this task we devote the rest of the paper.

Since $g(u_t + \hat{w}_t) - g(\hat{w}_t)$ is monotone, we can appeal to Lemma 3.2 in Section 2 in order to conclude that $|U(t)|_{\mathcal{H}} \leqslant \mathcal{S}(t) \to 0,\ t \to \infty$ for all $W(0) \in \mathcal{A}$. We recall that $\mathcal{S}(t)$ is given by (32) and the convergence is uniform for all $W(0) \in \mathcal{A}$. This, in particular, implies that $\forall \epsilon > 0$ there exist $t_0 > 0$ such that

$$|U(t_0)|_{\mathcal{H}} \leqslant \epsilon \big|W(0) - \widehat{W}(0)\big|_{\mathcal{H}}. \tag{113}$$

Thus, in order to show that Lipschitz condition (109) holds, it suffices to prove this condition for $Z(t)$. The main estimate responsible for this is the following: Lipschitz regularity for the the variable z. In what follows we will use the notation $A = -\Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

Lemma 5.1. *For all* $0 \leqslant r < 1/2$ *we have*

- $|Z_t(t)|_{\mathcal{H}} \leqslant C_T(\mathcal{A}),\ t \leqslant T,$
- $|Z(t)|_{D(A^{1/2+r}) \times D(A^r)} \leqslant C_T |W(0) - \widehat{W}(0)|_{\mathcal{H}},\ t \leqslant T.$

Proof of the Lemma 5.1. This proof is based on the following propositions.

Proposition 5.2.

- $|f(w(t)) - f(\hat{w}(t))|_{1,\Omega} \leqslant C(\mathcal{A})|W(0) - \widehat{W}(0)|_{\mathcal{H}},$
- $|(d/dt)[f(w(t)) - f(\hat{w}(t))]|_{\mathcal{H}} \leqslant C(\mathcal{A}).$

Proof of Proposition 5.2. This proof follows from the growth condition assumed on f along with critical use of additional regularity of the attractor established in Theorem 5, i.e., $W(t) \in H^2(\Omega) \times H^1(\Omega)$. $\square$

Returning to the proof of Lemma 5.1, in order to prove the first statement we consider the new variable $\bar{z} \equiv z_t$. It is straightforward to verify that function $\bar{z}$ satisfies the following equation, where time derivatives are understood in the distributional sense:

$$\bar{z}_{tt} - \Delta \bar{z} + g'(w_t - z_t)\bar{z}_t$$
$$= [g'(w_t - z_t) - g'(w_t)]w_{tt} + \frac{d}{dt}[f(w) - f(\hat{w})] \quad \text{in } Q,$$

$$\bar{z}(0) = 0, \quad \bar{z}_t(0) = f(w(0)) - f(\hat{w}(0)) \quad \text{in } \Omega,$$
$$\bar{z} = 0 \quad \text{on } \Sigma. \tag{114}$$

By virtue of the additional regularity stated in Theorem 5 and the assumption $g'(s) \leqslant M$, we infer that

$$\left|[g'(w_t - z_t) - g'(w_t)]w_{tt}\right|_{0,\Omega} \leqslant M|w_{tt}|_{0,\Omega} \leqslant C_M(\mathcal{A})$$

and combining the above with the second statement in Proposition 5.2 yields via the standard energy estimate the inequality in the first part of Lemma 5.1.

The second part is much more involved and requires the following estimate:

Proposition 5.3. *Under the assumptions imposed on function g we obtain with any $r < 1/2$*

$$\left|g(w_t(t)) - g(w_t(t) - z_t(t))\right|_{2r,\Omega} \leqslant C_T(\mathcal{A})|z_t(t)|_{2r,\Omega}, \quad t \leqslant T.$$

Proof of Proposition 5.3. We shall first prove the inequality

$$|g(u+v) - g(u)|_{\alpha,\Omega} \leqslant C[1 + |u|_{1,\Omega} + |v|_{1,\Omega}]|v|_{\alpha,\Omega},$$
$$0 \leqslant \alpha < 1. \tag{115}$$

We begin by quoting from [25, Section 2.3.1] the following result on multipliers in Besov potential spaces $H_p^l(\Omega)$. For $l =$ integer or $p = 2$ $H_p^l(\Omega)$ spaces coincide with classical Sobolev's spaces $W_p^l(\Omega)$ [25,30,32]. By the theorem in [25] with

$$lp < n, \qquad \gamma \in H_{n/l}^l(\Omega) \cap L_\infty(\Omega)$$

we have

$$\gamma \in M\big(H_p^l(\Omega)\big),$$

where $M(H)$ are spaces of multipliers [25]. In addition, we have control of the norms, i.e.,

$$|\gamma v|_{H_p^l(\Omega)} \leqslant C\big[|\gamma|_{L_\infty(\Omega)} + |\gamma|_{H_{n/l}^l(\Omega)}\big]|v|_{H_p^l(\Omega)}.$$

Applying the above result with $l = \alpha < 1$, $p = 2$, $n = 2$ (so that $lp < n$ holds true), we obtain

$$|\gamma v|_{H_2^\alpha(\Omega)} \leqslant C|v|_{H_2^\alpha(\Omega)}\big[|\gamma|_{L_\infty(\Omega)} + |\gamma|_{H_{n/\alpha}^\alpha(\Omega)}\big]. \tag{116}$$

From [30, p. 206, formula (15)] we have

$$H_2^1(\Omega) \subset H_{n/\alpha}^\alpha(\Omega), \quad \alpha < 1.$$

Hence

$$|\gamma|_{H_{n/\alpha}^\alpha(\Omega)} \leqslant C|\gamma|_{H_2^1(\Omega)} \tag{117}$$

and

$$|\gamma v|_{H_2^\alpha(\Omega)} \leqslant C|v|_{H_2^\alpha(\Omega)}\big[|\gamma|_{L_\infty(\Omega)} + |\gamma|_{H_2^1(\Omega)}\big]. \tag{118}$$

Since Besov potential spaces coincide with Sobolev's spaces when $p = 2$ by combining (118) and (117) we obtain

$$|\gamma v|_{\alpha,\Omega} \leqslant C|v|_{\alpha,\Omega}\big[|\gamma|_{L_\infty(\Omega)} + |\gamma|_{1,\Omega}\big]. \tag{119}$$

We apply next the multiplier's inequality (119) with the following choice of the multiplier γ:

$$\gamma(x) \equiv \int_0^1 g'\big(s(u+v)(x) + (1-s)u(x)\big)\,ds.$$

Then by the integral form of the Mean Value Theorem [28]

$$g(u+v) - g(u) = \gamma v$$

and by (119)

$$|g(u+v) - g(u)|_{\alpha,\Omega} \leqslant C|v|_{\alpha,\Omega}\big[|\gamma|_{L_\infty(\Omega)} + |\gamma|_{1,\Omega}\big]. \tag{120}$$

By using growth conditions imposed on g' and g'' we easily obtain

$$|\gamma|_{L_\infty(\Omega)} \leqslant M,$$
$$|\gamma|_{1,\Omega} \leqslant C\big[|u|_{1,\Omega} + |v|_{1,\Omega} + 1\big].$$

Therefore,

$$|\gamma|_{L_\infty(\Omega)} + |\gamma|_{1,\Omega} \leqslant C\big[1 + |u|_{1,\Omega} + |v|_{1,\Omega}\big], \tag{121}$$

which combined with (120) yields (115).

To complete the proof of Proposition 5.3 it suffices to apply the inequality in (115) with $\alpha = 2r$, to replace v with $z_t(t)$ and to replace u with $w_t(t) - z_t(t)$. This gives

$$\begin{aligned}&\big|g(w_t)(t) - g(w_t(t) - z_t(t))\big|_{2r,\Omega}\\ &\quad\leqslant C|z_t(t)|_{2r,\Omega}\big[|w_t(t)|_{1,\Omega} + |z_t(t)|_{1,\Omega} + 1\big].\end{aligned} \tag{122}$$

Applying the result of Theorem 5 and the estimate in the first part of the Lemma 5.1 we obtain

$$|w_t(t)|_{1,\Omega} + |z_t(t)|_{1,\Omega} \leqslant C_T(\mathcal{A}),$$

which combined with (122) yields Proposition 5.3. □

To continue with the proof of the second part of Lemma 5.1, we take an inner product of equation for z with an element $A^{2r}z_t$. The energy estimate gives

$$\left|A^r z_t(t)\right|^2_{0,\Omega} + \left|A^{1/2+r} z(t)\right|^2_{0,\Omega} \tag{123}$$

$$+ \int_0^t \left(A^r(g(w_t) - g(w_t - z_t)), A^r z_t\right)_\Omega$$

$$+ \left(A^r(f(w) - f(\hat{w})), A^r z_t\right)_\Omega \, dt = 0. \tag{124}$$

Since $D(A^r) = H_0^{2r}$ for $r < 1/4$ [33], we obtain

$$\left|A^r z_t(t)\right|^2_{0,\Omega} + \left|A^{1/2+r} z(t)\right|^2_{0,\Omega} \tag{125}$$

$$\leqslant C \int_0^t |g(w_t) - g(w_t - z_t)|^2_{2r,\Omega} + |f(w) - f(\hat{w})|^2_{2r,\Omega} \, dt. \tag{126}$$

Applying the result of Proposition 5.3 and Proposition 5.2 with $r < 1/4$ we obtain

$$\left|A^r z_t(t)\right|^2_{0,\Omega} + \left|A^{1/2+r} z(t)\right|^2_{0,\Omega} \tag{127}$$

$$\leqslant C_T(\mathcal{A}) \int_0^t \left[|z_t|^2_{2r,\Omega} + C(\mathcal{A})|f(w) - f(\hat{w})|^2_{1,\Omega}\right] dt \tag{128}$$

$$\leqslant C_T(\mathcal{A}) \int_0^t \left|A^r z_t\right|^2_{0,\Omega} dt + C_T(\mathcal{A})\left|W(0) - \widehat{W}(0)\right|^2_{\mathcal{H}}, \tag{129}$$

where in the last inequality we again used the fact [33] that $D(A^r) \subset H^{2r}(\Omega)$, $r < 1/4$.

Gronwall's inequality applied to (127) completes the proof of the second inequality in the lemma. □

Proper Proof of Theorem 8. To complete the proof of (109) (and hence of the theorem) we proceed now as in (e.g., see [34]). We denote by $\mathcal{Q}_n$ a coprojection of $L^2(\Omega)$ onto $V_n \equiv \operatorname{span}(\phi_1, \ldots, \phi_n)$, where ϕ_i are the eigenvectors corresponding to the eigenvalues λ_n of A. Since A is a positive, self-adjoint operator on $L_2(\Omega)$, the following relations are well known [30]:

$$\left|\mathcal{Q}_n A^{-r}\right|_{\mathcal{L}((L_2(\Omega))} = \sup_{|x|_{L_2(\Omega)}=1} \left|\mathcal{Q}_n A^{-r} x\right|_{L_2(\Omega)} \leqslant C{\lambda_n}^{-r}, \tag{130}$$

$$D\left(A^\theta\right) = H_0^{2\theta}(\Omega), \quad \text{for } \theta < \frac{1}{4}. \tag{131}$$

Denoting $Q_n(x, y) \equiv (\mathcal{Q}_n x, \mathcal{Q}_n y)$, and using the commutativity of fractional powers A^α with coprojections $\mathcal{Q}_n$, we obtain the following inequality valid for all $t \geqslant 0$ and $\alpha < 1/4$:

$$\begin{aligned}
&\left|Q_n(z(t), z_t(t))\right|_{\mathcal{H}}^2 \\
&\quad \leqslant C|\mathcal{Q}_n z(t)|_{1,\Omega}^2 + |\mathcal{Q}_n z_t(t)|_{0,\Omega}^2 \\
&\quad \leqslant C\left|\mathcal{Q}_n A^{-r}\right|_{\mathcal{L}(L_2(\Omega))}^2 \left[\left|\mathcal{Q}_n A^{1/2+r} z(t)\right|_{0,\Omega}^2 + \left|\mathcal{Q}_n A^r z_t(t)\right|_{0,\Omega}^2\right] \\
&\quad \leqslant C\left|\mathcal{Q}_n A^{-r}\right|_{\mathcal{L}(L_2(\Omega))}^2 \left[\left|A^{1/2+r} z(t)\right|_{0,\Omega}^2 + \left|A^r z_t(t)\right|_{0,\Omega}^2\right].
\end{aligned} \tag{132}$$

Hence

$$\begin{aligned}
\left|Q_n(z(t), z_t(t))\right|_{\mathcal{H}} &\leqslant C\left|\mathcal{Q}_n A^{-r}\right|_{\mathcal{L}(L_2(\Omega))} |Z(t)|_{D(A^{1/2+r})\times D(A^r)} \\
&\leqslant C_{T,m,M}(\mathcal{A})\left|W(0) - \widehat{W}(0)\right|_{\mathcal{H}} \lambda_n^{-r}, \quad t \leqslant T,
\end{aligned} \tag{133}$$

where we have used the result of Lemma 5.1.

The estimate in (109) follows now from (113) and (133). Indeed, it suffices to take t_0 large enough in (113), then select a sufficiently large n in (133). □

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