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Multiplicity of ground states in quantum field models: applications of asymptotic fields ☆

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Abstract

Ground states of Hamiltonian H of quantum field models are investigated. The infimum of the spectrum of H is in the edge of its essential spectrum. By means of the asymptotic field theory, we give a necessary and sufficient condition for that the expectation value of the number operator of ground states is finite, from which we give an upper bound of the multiplicity of ground states of H. Typical examples are massless GSB models and the Pauli–Fierz model with spin $1/2$.

Keywords: Ground states; Multiplicity; Asymptotic fields; Number operators

1. Preliminaries

1.1. Boson Fock spaces

Let $\mathcal{W}$ be a Hilbert space over $\mathbb{C}$ with a conjugation$^-$. The boson Fock space $\mathcal{F}_{\rm b}$ over $\mathcal{W}$ is defined by

$$\begin{aligned}\mathcal{F}_{\rm b} = \mathcal{F}_{\rm b}(\mathcal{W}) &:= \bigoplus_{n=0}^{\infty}[\otimes_s^n \mathcal{W}]\\ &= \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \middle| \Psi^{(n)} \in \otimes_s^n \mathcal{W}, \ \|\Psi\|_{\mathcal{F}_{\rm b}}^2 := \sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\otimes^n \mathcal{W}}^2 < \infty \right\},\end{aligned}$$

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where $\otimes_s^n \mathcal{W}$ denotes the n-fold symmetric tensor product of $\mathcal{W}$ with $\otimes_s^0 \mathcal{W} := \mathbb{C}$. In this paper $(f, g)_{\mathcal{K}}$ and $\|f\|_{\mathcal{K}}$ denote the scalar product and the norm on Hilbert space $\mathcal{K}$ over $\mathbb{C}$, respectively, where $(f, g)_{\mathcal{K}}$ is linear in g and antilinear in f. Unless confusions arise we omit $\mathcal{K}$ of $(\cdot, \cdot)_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}$. $D(T)$ denotes the domain of operator T. Moreover, for a bounded operator S, we denote its operator norm by $\|S\|$.

The Fock vacuum $\Omega \in \mathcal{F}_{\mathrm{b}}$ is given by $\Omega = \{1, 0, 0, \ldots\}$. The finite particle subspace of $\mathcal{F}_{\mathrm{b}}$ is defined by

$$\mathcal{F}_{\mathrm{fin}} := \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}} | \Psi^{(m)} = 0 \text{ for all } m \geqslant n \text{ with some } n\}.$$

It is known that $\mathcal{F}_{\mathrm{fin}}$ is dense in $\mathcal{F}_{\mathrm{b}}$. The creation operator $a^\dagger(f) : \mathcal{F}_{\mathrm{b}} \to \mathcal{F}_{\mathrm{b}}$ with test function $f \in \mathcal{W}$ is the densely defined linear operator in $\mathcal{F}_{\mathrm{b}}$ defined by

$$(a^\dagger(f)\Psi)^{(0)} = 0, \quad (a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geqslant 1,$$

where S_n is the symmetrization operator on $\otimes^n \mathcal{W}$, i.e., $S_n[\otimes^n \mathcal{W}] = \otimes_s^n \mathcal{W}$. The annihilation operator $a(f)$, $f \in \mathcal{W}$, is defined by $a(f) = (a^\dagger(\overline{f}))^* \lceil_{\mathcal{F}_{\mathrm{fin}}}$. Since it is seen that $a(f)$ and $a^\dagger(f)$ are closable operators, their closures are denoted by the same symbols, respectively. Note that $a^\sharp(f)$ ($a^\sharp = a$ or $a^\dagger$) is linear in f. On $\mathcal{F}_{\mathrm{fin}}$ the annihilation operator and the creation operator obey canonical commutation relations,

$$[a(f), a^\dagger(g)] = (\overline{f}, g)_{\mathcal{W}}, \quad [a(f), a(g)] = 0, \quad [a^\dagger(f), a^\dagger(g)] = 0,$$

where $[A, B] := AB - BA$. Define

$$\mathcal{F}_{\mathrm{fin}}^D := \text{the linear hull of } \{a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega, \Omega | f_j \in D, j = 1, \ldots,, n \geqslant 1\}.$$

Let S be a self-adjoint operator acting in $\mathcal{W}$. The second quantization of S, $d\Gamma(S) : \mathcal{F}_{\mathrm{b}} \to \mathcal{F}_{\mathrm{b}}$, is defined by

$$d\Gamma(S) := \bigoplus_{n=0}^{\infty} \left(\sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes \overset{j_{\mathrm{th}}}{S} \otimes \cdots \otimes 1}_{n} \right)$$

with $D(d\Gamma(S)) := \mathcal{F}_{\mathrm{fin}}^{D(S)}$. Here we define $(d\Gamma(S)\Psi)^{(0)} := 0$. In particular it follows that

$$d\Gamma(S)\Omega = 0. \tag{1.1}$$

Note that

$$d\Gamma(S) a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega = \sum_{j=1}^{n} a^\dagger(f_1) \cdots a^\dagger(Sf_j) \cdots a^\dagger(f_n)\Omega. \tag{1.2}$$

From (1.2) it follows that, for $f \in D(S)$,

$$[d\Gamma(S), a(f)] = -a(Sf), \tag{1.3}$$

$$[d\Gamma(S), a^\dagger(f)] = a^\dagger(Sf) \tag{1.4}$$

on $\mathcal{F}_{\rm fin}^{D(S)}$. It is known that $d\Gamma(S)$ is essentially self-adjoint. The self-adjoint extension of $d\Gamma(S)$ is denoted by the same symbol $d\Gamma(S)$. It can be seen that unitary operator $e^{itd\Gamma(S)}$ acts as

$$e^{itd\Gamma(S)}a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega = a^\dagger(e^{itS}f_1)\cdots a^\dagger(e^{itS}f_n)\Omega.$$

Thus we see that

$$e^{itd\Gamma(S)}a(f)e^{-itd\Gamma(S)} = a(e^{-itS}f), \tag{1.5}$$

$$e^{itd\Gamma(S)}a^\dagger(f)e^{-itd\Gamma(S)} = a^\dagger(e^{itS}f) \tag{1.6}$$

on $\mathcal{F}_{\rm fin}$. For a self-adjoint operator T, we write its spectrum (resp. essential spectrum, point spectrum) as $\sigma(T)$ (resp. $\sigma_{\rm ess}(T)$, $\sigma_{\rm p}(T)$). The second quantization of the identity operator 1 on $\mathcal{W}$, $d\Gamma(1)$, is referred to as the number operator, which is written as

$$N := d\Gamma(1).$$

We note that

$$D(N^k) = \left\{\Psi = \left\{\Psi^{(n)}\right\}_{n=0}^\infty \mid \sum_{n=0}^\infty n^{2k}\|\Psi^{(n)}\|^2 < \infty\right\}$$

and

$$\sigma(N) = \sigma_{\rm p}(N) = \mathbb{N} \cup \{0\}.$$

1.2. Abstract interaction systems

Let $\mathcal{H}$ be a Hilbert space. A Hilbert space for an abstract coupled system is given by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\rm b}$$

and a decoupled Hamiltonian H_0 acting in $\mathcal{F}$ is of the form

$$H_0 = A \otimes 1 + 1 \otimes d\Gamma(S).$$

Assumptions (A1) and (A2) are as follows.

(A1) Operator A is a self-adjoint operator acting in $\mathcal{H}$, and bounded from below.
(A2) Operator S is a nonnegative self-adjoint operator acting in $\mathcal{W}$.
Total Hamiltonians under consideration are of the form

$$H = H_0 + gH_{\rm I}, \tag{1.7}$$

where $g \in \mathbb{R}$ denotes a coupling constant and $H_{\rm I}$ a symmetric operator. Assumption (A3) is as follows.
(A3) $H_{\rm I}$ is H_0-bounded with

$$\|H_{\rm I}\Psi\| \leqslant a\|H_0\Psi\| + b\|\Psi\|, \quad \Psi \in D(H_0),$$

where a and b are nonnegative constants.

Under (A3), by the Kato–Rellich theorem, H is self-adjoint on $D(H_0)$ and bounded from below for g with $|g| < 1/a$. Moreover H is essentially self-adjoint on any core of H_0. The bottom of $\sigma(H)$ is denoted by

$$E(H) := \inf \sigma(H),$$

which is referred to as the ground state energy of H. If an eigenvector Ψ associated with $E(H)$ exists, i.e.,

$$H\Psi = E(H)\Psi,$$

then Ψ is called a ground state of H. Let $E_T(B)$ be the spectral projection of self-adjoint operator T onto a Borel set $B \subset \mathbb{R}$. We set

$$P_T := E_T(\{E(T)\}).$$

Then P_H denotes the projection onto the subspace spanned by ground states of H. The dimension of $P_H\mathcal{F}$ is called the multiplicity of ground states of H, and it is denoted by

$$\mathrm{m}(H) := \dim P_H\mathcal{F}.$$

If $\mathrm{m}(H) = 1$, then we call that the ground state of H is unique.

1.3. Expectation values of the number operator

For Hamiltonians like as (1.7), the existence of a ground state $\varphi_{\rm g}$ such that

$$\varphi_{\rm g} \in D(1 \otimes N^{1/2}) \tag{1.8}$$

has been shown by many authors, e.g., [4,9,10,15,17,23,38]. Conversely, if $\varphi_{\rm g}$ exists, little attention, however, has been given to investigate whether (1.8) holds or not. Then the first task in this paper is to give a necessary and sufficient condition for

$$P_H \mathcal{F} \subset D(1 \otimes N^{1/2}). \tag{1.9}$$

As we will see later, to show (1.9) is also the primary problem in estimating an upper bound of $\mathrm{m}(H)$.

1.4. Massive and massless cases

Typical examples of Hilbert space $\mathcal{W}$ and nonnegative self-adjoint operator S are

$$\mathcal{W} = L^2(\mathbb{R}^d), \tag{1.10}$$

$$S = \text{ the multiplication operator by } \omega_\nu(k) := \sqrt{|k|^2 + \nu^2}. \tag{1.11}$$

In the case of $\nu > 0$ (resp. $\nu = 0$), a model is referred to as a *massive* (resp. *massless*) model. Note that under (A1) and (A3),

$$D(H) = D(H_0) = D(A \otimes 1) \cap D(1 \otimes d\Gamma(\omega_\nu)). \tag{1.12}$$

In a massive case, one can see that (1.9) is always satisfied. Actually in a massive case, we have $D(d\Gamma(\omega_\nu)) \subset D(N)$ and

$$\frac{1}{\nu}\|d\Gamma(\omega_\nu)\Psi\| \geqslant \|N\Psi\|, \qquad \Psi \in D(d\Gamma(\omega_\nu)).$$

Together with (1.12) we obtain that

$$P_H \mathcal{F} \subset D(H) \subset D(1 \otimes d\Gamma(\omega_\nu)) \subset D(1 \otimes N) \subset D(1 \otimes N^{1/2}).$$

Hence (1.9) follows. Kernel $a(k)$ of $a(f)$, $f \in L^2(\mathbb{R}^d)$, is defined for each $k \in \mathbb{R}^d$ as

$$(a(k)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n+1}\,\Psi^{(n+1)}(k, k_1, \ldots, k_n)$$

and

$$(a(f)\Psi)^{(n)} = \int f(k)(a(k)\Psi)^{(n)}\, dk$$

for $\Psi \in \mathcal{F}_{\rm fin}^{C_0^\infty(\mathbb{R}^d)}$, and it is directly seen that

$$\int_{\mathbb{R}^d} \|a(k)\Psi\|^2\, dk = \|N^{1/2}\Psi\|^2, \quad \Psi \in \mathcal{F}_{\rm fin}^{C_0^\infty(\mathbb{R}^d)}. \tag{1.13}$$

From (1.13), $a(\cdot)\Psi$ for $\Psi \in D(N^{1/2})$ can be defined as an $\mathcal{F}_{\rm b}$-valued L^2 function on $\mathbb{R}^d$ by

$$a(\cdot)\Psi := \text{s-}\lim_{m\to\infty} a(\cdot)\Psi_m \quad \text{in} \quad L^2(\mathbb{R}^d; \mathcal{F}_{\rm b}),$$

where s-$\lim_{m\to\infty}$ denotes the strong limit in $L^2(\mathbb{R}^d; \mathcal{F}_{\rm b})$ and sequence $\Psi_m \in \mathcal{F}_{\rm fin}^{C_0^\infty(\mathbb{R}^d)}$ is such that $\Psi_m \to \Psi$ and $N^{1/2}\Psi_m \to N^{1/2}\Psi$ strongly as $m \to \infty$. By an *informal* calculation, it can be derived *pointwise* that

$$(1 \otimes a(k))\varphi_{\rm g} = g(H - E(H) + \omega(k))^{-1}[H_{\rm I}, 1 \otimes a(k)]\varphi_{\rm g}. \tag{1.14}$$

Note that at least we have to assume $\varphi_{\rm g} \in D(1 \otimes N^{1/2})$ for (1.14) to make a sense, and the right-hand side of (1.14) is also delicate. See e.g., [37, Lemma 2.6, 13, p. 170, Conclusion] for this point. For massive cases, $(1 \otimes a(\cdot))\varphi_{\rm g}$ is well defined as an $\mathcal{F}$-valued L^2 function on $\mathbb{R}^d$, since $\varphi_{\rm g} \in D(1 \otimes N^{1/2})$, but of course it does not make sense pointwise. From (1.13) and (1.14) it follows that

$$\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = g^2 \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}[H_{\rm I}, 1 \otimes a(k)]\varphi_{\rm g}\|^2\, dk. \tag{1.15}$$

We may say under some conditions that

$$\varphi_{\rm g} \in D(1 \otimes N^{1/2}) \text{ and } \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}[H_{\rm I}, 1 \otimes a(k)]\varphi_{\rm g}\|^2\, dk < \infty$$
$$\Longrightarrow \|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = g^2 \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}[H_{\rm I}, 1 \otimes a(k)]\varphi_{\rm g}\|^2\, dk.$$

Although (1.15) has been applied to study $\|(1 \otimes N^{1/2})\varphi_{\rm g}\|$ by many authors, it must be noted again that (1.15) is derived from *informal* formula (1.14).

We are most interested in analysis of ground states for massless cases. In this case $\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2})$ is not clear, and it is also not clear a priori that $(1 \otimes a(k))\varphi_{\mathrm{g}}$ makes a sense. Then it is uncertain that identity (1.14) holds true for massless cases.

Because of the tedious argument involved in establishing (1.14) pointwise, a quite different method is taken to show (1.15) in this paper. We will show under some conditions that

$$\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2}) \Longleftrightarrow \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1}[H_{\mathrm{I}}, 1 \otimes a(k)]\varphi_{\mathrm{g}}\|^2 \, dk < \infty, \quad (1.16)$$

and (1.15) follows when the right or left-hand side of (1.16) holds. The method is an application of the fact that asymptotic annihilation operators vanish arbitrary ground states. See (1.21). As a result, (1.15) and (1.16) can be valid rigorously for both massive and massless cases without using (1.14). As far as we know, this method is new, cf., see [6,7,20,21]. By means of (1.16) we can find a condition for $P_H \mathcal{F} \subset D(1 \otimes N^{1/2})$.

1.5. Multiplicity

Generally, in the case where $E(H)$ is discrete, the min-max principle [35] is available to estimate the multiplicity of ground states. Actually the ground state energy of a *massive* generalized-spin-boson (GSB) model with a sufficiently weak coupling is discrete. Hence the min-max principle can be applied for this model [4]. However, for some typical models, e.g., massless GSB models, the Pauli–Fierz model, and the Nelson model [33], etc., their ground state energy is the edge of the essential spectrum, namely it is not discrete. See also [3,26]. Then the min-max principle does not work at all.

Instead of the min-max principle, we can apply an infinite dimensional version of the Perron–Frobenius theorem [16,18,19] to show the uniqueness of its ground state. I.e., in a Schrödinger representation,

$$(\Psi, e^{-tH}\Phi) > 0, \quad \Psi \geqslant 0 \ (\not\equiv 0), \quad \Phi \geqslant 0 \ (\not\equiv 0), \quad (1.17)$$

implies $\mathrm{m}(H) = 1$. Property (1.17) is called that e^{-tH} is positivity-improving. The Perron–Frobenius theorem has been applied for some models, e.g., the Nelson model in [9], and the spinless Pauli–Fierz model in [24]. It is, however, for, e.g., the Pauli–Fierz model with spin $1/2$, H_{PF}, we cannot apply the Perron–Frobenius theorem, since, as far as we know, a suitable representation for $e^{-tH_{\mathrm{PF}}}$ to be positivity-improving cannot be constructed.

In this paper, applying the fact $P_H \mathcal{F} \subset D(1 \otimes N^{1/2})$, we establish a wide-usable method to estimate an upper bound of the multiplicity of ground states under some conditions.

1.6. Main results and strategies

The main results are (m1) and (m2).

(m1) We give a necessary and sufficient condition for $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$.
(m2) We prove $\mathrm{m}(H) \leqslant \mathrm{m}(A)$ under some conditions.

Strategies are as follows. It is proven that

$$\varphi_{\mathrm{g}} \in D(1 \otimes N^{1/2}) \Longleftrightarrow \sum_{m=1}^{\infty} \|(1 \otimes a(e_m))\varphi_{\mathrm{g}}\|^2 < \infty, \tag{1.18}$$

where $\{e_m\}_{m=1}^{\infty}$ is an arbitrary complete orthonormal system of $\mathcal{W}$. When the left or right-hand side of (1.18) holds, it follows that

$$\sum_{m=1}^{\infty} \|(1 \otimes a(e_m))\varphi_{\mathrm{g}}\|^2 = \|(1 \otimes N^{1/2})\varphi_{\mathrm{g}}\|^2. \tag{1.19}$$

Let us define an asymptotic annihilation operator by

$$a_+(f)\Psi := \operatorname{s-}\lim_{t\to\infty} e^{-itH} e^{itH_0}(1 \otimes a(f)) e^{-itH_0} e^{itH}\Psi. \tag{1.20}$$

Of course some conditions on Ψ and f are required to show the existence of $a_+(f)\Psi$. It is well known [1,29], however, that (1.20) exists for an arbitrary ground state of H, $\Psi = \varphi_{\mathrm{g}}$, and $a_+(f)$ vanishes φ_{g}, i.e.,

$$a_+(f)\varphi_{\mathrm{g}} = 0 \tag{1.21}$$

for $f \in \mathcal{D}$ with some dense subspace $\mathcal{D}$, (1.21) is applied for (m1). We decompose $a_+(f)\Psi$ as

$$a_+(f)\Psi = (1 \otimes a(f))\Psi - gG(f)\Psi, \quad f \in \mathcal{D}.$$

with some operator $G(f) : \mathcal{F} \to \mathcal{F}$. From (1.21) it follows that

$$(1 \otimes a(f))\varphi_{\mathrm{g}} = gG(f)\varphi_{\mathrm{g}}, \quad f \in \mathcal{D}. \tag{1.22}$$

We define the operator $T_{\varphi_{\mathrm{g}}} : \mathcal{W} \to \mathcal{F}$ by

$$T_{\varphi_{\mathrm{g}}} f := G(f)\varphi_{\mathrm{g}}, \quad f \in \mathcal{D}. \tag{1.23}$$

I.e.,

$$(1 \otimes a(f))\varphi_g = gT_{\varphi_g} f. \tag{1.24}$$

It is seen that the closure of T_{φ_g}, $\overline{T}_{\varphi_g}$, is a Hilbert Schmidt operator and

$$\overline{T}_{\varphi_g} f = \int_{\mathbb{R}^d} f(k)\kappa_{\varphi_g}(k)\,dk, \qquad f \in \mathcal{W},$$

with some kernel $\kappa_{\varphi_g}(k) \in \mathcal{F}$. See (2.17) for details. Note that

$$\sum_{m=1}^{\infty} \|\overline{T}_{\varphi_g} e_m\|^2 = \mathrm{Tr}\big((\overline{T}_{\varphi_g})^* \overline{T}_{\varphi_g}\big) = \int_{\mathbb{R}^d} \|\kappa_{\varphi_g}(k)\|^2\,dk. \tag{1.25}$$

Using (1.18), (1.24) and (1.25), we see that

$$\varphi_g \in D(1 \otimes N^{1/2}) \Longleftrightarrow g^2 \int_{\mathbb{R}^d} \|\kappa_{\varphi_g}(k)\|^2\,dk < \infty$$

and by (1.19),

$$\|(1 \otimes N^{1/2})\varphi_g\|^2 = g^2 \int_{\mathbb{R}^d} \|\kappa_{\varphi_g}(k)\|^2\,dk. \tag{1.26}$$

Thus we can obtain that

$$P_H\mathcal{F} \subset D(1 \otimes N^{1/2}) \Longleftrightarrow \int \|\kappa_{\varphi_g}(k)\|^2\,dk < \infty \text{ for all } \varphi_g \in P_H\mathcal{F}.$$

To show (m2) we apply the method in [28], by which we can prove that

$$\dim(P_H\mathcal{F} \cap D(1 \otimes N^{1/2})) \leqslant \frac{1}{1-\delta(g)}\mathrm{m}(A),$$

where $\delta(g) = \sup_{\varphi_g \in P_H\mathcal{F} \cap D(1 \otimes N^{1/2})} \dfrac{\|(1 \otimes N^{1/2})\varphi_g\|^2}{\|\varphi_g\|^2} + o(g)$. By (1.26) and the fact

$$\lim_{g \to 0} \sup_{\varphi_g \in P_H\mathcal{F}} \frac{\int_{\mathbb{R}^d} \|\kappa_{\varphi_g}(k)\|^2\,dk}{\|\varphi_g\|^2} < \infty,$$

we see that $\lim_{g\to 0}\delta(g) = 0$. Hence for a sufficiently small g, it is proven that $\dim(P_H\mathcal{F}\cap D(1\otimes N^{1/2}))\leqslant \mathrm{m}(A)$. Together with the fact $P_H\mathcal{F}\subset D(1\otimes N^{1/2})$ under some conditions, we get

$$\mathrm{m}(H) = \dim P_H\mathcal{F}\leqslant \mathrm{m}(A).$$

We organize this paper as follows.

Section 2 is devoted to show $P_H\mathcal{F}\subset D(1\otimes N^{1/2})$. In Section 3, we estimate the multiplicity of ground states. In Sections 4, we give examples including massless GSB models, the Pauli–Fierz model and Coulom–Dirac system.

2. Equivalent conditions to $P_H\mathcal{F}\subset D(1\otimes N^{1/2})$

2.1. The number operator

Let $\{e_m\}_{m=1}^\infty$ be a complete orthonormal system of $\mathcal{W}$. We define A_M, by

$$A_M := (N+1)^{-1/2}\left(\sum_{m=1}^M a^\dagger(e_m)a(\overline{e_m})\right)(N+1)^{-1/2},\qquad M=1,2,\dots\ .$$

Lemma 2.1. *It follows that* (1) A_M *can be uniquely extended to bounded operator* $\overline{A_M}$, (2) $\overline{A_M}$ *is uniformly bounded in* M *as* $\|\overline{A_M}\|\leqslant 1$, *and* (3) $\text{s-}\lim_{M\to\infty}\overline{A_M} = N(N+1)^{-1}$.

Proof. Let us define

$$\mathcal{F}_\omega := \left[\bigoplus_{n=0}^\infty\left\{\sum_{i_1\leqslant\cdots\leqslant i_n}^{\text{finite}}\alpha_{i_1,\dots,i_n}a^\dagger(e_{i_1})\cdots a^\dagger(e_{i_n})\Omega\,\middle|\,\alpha_{i_1,\dots,i_n}\in\mathbb{C}\right\}\right]\bigcap\mathcal{F}_{\text{fin}}.$$

Note that $\mathcal{F}_\omega$ is dense in $\mathcal{F}_\mathrm{b}$. Let $\phi = a^\dagger(e_{i_1})\cdots a^\dagger(e_{i_n})\Omega$, $i_1\leqslant\cdots\leqslant i_n$. Then

$$A_M\phi = \beta_{i_1,\dots,i_n}(M)\phi, \tag{2.1}$$

where

$$\beta_{i_1,\dots,i_n}(M) := \begin{cases} \frac{n}{n+1}, & i_n\leqslant M,\\ \frac{n-1}{n+1}, & i_{n-1}\leqslant M < i_n,\\ \vdots & \vdots\\ \frac{1}{n+1}, & i_1\leqslant M < i_2,\\ 0, & M < i_1.\end{cases}$$

Let $\Psi \in \mathcal{F}_\omega$ be such that $\Psi = \sum_{i_1 \leqslant \cdots \leqslant i_n}^{\text{finite}} \alpha_{i_1,\ldots,i_n} a^\dagger(e_{i_1}) \cdots a^\dagger(e_{i_n})\Omega$. We see that $\|\Psi\|^2 = \sum_{i_1 \leqslant \cdots \leqslant i_n}^{\text{finite}} |\alpha_{i_1,\ldots,i_n}|^2$. From (2.1) it follows that

$$A_M \Psi = \sum_{i_1 \leqslant \cdots \leqslant i_n}^{\text{finite}} \alpha_{i_1,\ldots,i_n} \beta_{i_1,\ldots,i_n}(M) a^\dagger(e_{i_1}) \cdots a^\dagger(e_{i_n}).$$

Then

$$\|A_M \Psi\|^2 = \sum_{i_1 \leqslant \cdots \leqslant i_n}^{\text{finite}} |\alpha_{i_1,\ldots,i_n}|^2 |\beta_{i_1,\ldots,i_n}(M)|^2 \leqslant \left(\frac{n}{n+1}\right)^2 \|\Psi\|^2.$$

Note that A_M leaves $\otimes_s^n \mathcal{W}$ invariant. Hence for an arbitrary $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_\omega$, we have

$$\|A_M \Psi\|^2 = \sum_{n=0}^\infty \|(A_M \Psi)^{(n)}\|^2 = \sum_{n=0}^\infty \|A_M \Psi^{(n)}\|^2 \leqslant \sum_{n=0}^\infty \left(\frac{n}{n+1}\right)^2 \|\Psi^{(n)}\|^2 \leqslant \|\Psi\|^2.$$

Since $\mathcal{F}_\omega$ is dense in $\mathcal{F}_{\mathrm{b}}$, (1) and (2) follow. Let $\Psi \in \mathcal{F}_\omega$ be as above. We see that $\text{s-}\lim_{M\to\infty} A_M \Psi = \frac{n}{n+1}\Psi$. Hence for an arbitrary $\Phi \in \mathcal{F}_\omega$,

$$\text{s-}\lim_{M\to\infty} A_M \Phi = N(N+1)^{-1}\Phi. \tag{2.2}$$

Since $\|\overline{A_M}\| \leqslant 1$, we obtain (2.2) for $\Phi \in \mathcal{F}_{\mathrm{b}}$ by a limiting argument. Thus (3) follows. □

Lemma 2.2. *Let* $\{e_m\}_{m=1}^\infty$ *be an arbitrary complete orthonormal system in* $\mathcal{W}$. *Then* (1) *and* (2) *are equivalent*.

(1) $\Psi \in D(N^{1/2})$.
(2) $\Psi \in \cap_{m=1}^\infty D(a(\overline{e_m}))$ *and* $\sum_{m=1}^\infty \|a(\overline{e_m})\Psi\|^2 < \infty$.

Moreover when (1) *or* (2) *holds, it follows that* $\|N^{1/2}\Psi\|^2 = \sum_{m=1}^\infty \|a(\overline{e_m})\Psi\|^2$.

Proof. (1) ⇒ (2) Since $\mathcal{F}_\omega$ is a core of $N^{1/2}$, for $\Psi \in D(N^{1/2})$, there exists a sequence $\Psi_\varepsilon \in \mathcal{F}_\omega$ such that $\text{s-lim}_{\varepsilon\to 0} \Psi_\varepsilon = \Psi$ and $\text{s-lim}_{\varepsilon\to 0} N^{1/2}\Psi_\varepsilon = N^{1/2}\Psi$. It is well known that $\|a(f)\Phi\| \leqslant \|f\| \|N^{1/2}\Phi\|$ for $\Phi \in D(N^{1/2})$. Hence from the fact $\Psi \in D(N^{1/2})$, it follows that $\Psi \in D(a(\overline{e_m}))$. We have

$$\sum_{m=1}^M \|a(\overline{e_m})\Psi_\varepsilon\|^2 = ((N+1)^{1/2}\Psi_\varepsilon, A_M (N+1)^{1/2}\Psi_\varepsilon) \leqslant \|N^{1/2}\Psi_\varepsilon\|^2 + \|\Psi_\varepsilon\|^2. \tag{2.3}$$

From this it follows that $a(\overline{e_m})\Psi_\varepsilon$ is a Cauchy sequence in ε. Since $a(\overline{e_m})$ is a closed operator, $\text{s-lim}_{\varepsilon\to 0} a(\overline{e_m})\Psi_\varepsilon = a(\overline{e_m})\Psi$ follows. Hence we obtain that, as $\varepsilon \to 0$ and then $M \to \infty$ on the both sides of (2.3), we have

$$\sum_{m=1}^{\infty} \|a(\overline{e_m})\Psi\|^2 \leqslant \|N^{1/2}\Psi\|^2 + \|\Psi\|^2.$$

Thus the desired results follow.

(2) ⇒ (1) We see that

$$\sum_{m=1}^{\infty} \|a(\overline{e_m})\Psi\|^2 = \lim_{M\to\infty} \sum_{n=0}^{\infty}\sum_{m=1}^{M} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)}).$$

Since $\sum_{m=1}^{M}(a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)})$ is monotonously increasing as $M \uparrow \infty$ and by the fact that $\lim_{M\to\infty}\sum_{n=0}^{\infty}\sum_{m=1}^{M}(a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)}) < \infty$, we have by the Lebesgue monotone convergence theorem and (3) of Lemma 2.1,

$$\infty > \lim_{M\to\infty} \sum_{n=0}^{\infty}\sum_{m=1}^{M} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)}) = \sum_{n=0}^{\infty} \lim_{M\to\infty} \sum_{m=1}^{M} (a(\overline{e_m})\Psi^{(n)}, a(\overline{e_m})\Psi^{(n)})$$
$$= \sum_{n=0}^{\infty} \lim_{M\to\infty} ((N+1)^{1/2}\Psi^{(n)}, \overline{A_M}(N+1)^{1/2}\Psi^{(n)}) = \sum_{n=0}^{\infty} n\|\Psi^{(n)}\|^2.$$

This yields that $\Psi \in D(N^{1/2})$. □

2.2. Weak commutators

In Sections 2.2–2.4, we consider the case where $\mathcal{W} = \oplus^D L^2(\mathbb{R}^d) \cong L^2(\mathbb{R}^d \times \{1,\dots,D\})$ and $S = [\omega]$ where $[\omega]\colon \oplus^D L^2(\mathbb{R}^d) \to \oplus^D L^2(\mathbb{R}^d)$ is the multiplication operator defined by

$$[\omega](\oplus_{j=1}^{D} f_j) = \oplus_{j=1}^{D} \omega f_j \tag{2.4}$$

with $\omega(\cdot) : \mathbb{R}^d \to [0,\infty)$ and $(\omega f)(k) = \omega(k) f(k)$. The creation operator and the annihilation operator of $\mathcal{F}_{\rm b}(\mathcal{W})$ are denoted by

$$a^\sharp(f,j) := a(0 \oplus \cdots \oplus \overset{j\text{th}}{f} \oplus \cdots \oplus 0), \quad f \in L^2(\mathbb{R}^d), \quad j = 1,\dots,D,$$

which satisfy on $\mathcal{F}_{\rm fin}$,

$$[a(f,j), a^\dagger(g,j')] = (\bar{f}, g)\delta_{jj'}, \quad [a^\dagger(f,j), a^\dagger(g,j')] = 0, \quad [a(f,j), a(g,j')] = 0.$$

Let S and T be operators acting in a Hilbert space $\mathcal{K}$. We define a quadratic form $[S,T]_W^{\mathcal{D}}$ with a form domain $\mathcal{D}$ such that $\mathcal{D} \subset D(S^*) \cap D(S) \cap D(T^*) \cap D(T)$ by

$$[S,T]_W^D(\Psi,\Phi) := (S^*\Psi, T\Phi) - (T^*\Psi, S\Phi), \quad \Psi,\Phi \in \mathcal{D}.$$

The proposition below is fundamental.

Proposition 2.3. *Let* $\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. *Then*

$$[1\otimes d\Gamma([\omega]), 1\otimes a(f,j)]_W^{D(1\otimes d\Gamma([\omega]))}(\Psi,\Phi) = (\Psi, -(1\otimes a(\omega f, j))\Phi), \quad (2.5)$$

$$[1\otimes d\Gamma([\omega]), 1\otimes a^\dagger(f,j)]_W^{D(1\otimes d\Gamma([\omega]))}(\Psi,\Phi) = (\Psi, (1\otimes a^\dagger(\omega f, j))\Phi). \quad (2.6)$$

2.3. Asymptotic fields

Define on $D(H)$,

$$a_t(f,j) := e^{-itH}e^{itH_0}(1\otimes a(f,j))e^{-itH_0}e^{itH} = e^{-itH}(1\otimes a(e^{-it\omega}f,j))e^{itH}.$$

Note that $H_0 = A\otimes 1 + 1\otimes d\Gamma([\omega])$. Assumption (B1) is as follows.

(B1) ω satisfies that (1) the Lebesgue measure of $K_\omega := \{k \in \mathbb{R}^d | \omega(k) = 0\}$ is zero, (2) there exists a subset $K \subset \mathbb{R}^d$ with Lebesgue measure zero such that $\omega \in C^3(\mathbb{R}^d \setminus K)$ and $\dfrac{\partial\omega}{\partial k_n}(k) \neq 0$ for $n = 1,\ldots,d$, $k = (k_1,\ldots,k_d) \in \mathbb{R}^d \setminus K$.

Example 2.4. A typical example of ω is $\omega(k) = |k|^p$ with $p > 0$. In this case $K_\omega = \{0\}$ and $K = \bigcup_{n=1}^d \{(k_1,\ldots,k_d) \in \mathbb{R}^d | k_n = 0\}$.

Lemma 2.5. *Suppose* (2) *of* (B1). *Then* $\left|\int_{\mathbb{R}^d} e^{is\omega(k)} f(k)\,dk\right| \leqslant \dfrac{c}{s^2}$ *for* $f \in C_0^2(\mathbb{R}^d \setminus K)$ *with some constant* c.

Proof. We have, for $1 \leqslant m,n \leqslant d$, $e^{is\omega} = -\dfrac{1}{s^2}\left(\dfrac{\partial\omega}{\partial k_n}\right)^{-1}\dfrac{\partial}{\partial k_n}\left(\left(\dfrac{\partial\omega}{\partial k_m}\right)^{-1}\dfrac{\partial e^{is\omega}}{\partial k_m}\right)$ on $\mathbb{R}^d \setminus K$. Hence it follows that by integration by parts,

$$\left|\int_{\mathbb{R}^d} e^{is\omega(k)} f(k)\,dk\right| \leqslant \frac{1}{s^2}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial k_m}\left(\left(\frac{\partial\omega}{\partial k_m}\right)^{-1}\frac{\partial}{\partial k_n}\left(\left(\frac{\partial\omega}{\partial k_n}\right)^{-1} f(k)\right)\right)\right| dk.$$

Since the integrand of the right-hand side above is integrable, the lemma follows. □

Proposition 2.6. *Suppose* (B1). *Let* $f \in C^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *and* $f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. *Then*

$$\text{s-}\lim_{t\to\infty} a_t(f, j)\varphi_{\rm g} = 0, \qquad j = 1, \dots, D. \tag{2.7}$$

Proof. Note that it follows that $\|a_t(f, j)\Psi\| \leqslant \|f/\sqrt{\omega}\| \|(1 \otimes d\Gamma([\omega])^{1/2})e^{itH}\Psi\|$, $j = 1, \dots, D$. Thus it is seen that

$$\|a_t(f, j)\Psi\| \leqslant c_1 \|f/\sqrt{\omega}\| \|(H + 1)\Psi\| \tag{2.8}$$

with some constant c_1. Let $\mathcal{D}$ be a core of A and $\Psi = G \otimes a^\dagger(f_1, j_1) \cdots a^\dagger(f_n, j_n)\Omega$, where $G \in \mathcal{D}$ and $f_l \in C_0^\infty(\mathbb{R}^d \setminus K)$, $l = 1, \dots, n$. We see that for an arbitrary $\delta \in \mathbb{R}$,

$$a(e^{-it(\omega-\delta)} f, j)\Psi = \sum_{l=1}^n (e^{it(\omega-\delta)} \bar{f}, f_l) G \otimes a^\dagger(f_1, j_1) \cdots a^\dagger(\widehat{f_l}, j_l) \cdots a^\dagger(f_n, j_n)\Omega,$$

where $\widehat{X}$ means neglecting X. Since $f f_l \in C_0^2(\mathbb{R}^d \setminus K)$, by Lemma 2.5 we see that $|(e^{it(\omega-\delta)}\bar{f}, f_l)| \leqslant c_2/|t|^2$ with some constant c_2. Hence $\text{s-}\lim_{t\to\infty} a(e^{it(\omega-\delta)} f, j)\Psi = 0$ follows. Let $\mathcal{E}$ be the set of the linear hull of vectors such as Ψ above, which is a core of H_0. Thus there exists $\Psi_\varepsilon \in \mathcal{E}$ such that $\Psi_\varepsilon \to \varphi_{\rm g}$, $H_0\Psi_\varepsilon \to H_0\varphi_{\rm g}$ strongly as $\varepsilon \to 0$, which yields that $\lim_{\varepsilon\to 0} \|(H_0 + 1)^{1/2}(\Psi_\varepsilon - \varphi_{\rm g})\| = 0$. Let $\|(H_0 + 1)^{1/2}(\Psi_\varepsilon - \varphi_{\rm g})\| < \varepsilon$. We obtain that

$$\begin{aligned}
&\|a_t(f, j)\varphi_{\rm g}\| \\
&\quad \leqslant \|(1 \otimes a(e^{-it(\omega-E(H))} f, j))\Psi_\varepsilon\| + \|(1 \otimes a(e^{-it(\omega-E(H))} f, j))(\Psi_\varepsilon - \varphi_{\rm g})\| \\
&\quad \leqslant \|(1 \otimes a(e^{-it(\omega-E(H))} f, j))\Psi_\varepsilon\| + C\varepsilon.
\end{aligned}$$

Then $\lim_{t\to\infty} \|a_t(f, j)\varphi_{\rm g}\| < C\varepsilon$ for an arbitrary ε. Then the proposition follows. □

In addition to (B1), we introduce assumptions (B2)–(B4).

(B2) There exists an operator $T_j(k) : \mathcal{F} \to \mathcal{F}$, $k \in \mathbb{R}^d$, $j = 1, \dots, D$, such that $D(T_j(k)) \supset D(H)$ for alomst everywhere $k \in \mathbb{R}^d$ and

$$[1 \otimes a(f, j), H_{\rm I}]_W^{D(H)}(\Psi, \Phi) = \int_{\mathbb{R}^d} f(k)(\Psi, T_j(k)\Phi)\, dk.$$

(B3) Let $\Psi \in D(H)$ and $f \in C_0^2(\mathbb{R}^d \setminus \widetilde{K})$ with some measurable set $\widetilde{K} \subset \mathbb{R}^d$ such that $K \subset \widetilde{K}$ and its Lebesgue measure is zero. Then

$$\left|\int_{\mathbb{R}^d} dk f(k)(\Psi, e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_{\rm g})\right| \in L^1([0,\infty), ds).$$

(B4) $\|T_j(\cdot)\varphi_{\rm g}\| \in L^2(\mathbb{R}^d)$.

Lemma 2.7. *Suppose* (B1)–(B4). *Let* $f, f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. *Then it follows that*

$$\int_{\mathbb{R}^d} \|f(k)(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_{\rm g}\|\, dk < \infty \tag{2.9}$$

and

$$(1 \otimes a(f,j))\varphi_{\rm g} = -g\int_{\mathbb{R}^d} f(k)(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_{\rm g}\, dk. \tag{2.10}$$

Proof. Noting that $\|(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_{\rm g}\| \leqslant \|T_j(k)\varphi_{\rm g}\|/\omega(k)$ for $k \notin K_\omega$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \|f(k)(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_{\rm g}\|\, dk \\
&\quad\leqslant \left(\int_{|k|<1} \frac{|f(k)|^2}{\omega(k)}\, dk\right)^{1/2} \left(\int_{|k|<1} \omega(k)\|(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_{\rm g}\|^2\, dk\right)^{1/2} \\
&\qquad + \left(\int_{|k|\geqslant 1} |f(k)|^2\, dk\right)^{1/2} \left(\int_{|k|\geqslant 1} \|(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_{\rm g}\|^2\, dk\right)^{1/2} \\
&\quad\leqslant (\|f/\sqrt{\omega}\| + \|f\|)\|T_j(\cdot)\varphi_{\rm g}\| < \infty.
\end{aligned} \tag{2.11}$$

Then (2.9) follows. We divide a proof of (2.10) into three steps.

Step 1: Let $f \in C_0^2(\mathbb{R}^d \setminus \widetilde{K})$, $f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$, and $\Psi, \Phi \in D(H)$. Then

$$(\Psi, (1 \otimes a(f,j))\varphi_{\rm g}) = -ig\int_0^\infty \left(\int_{\mathbb{R}^d} (\Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_{\rm g})\, dk\right) ds. \tag{2.12}$$

Proof. Let $\Psi, \Phi \in \mathcal{D} := C_0^\infty(\mathbb{R}^d) \otimes D(d\Gamma([\omega]))$. Note that $\mathcal{D}$ is a core of H. We see that by (2.5) of Proposition 2.3 and (B2),

$$\frac{d}{dt}(\Psi, a_t(f,j)\Phi) = ig\int_{\mathbb{R}^d} f(k)e^{-it\omega(k)}(\Psi, e^{-itH}T_j(k)e^{itH}\Phi)\, dk.$$

Then we obtain that for $\Psi, \Phi \in \mathcal{D}$,

$$
\begin{aligned}
&(\Psi, a_t(f,j)\Phi) \\
&\quad = (\Psi, (1 \otimes a(f,j))\Phi) + ig \int_0^t \left(\int_{\mathbb{R}^d} f(k) e^{-is\omega(k)} (\Psi, e^{-isH} T_j(k) e^{isH} \Phi)\, dk \right) ds.
\end{aligned}
\tag{2.13}
$$

Let $\Psi, \Phi \in D(H)$. There exist sequences $\Psi_m, \Phi_n \in \mathcal{D}$ such that $\lim_{m\to\infty} \Psi_m = \Psi$ and $\lim_{n\to\infty} \Phi_n = \Phi$ strongly. Eq. (2.13) holds true for Ψ, Φ replaced by Ψ_m, Φ_n, respectively. By a simple limiting argument as $m \to \infty$ and then $n \to \infty$, we get (2.13) for $\Psi, \Phi \in D(H)$. By Proposition 2.6 and (2.13) we have

$$
\begin{aligned}
0 &= \lim_{t\to\infty} (\Psi, a_t(f,j)\varphi_g) \\
&= (\Psi, (1 \otimes a(f,j))\varphi_g) + ig \int_0^\infty \left(\int_{\mathbb{R}^d} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k) \varphi_g)\, dk \right) ds.
\end{aligned}
$$

Thus (2.12) follows. □

Step 2: (2.10) holds true for f such that $f \in C_0^2(\mathbb{R}^d \setminus \widetilde{K})$ and $f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.

Proof. By (B3) and the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&-ig \int_0^\infty \left(\int_{\mathbb{R}^d} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k) \varphi_g)\, dk \right) ds \\
&\qquad = -ig \lim_{\varepsilon\to 0} \int_0^\infty ds e^{-\varepsilon s} \left(\int_{\mathbb{R}^d} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k) \varphi_g)\, dk \right).
\end{aligned}
$$

By (B4),

$$
\begin{aligned}
&\int_{\mathbb{R}^d} dk \int_0^\infty \left| e^{-s\varepsilon} (\Psi, f(k) e^{-is(H-E(H)+\omega(k))} T_j(k) \varphi_g) \right| ds \\
&\quad \leqslant \|\Psi\| \left(\int_{\mathbb{R}^d} |f(k)| \|T_j(k)\varphi_g\|\, dk \right) \int_0^\infty e^{-s\varepsilon} ds < \infty.
\end{aligned}
$$

Hence Fubini's theorem yields that $\int dk$ and $\int ds$ can be exchanged, i.e.,

$$
\begin{aligned}
&-ig \lim_{\varepsilon\to 0}\int_0^\infty e^{-\varepsilon s}\left(\int_{\mathbb{R}^d}(\Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_{\rm g})\,dk\right)ds\\
&= -ig \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\left(\int_0^\infty(\Psi, f(k)e^{-is(H-E(H)+\omega(k)-i\varepsilon)}T_j(k)\varphi_{\rm g})\,ds\right)dk\\
&= -g \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}(\Psi, f(k)(H-E(H)+\omega(k)-i\varepsilon)^{-1}T_j(k)\varphi_{\rm g})\,dk.
\end{aligned}
$$

We can check that, for $k \notin K_\omega$,

$$
\begin{aligned}
&|(\Psi, f(k)(H-E(H)+\omega(k)-i\varepsilon)^{-1}T_j(k)\varphi_{\rm g})|\\
&\quad\leqslant \|\Psi\||f(k)|\|(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g}\|,
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^d}|f(k)|\|(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g}\|\,dk\\
&\quad\leqslant (\|f/\sqrt{\omega}\|+\|f\|)\|T_j(\cdot)\varphi_{\rm g}\| < \infty
\end{aligned}
\tag{2.15}
$$

and

$$
\text{s-}\lim_{\varepsilon\to 0}(H-E(H)+\omega(k)-i\varepsilon)^{-1}\varphi_{\rm g} = (H-E(H)+\omega(k))^{-1}\varphi_{\rm g}. \tag{2.16}
$$

Eqs. (2.14)–(2.16) imply that by the Lebesgue dominated convergence theorem,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}(\Psi, f(k)(H-E(H)+\omega(k)-i\varepsilon)^{-1}T_j(k)\varphi_{\rm g})\,dk\\
&= \int_{\mathbb{R}^d}(\Psi, f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g})\,dk.
\end{aligned}
$$

Since, by (2.15) we have

$$
\begin{aligned}
(\Psi, a(f,j)\varphi_{\rm g}) &= -g\int_{\mathbb{R}^d}(\Psi, f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g})\,dk\\
&= (\Psi, -g\int_{\mathbb{R}^d} f(k)(H-E(H)+\omega(k))^{-1}T_j(k)\varphi_{\rm g}\,dk),
\end{aligned}
$$

we obtain (2.10). □

Step 3: Eq. (2.10) holds true for f such that $f, f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.

Proof. Set $g(k) := \begin{cases} f(k)/\sqrt{\omega(k)}, & |k| < 1, \\ f(k), & |k| \geqslant 1. \end{cases}$ Since $g \in L^2(\mathbb{R}^d)$, there exists a sequence $g_\varepsilon \in C_0^\infty(\mathbb{R}^d \setminus \widetilde{K})$ such that $g_\varepsilon \to g$ strongly as $\varepsilon \to 0$. Define

$$f_\varepsilon(k) := \begin{cases} \sqrt{\omega(k)} g_\varepsilon(k), & |k| < 1, \\ g_\varepsilon(k), & |k| \geqslant 1. \end{cases}$$

Hence $f_\varepsilon \in C_0^3(\mathbb{R}^d \setminus \widetilde{K})$ by (2) of (B1), and $\int_{\mathbb{R}^d} |f(k) - f_\varepsilon(k)|^2 / \omega(k)\, dk \to 0$ and $\int_{|k|>1} |f(k) - f_\varepsilon(k)|^2\, dk \to 0$, as $\varepsilon \to 0$. We see that

$$\|(1 \otimes a(f))\varphi_{\mathrm{g}} - (1 \otimes a(f_\varepsilon))\varphi_{\mathrm{g}}\| \leqslant \|(f - f_\varepsilon)/\sqrt{\omega}\| \, \|(1 \otimes d\Gamma([\omega])^{1/2})\varphi_{\mathrm{g}}\| \to 0$$

and

$$\left\| \int_{\mathbb{R}^d} (f(k) - f_\varepsilon(k))(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\mathrm{g}}\, dk \right\|$$
$$\leqslant \left\{ \left(\int_{|k|<1} \frac{|f(k) - f_\varepsilon(k)|^2}{\omega(k)}\, dk \right)^{1/2} + \left(\int_{|k|\geqslant 1} |f(k) - f_\varepsilon(k)|^2\, dk \right)^{1/2} \right\} \|T_j(\cdot)\varphi_{\mathrm{g}}\|$$
$$\to 0$$

as $\varepsilon \to 0$. Then we can extend (2.10) to *f* such that $f, f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. $\square$

2.4. Main theorem I

Set

$$\kappa_{\varphi_{\mathrm{g}}\, j}(k) := (H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\mathrm{g}}, \quad k \notin K_\omega.$$

We define $T_{\varphi_{\mathrm{g}}\, j} : L^2(\mathbb{R}^d) \to \mathcal{F}$, $j = 1, \ldots, D$, by

$$T_{\varphi_{\mathrm{g}}\, j} f := \int_{\mathbb{R}^d} f(k) \kappa_{\varphi_{\mathrm{g}}\, j}(k)\, dk$$

with the domain

$$D(T_{\varphi_{\mathrm{g}}\, j}) := \left\{ f \in L^2(\mathbb{R}^d) \middle| \left\| \int_{\mathbb{R}^d} f(k) \kappa_{\varphi_{\mathrm{g}}\, j}(k)\, dk \right\| < \infty \right\},$$

where the integral is taken in the strong sense in $\mathcal{F}$. Note that generally $T_{\varphi_{g\,j}}$ is an unbounded operator, and

$$(1\otimes a(f,j))\varphi_{\rm g} = -gT_{\varphi_{g\,j}}f,\quad f, f/\sqrt{\omega}\in L^2(\mathbb{R}^d). \tag{2.17}$$

Actually

$$\|(1\otimes a(f,j))\varphi_g\| \leqslant \| f/\sqrt{\omega}\|\ \|(1\otimes d\Gamma([\omega])^{1/2})\varphi_g\|$$

holds, since $\varphi_g \in D(1\otimes d\Gamma([\omega])^{1/2})$.

Lemma 2.8. (1) $\int_{\mathbb{R}^d}\|\kappa_{\varphi_{g\,j}}(k)\|^2\,dk<\infty$ *if and only if the closure of* $T_{\varphi_{g\,j}}$, $\overline{T}_{\varphi_{g\,j}}$, *is a Hilbert–Schmidt operator*. (2) *Suppose that* $T_{\varphi_{g\,j}}$ *is a Hilbert–Schmidt operator, then*

$$\sum_{m=1}^{\infty}\|\overline{T}_{\varphi_{g\,j}}e_m\|^2 = \int_{\mathbb{R}^d}\|\kappa_{\varphi_{g\,j}}(k)\|^2\,dk$$

for an arbitrary complete orthonormal system $\{e_m\}_{m=1}^{\infty}$ *in* $L^2(\mathbb{R}^d)$.

Proof. The adjoint of $T_{\varphi_{g\,j}}$, $T^*_{\varphi_{g\,j}}:\mathcal{F}\to L^2(\mathbb{R}^d)$, with the domain

$$D(T^*_{\varphi_{g\,j}}) = \{\Phi\in\mathcal{F}|(\kappa(\cdot),\Phi)\in L^2(\mathbb{R}^d)\}$$

is referred to as Carleman operator...

It is known [39, Theorem 6.12] that $\int_{\mathbb{R}^d}\|\kappa_{\varphi_{g\,j}}(k)\|^2\,dk<\infty$ if and only if $T^*_{\varphi_{g\,j}}$ is a Hilbert–Schmidt operator. When $T^*_{\varphi_{g\,j}}$ is a Hilbert–Schmidt operator, it is also known that

$$\mathrm{Tr}(T^{**}_{\varphi_{g\,j}}T^*_{\varphi_{g\,j}}) = \int_{\mathbb{R}^d}\|\kappa_{\varphi_{g\,j}}(k)\|^2\,dk,$$

which implies that $\int_{\mathbb{R}^d}\|\kappa_{\varphi_{g\,j}}(k)\|^2\,dk<\infty$ if and only if $\overline{T}_{\varphi_{g\,j}}(=T^{**}_{\varphi_{g\,j}})$ is a Hilbert–Schmidt operator,

$$\mathrm{Tr}((\overline{T}_{\varphi_{g\,j}})^*\overline{T}_{\varphi_{g\,j}}) = \mathrm{Tr}(T^{**}_{\varphi_{g\,j}}T^*_{\varphi_{g\,j}} = \int_{\mathbb{R}^d}\|\kappa_{\varphi_{g\,j}}(k)\|^2\,dk.$$

Thus the proposition follows. □

The main theorem in this section is as follows.

Theorem 2.9. *Suppose* (B1)–(B4). *Then* (1), (2) *and* (3) *are equivalent.*

(1) $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$,
(2) $\overline{T}_{\varphi_{\rm g}\, j}$ *is a Hilbert–Schmidt operator for all* $j = 1, \ldots, D$ *and all* $\varphi_{\rm g} \in P_H\mathcal{F}$,
(3) $\int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\rm g}\|^2 \, dk < \infty$ *for all* $j = 1, \ldots, D$ *and all* $\varphi_{\rm g} \in P_H\mathcal{F}$.

Suppose that one of (1), (2) *and* (3) *holds, it follows that for an arbitrary ground state* $\varphi_{\rm g}$,

$$\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = g^2 \sum_{j=1}^{D} \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_{\rm g}\|^2 \, dk. \tag{2.18}$$

Proof. Let $\{e_m\}_{m=1}^{\infty}$ be a complete orthonormal system of $L^2(\mathbb{R}^d)$ such that $e_m/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. It is proven in Lemma 2.2 that $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ if and only if

$$\sum_{j=1}^{D} \sum_{m=1}^{\infty} \|(1 \otimes a(\overline{e_m}, j))\varphi_{\rm g}\|^2 < \infty \tag{2.19}$$

for an arbitrary $\varphi_{\rm g} \in P_H\mathcal{F}$. By (2.17), $(1 \otimes a(\overline{e_m}, j))\varphi_{\rm g} = -gT_{\varphi_{\rm g}\, j}\overline{e_m} = -g\overline{T}_{\varphi_{\rm g}\, j}\overline{e_m}$. Hence $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ if and only if

$$g^2 \sum_{j=1}^{D} \sum_{m=1}^{\infty} \|\overline{T}_{\varphi_{\rm g}\, j}\overline{e_m}\|^2 < \infty, \qquad \varphi_{\rm g} \in P_H\mathcal{F}.$$

That is to say, $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ if and only if $\overline{T}_{\varphi_{\rm g}\, j}$ is a Hilbert–Schmidt operator for all $j = 1, \ldots, D$, and all $\varphi_{\rm g} \in P_H\mathcal{F}$, i.e., by Lemma 2.8, $P_H\mathcal{F} \subset D(1 \otimes N^{1/2})$ if and only if

$$g^2 \sum_{j=1}^{D} \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}\, j}(k)\|^2 \, dk < \infty, \qquad \varphi_{\rm g} \in P_H\mathcal{F}.$$

Then the first half of the theorem is proven. Moreover by Lemma 2.2, when $\varphi_{\rm g} \in D(1 \otimes N^{1/2})$, $\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = \sum_{j=1}^{D} \sum_{m=1}^{\infty} \|(1 \otimes a(\overline{e_m}, j))\varphi_{\rm g}\|^2$, which yields that

$$\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = g^2 \sum_{j=1}^{D} \mathrm{Tr}((\overline{T}_{\varphi_{\rm g}\, j})^* \overline{T}_{\varphi_{\rm g}\, j}) = g^2 \sum_{j=1}^{D} \int_{\mathbb{R}^d} \|\kappa_{\varphi_{\rm g}\, j}(k)\|^2 \, dk.$$

Thus the proof is complete. □

Remark 2.10. In [7] a more general formula than (2.18) is obtained.

3. Proof of $\mathbf{m}(H) \leqslant \mathbf{m}(A)$

3.1. Quadratic forms

We revive $H = H_0 + gH_{\mathrm{I}}$, where $H_0 = A \otimes 1 + 1 \otimes d\Gamma(S)$, and (A.1)–(A.3) are assumed. Set

$$\overline{H}_0 := H_0 - E(H_0).$$

Actually $E(H_0) = E(A)$. The quadratic form β_0 associated with $\overline{H}_0$ is defined by

$$\beta_0(\Psi, \Phi) := (\overline{H}_0^{1/2}\Psi, \overline{H}_0^{1/2}\Phi), \quad \Psi, \Phi \in D(\overline{H}_0^{1/2}).$$

Define a symmetric form by

$$\beta_{H_{\mathrm{I}}}(\Psi, \Phi) := (\Psi, H_{\mathrm{I}}\Phi), \quad \Psi, \Phi \in D(\overline{H}_0).$$

Since $\|H_{\mathrm{I}}\Psi\| \leqslant a\|H_0\Psi\| + b\|\Psi\|$, it follows that $\|H_{\mathrm{I}}\Psi\| \leqslant a\|\overline{H}_0\Psi\| + b'\|\Psi\|$, where $b' = b + a|E(H_0)|$. By an interpolation argument [34, Section IX], it obeys that

$$\|(\overline{H}_0 + \mu)^{-1/2} H_{\mathrm{I}} (\overline{H}_0 + \mu)^{-1/2}\| \leqslant a + b'/\mu.$$

Then

$$|\beta_{H_{\mathrm{I}}}(\Psi, \Psi)| \leqslant (a + b'/\mu)\beta_0(\Psi, \Psi) + (a + b'/\mu)\|\Psi\|^2, \quad \Psi \in D(\overline{H}_0), \tag{3.1}$$

for an arbitrary $\mu > 0$. By (3.1), a polarization identity and a limiting argument, $\beta_{H_{\mathrm{I}}}(\Psi, \Phi)$ can be extended to $\Psi, \Phi \in D(\overline{H}_0^{1/2})$. The extension of $\beta_{H_{\mathrm{I}}}$ is denoted by $\widetilde{\beta}_{H_{\mathrm{I}}}$, and which satisfies

$$|\widetilde{\beta}_{H_{\mathrm{I}}}(\Psi, \Psi)| \leqslant (a + b'/\mu)\beta_0(\Psi, \Psi) + (a + b'/\mu)\|\Psi\|^2, \quad \Psi \in D(\overline{H}_0^{1/2}). \tag{3.2}$$

Thus we see that, for a sufficiently small g,

$$\beta_H := \beta_0 + g\widetilde{\beta}_{H_{\mathrm{I}}}$$

is a semibounded closed quadratic form on $D(\overline{H}_0^{1/2}) \times D(\overline{H}_0^{1/2})$. Then by the representation theorem for forms [30, p.322, Theorem 2.1], there exists a unique self-adjoint operator H' such that $D(H') \subset D(\overline{H}_0^{1/2})$ and

$$\beta_H(\Psi, \Phi) = (\Psi, H'\Phi), \quad \Psi \in D(\overline{H}_0^{1/2}), \quad \Phi \in D(H').$$

On the other hand, we can see directly that $D(H) \subset D(\overline{H}_0^{1/2})$ and

$$\beta_H(\Psi, \Phi) = (\Psi, H\Phi), \quad \Psi \in D(\overline{H}_0^{1/2}), \quad \Phi \in D(\overline{H}_0),$$

which yields that $H' = H$. I.e., H is a unique self-adjoint operator associated with the quadratic form β_H. We generalize this fact in the next subsection.

3.2. Abstract results

As was seen in the previous subsection, self-adjoint operator $H = H_0 + gH_{\rm I}$ is defined through the quadratic form β_H. In this subsection, as a mathematical generalization, we define a total Hamiltonian $H_{\rm q}$ through an abstract quadratic form, and estimate an upper bound of $\dim\left\{P_{H_{\rm q}}\mathcal{F} \cap D(1 \otimes N^{1/2})\right\}$.

Remark 3.1. The Nelson Hamiltonians without ultraviolet cutoffs are defined as the self-adjoint operator associated with a semibounded quadratic form. See [2,22,33]. As far as we know, it cannot be represented as the form $H_0 + gH_{\rm I}$.

Let $\beta_{\rm int}$ be a symmetric quadratic form with form domain $D(\overline{H}_0^{1/2})$ such that

$$|\beta_{\rm int}(\Psi, \Psi)| \leqslant a\beta_0(\Psi, \Psi) + b(\Psi, \Psi), \qquad \Psi \in D(\overline{H}_0^{1/2}) \tag{3.3}$$

with some nonnegative constants a and b. Define the quadratic form β on $D(\overline{H}_0^{1/2})$ by

$$\beta := \beta_0 + g\beta_{\rm int}.$$

Proposition 3.2. *Let $|g| < 1/a$. Then there exists a unique self-adjoint operator $H_{\rm q}$ associated with β such that its form domain is $D(\overline{H}_0^{1/2})$,*

$$\beta(\Psi, \Phi) = (\Psi, H_{\rm q}\Phi), \qquad \Psi \in D(\overline{H}_0^{1/2}), \Phi \in D(H_{\rm q})$$

and

$$\beta(\Psi, \Phi) = (H_{{\rm q}+}^{1/2}\Psi, H_{{\rm q}+}^{1/2}\Phi) - (H_{{\rm q}-}^{1/2}\Psi, H_{{\rm q}-}^{1/2}\Phi), \qquad \Psi, \Phi \in D(\overline{H}_0^{1/2}),$$

where $H_{{\rm q}+} := H_{\rm q}E_{H_{\rm q}}((0, \infty))$ and $H_{{\rm q}-} := -H_{\rm q}E_{H_{\rm q}}((-\infty, 0])$.

Proof. From (3.3) it follows that $|g\beta_{\rm int}(\Psi,\Psi)| \leqslant |g| a\beta_0(\Psi,\Psi) + |g| b(\Psi,\Psi)$. Hence by the KLMN theorem [34, Theorem X.17], the proposition follows. □

Assumptions (Gap) and (N) are as follows.

(Gap) $\inf \sigma_{\rm ess}(A) - E(A) > 0$.

(N) $\displaystyle \lim_{g\to 0} \sup_{\Psi\in(P_{H_{\rm q}}\mathcal{F})\cap D(1\otimes N^{1/2})} \frac{\|(1\otimes N^{1/2})\Psi\|}{\|\Psi\|} = 0.$

Suppose that $\sigma_{\rm p}(S) \not\ni 0$. Then by the facts that $\inf \sigma(d\Gamma(S)\lceil_{\oplus_{n=1}^\infty[\otimes_s^n \mathcal{W}]}) \geqslant 0$, $\sigma_{\rm p}(d\Gamma(S)\lceil_{\oplus_{n=1}^\infty[\otimes_s^n \mathcal{W}]}) \not\ni 0$, and $\sigma(d\Gamma(S)\lceil_{\otimes_s^0 \mathcal{W}}) = \sigma_{\rm p}(d\Gamma(S)\lceil_{\otimes_s^0 \mathcal{W}}) = \{0\}$, it is seen that $d\Gamma(S)$ is a nonnegative self-adjoint operator, and has a unique ground state Ω with eigenvalue 0. We have a lemma.

Lemma 3.3. *Assume* (A1), (A2), (3.3), (Gap), (N) *and* $\sigma_{\rm p}(S) \not\ni 0$. *Then there exists* $\delta(g) > 0$ *such that* $\lim_{g\to 0}\delta(g) = 0$ *and, for* g *with* $\delta(g) < 1$,

$$\dim\left\{(P_{H_{\rm q}}\mathcal{F}) \cap D(1\otimes N^{1/2})\right\} \leqslant \frac{1}{1-\delta(g)} {\rm m}(A). \tag{3.4}$$

Proof. Let $\varepsilon > 0$ be such that $[E(A), E(A)+\varepsilon) \cap \sigma(A) = \{E(A)\}$ and we set $\mathcal{P}_\varepsilon := E_A([E(A), E(A)+\varepsilon))$ and $\mathcal{P}_\varepsilon^\perp := 1 - \mathcal{P}_\varepsilon$. Furthermore let $\mathcal{P}_\Omega := E_{d\Gamma(S)}(\{0\})$. We fix a $\varphi_{\rm g} \in (P_{H_{\rm q}}\mathcal{F}) \cap D(1\otimes N^{1/2})$. Using the inequality $1\otimes 1 \leqslant 1\otimes N + 1\otimes \mathcal{P}_\Omega$ in the sense of form, we have

$$(\varphi_{\rm g}, \varphi_{\rm g}) \leqslant \|(1\otimes N^{1/2})\varphi_{\rm g}\|^2 + \|(\mathcal{P}_\varepsilon \otimes \mathcal{P}_\Omega)\varphi_{\rm g}\|^2 + \|(\mathcal{P}_\varepsilon^\perp \otimes \mathcal{P}_\Omega)\varphi_{\rm g}\|^2. \tag{3.5}$$

Let $Q := \mathcal{P}_\varepsilon^\perp \otimes \mathcal{P}_\Omega$. It is checked that $\varphi_{\rm g} \in D(\overline{H}_0^{1/2})$, $Q\varphi_{\rm g} \in D(\overline{H}_0^{1/2})$ and $\overline{H}_0^{1/2} Q\varphi_{\rm g} = Q\overline{H}_0^{1/2}\varphi_{\rm g}$. Hence we have

$$\begin{aligned} 0 &= (Q\varphi_{\rm g}, (H_{\rm q} - E(H_{\rm q}))\varphi_{\rm g}) \\ &= \beta_0(Q\varphi_{\rm g}, \varphi_{\rm g}) + g\beta_{\rm int}(Q\varphi_{\rm g}, \varphi_{\rm g}) - E(H_{\rm q})(Q\varphi_{\rm g}, \varphi_{\rm g}). \end{aligned}$$

From this we have

$$-g\beta_{\rm int}(Q\varphi_{\rm g}, \varphi_{\rm g}) = (\overline{H}_0^{1/2} Q\varphi_{\rm g}, \overline{H}_0^{1/2}\varphi_{\rm g}) - E(H_{\rm q})(Q\varphi_{\rm g}, \varphi_{\rm g}). \tag{3.6}$$

Since

$$\begin{aligned} &(\overline{H}_0^{1/2} Q\varphi_{\rm g}, \overline{H}_0^{1/2}\varphi_{\rm g}) = (\overline{H}_0^{1/2} Q\varphi_{\rm g}, \overline{H}_0^{1/2} Q\varphi_{\rm g}) \\ &\quad = \int_{[E(A)+\varepsilon,\infty)\times\{0\}} (\lambda + \mu - E(A))\, d\|(E_A(\lambda)\otimes E_{d\Gamma(S)}(\mu))Q\varphi_{\rm g}\|^2 \geqslant \varepsilon(\varphi_{\rm g}, Q\varphi_{\rm g}), \end{aligned}$$

then (3.6) implies that

$$-g\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g})\geqslant(\varepsilon-E(H_{\rm q}))(Q\varphi_{\rm g},\varphi_{\rm g}). \tag{3.7}$$

We shall estimate $|\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g})|$.

$$\begin{aligned}\beta_0(\varphi_{\rm g},\varphi_{\rm g}) &= (\varphi_{\rm g},H_{\rm q}\varphi_{\rm g})-g\beta_{\rm int}(\varphi_{\rm g},\varphi_{\rm g})\\ &\leqslant E(H_{\rm q})\|\varphi_{\rm g}\|^2+|g|\left(a\beta_0(\varphi_{\rm g},\varphi_{\rm g})+b(\varphi_{\rm g}.\varphi_{\rm g})\right),\end{aligned}$$

which yields that, since $|g|<1/a$, $\beta_0(\varphi_{\rm g},\varphi_{\rm g})\leqslant(E(H_{\rm q})+|g|b)(\varphi_{\rm g},\varphi_{\rm g})/(1-a|g|)$ follows. Then we have

$$|\beta_{\rm int}(\varphi_{\rm g},\varphi_{\rm g})|\leqslant a\beta_0(\varphi_{\rm g},\varphi_{\rm g})+b(\varphi_{\rm g},\varphi_{\rm g})\leqslant c_{\rm int}(\varphi_{\rm g},\varphi_{\rm g}),$$

where $c_{\rm int}:=\dfrac{a(E(H_{\rm q})+|g|b)}{1-a|g|}+b$. From the polarization identity, it follows that

$$|\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g})|\leqslant 2c_{\rm int}(\varphi_{\rm g},\varphi_{\rm g}). \tag{3.8}$$

Note that

$$|\beta(\Psi,\Psi)-\beta_0(\Psi,\Psi)|=|g||\beta_{\rm int}(\Psi,\Psi)|\leqslant|g|(a+b)\|(\overline{H}_0+1)^{1/2}\Psi\|^2.$$

Then

$$\lim_{g\to 0}\sup_{\Psi\in D(\overline{H}_0^{1/2})}\frac{|\beta(\Psi,\Psi)-\beta_0(\Psi,\Psi)|}{\|(\overline{H}_0+1)^{1/2}\Psi\|^2}\leqslant\lim_{g\to 0}|g|(a+b)=0,$$

which implies that for $z\in\mathbb{C}$ with $\Im z\neq 0$,

$$\lim_{g\to 0}\|(H_{\rm q}-z)^{-1}-(\overline{H}_0-z)^{-1}\|=0. \tag{3.9}$$

See e.g., [33]. Thus it follows that

$$\lim_{g\to 0}E(H_{\rm q})=E(\overline{H}_0)=0. \tag{3.10}$$

Then there exists a constant $c>0$ such that for all g with $|g|<c$, it obeys that $\varepsilon-E(H_{\rm q})>0$. Then by (3.7) and (3.8), for g with $|g|<c$,

$$\|Q\varphi_{\rm g}\|^2\leqslant|g|\frac{|\beta_{\rm int}(Q\varphi_{\rm g},\varphi_{\rm g})|}{\varepsilon-E(H_{\rm q})}\leqslant 2|g|\frac{c_{\rm int}}{\varepsilon-E(H_{\rm q})}\|\varphi_{\rm g}\|^2.$$

Let $c(g) := \sup_{\Psi\in(P_{H_{\mathrm{q}}}\mathcal{F})\cap D(1\otimes N^{1/2})} \|(1\otimes N^{1/2})\Psi\|/\|\Psi\|$. Together with (3.5) we have

$$(\varphi_{\mathrm{g}},\varphi_{\mathrm{g}})\leqslant c(g)^2\|\varphi_{\mathrm{g}}\|^2+2|g|\frac{c_{\mathrm{int}}}{\varepsilon-E(H_{\mathrm{q}})}\|\varphi_{\mathrm{g}}\|^2+\|(\mathcal{P}_\varepsilon\otimes\mathcal{P}_\Omega)\varphi_{\mathrm{g}}\|^2. \tag{3.11}$$

Setting $\delta(g) := c(g)^2+2|g|\frac{c_{\mathrm{int}}}{\varepsilon-E(H_{\mathrm{q}})}$, we see that by (3.10) and (N), $\lim_{g\to 0}\delta(g)=0$. Then by (3.11) there exists $g_*\leqslant c$ such that for g with $|g|<g_*$,

$$(\varphi_{\mathrm{g}},\varphi_{\mathrm{g}})\leqslant(1-\delta(g))^{-1}(\varphi_{\mathrm{g}},(\mathcal{P}_\varepsilon\otimes\mathcal{P}_\Omega)\varphi_{\mathrm{g}}). \tag{3.12}$$

Let $\{\varphi_{\mathrm{g}}^j\}_{j=1}^M$, $M\leqslant\infty$, be a complete orthonormal system of $(P_{H_{\mathrm{q}}}\mathcal{F})\cap D(1\otimes N^{1/2})$. Then by (3.12),

$$(\varphi_{\mathrm{g}}^j,\varphi_{\mathrm{g}}^j)\leqslant(1-\delta(g))^{-1}(\varphi_{\mathrm{g}}^j,(\mathcal{P}_\varepsilon\otimes\mathcal{P}_\Omega)\varphi_{\mathrm{g}}^j). \tag{3.13}$$

Summing up from $j=1$ to M, we have

$$\dim\left\{(P_{H_{\mathrm{q}}}\mathcal{F})\cap D(1\otimes N^{1/2})\right\}\leqslant(1-\delta(g))^{-1}\sum_{j=1}^M(\varphi_{\mathrm{g}}^j,(\mathcal{P}_\varepsilon\otimes\mathcal{P}_\Omega)\varphi_{\mathrm{g}}^j).$$

Since

$$\sum_{j=1}^M(\varphi_{\mathrm{g}}^j,(\mathcal{P}_\varepsilon\otimes\mathcal{P}_\Omega)\varphi_{\mathrm{g}}^j)\leqslant\mathrm{Tr}(P_A\otimes\mathcal{P}_\Omega)=\mathrm{Tr}\,P_A\times\mathrm{Tr}\mathcal{P}_\Omega=\mathrm{m}(A),$$

we obtain (3.4). Thus the lemma is proven. □

From Lemma 3.3, corollaries immediately follow.

Corollary 3.4. *Suppose the same assumptions as in Lemma* 3.3 *and, in addition,* $P_{H_{\mathrm{q}}}\mathcal{F}\subset D(1\otimes N^{1/2})$. *Then* $\mathrm{m}(H_{\mathrm{q}})\leqslant(1-\delta(g))^{-1}\mathrm{m}(A)$. *Moreover suppose that* g *is such that* $\delta(g)<1/2$. *Then* $\mathrm{m}(H)\leqslant\mathrm{m}(A)$.

Proof. Since $P_{H_{\mathrm{q}}}\mathcal{F}\cap D(1\otimes N^{1/2})=P_{H_{\mathrm{q}}}\mathcal{F}$, the corollary follows from Lemma 3.3. □

Corollary 3.5 (Overlap). *Suppose the same assumptions as in Lemma* 3.3 *and, in addition,* $P_{H_{\mathrm{q}}}\mathcal{F}\subset D(1\otimes N^{1/2})$. *Let* g *be such that* $\delta(g)<1$. *Then for an arbitrary ground state* φ_{g}, *it follows that* $(\varphi_{\mathrm{g}},(P_A\otimes\mathcal{P}_\Omega)\varphi_{\mathrm{g}})\neq 0$.

Proof. By (3.12) it is seen that

$$0 < \|\varphi_g\|^2 \leqslant (1-\delta(g))^{-1}(\varphi_g, (\mathcal{P}_\varepsilon \otimes \mathcal{P}_\Omega)\varphi_g) = (1-\delta(g))^{-1}(\varphi_g, (P_A \otimes \mathcal{P}_\Omega)\varphi_g).$$

Hence the corollary follows. □

3.3. Main theorem II

We assume that $\mathcal{W} = \oplus^D L^2(\mathbb{R}^3)$ and $S = [\omega]$, i.e., $H = H_0 + gH_{\mathrm{I}}$ and $H_0 = A \otimes 1 + 1 \otimes d\Gamma([\omega])$. Now we are in the position to state the main theorem in this section.

Theorem 3.6. *Suppose that* (B1)–(B4), (A1), (A3), (Gap). *In addition assume that for arbitrary* $\varphi_g \in P_H \mathcal{F}$,

$$\int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_g\|^2 \, dk < \infty$$

and

$$\lim_{g\to 0} g^2 \sup_{\varphi_g \in P_H\mathcal{F}} \frac{\sum_{j=1}^D \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_g\|^2 \, dk}{\|\varphi_g\|^2} = 0 \tag{3.14}$$

Then there exists a constant g_* *such that for* g *with* $|g| < g_*$, $\mathrm{m}(H) \leqslant \mathrm{m}(A)$.

Proof. By Theorem 2.9, it follows that $P_H \mathcal{F} \subset D(1 \otimes N^{1/2})$ and

$$\|(1 \otimes N^{1/2})\varphi_g\|^2 = g^2 \sum_{j=1}^D \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k)\varphi_g\|^2 \, dk.$$

By this and (3.14) we have $\lim_{g\to 0} \sup_{\Psi \in P_H \mathcal{F}} \|(1 \otimes N^{1/2})\varphi_g\| / \|\varphi_g\| = 0$. From this and Corollary 3.4, the theorem follows. □

4. Examples

4.1. GSB models

GSB models are a generalization of the spin-boson model, which was introduced and investigated in [4]. Examples of GSB models are e.g., N-level systems coupled to a Bose field, lattice spin systems, the Pauli–Fierz model with the dipole approximation neglected A^2 term, a Fermi field coupled to a Bose field, etc. See [4, p. 457].

The Hilbert space on which GSB Hamiltonians act is

$$\mathcal{F}_{\rm GSB} := \mathcal{H} \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^d)),$$

where $\mathcal{H}$ is a Hilbert space. Let $a(f)$ and $a^\dagger(f)$, $f \in L^2(\mathbb{R}^d)$, be the annihilation operator and the creation operator on $\mathcal{F}_{\rm b}(L^2(\mathbb{R}^d))$, respectively. We use the same notations $a(f)$ and $a^\dagger(f)$ as those of Section 1.1. We set

$$\phi(\lambda) := \frac{1}{\sqrt{2}}(a^\dagger(\bar{\lambda}) + a(\lambda)), \quad \lambda \in L^2(\mathbb{R}^d).$$

GSB Hamiltonians are defined by

$$H_{\rm GSB} := H_{\rm GSB,0} + \alpha H_{\rm GSB,I}.$$

Here $\alpha \in \mathbb{R}$ is a coupling constant, and

$$H_{\rm GSB,0} := A \otimes 1 + 1 \otimes d\Gamma(\omega_{\rm GSB}), \quad H_{\rm GSB,I} := \overline{\sum_{j=1}^{J} B_j \otimes \phi(\lambda_j)},$$

where $\omega_{\rm GSB} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a multiplication operator by $\omega_{\rm GSB}(k)$ such that $\omega_{\rm GSB}(\cdot) : \mathbb{R}^d \to [0, \infty)$ and $\overline{X}$ denotes the closure of X. Assumption (GSB1)–(GSB5) are as follows.

(GSB1) Operator A satisfies (A1). Set $\overline{A} := A - E(A)$.

(GSB2) $\lambda_j, \lambda_j/\sqrt{\omega_{\rm GSB}} \in L^2(\mathbb{R}^d)$, $j = 1, \ldots, J$.

(GSB3) B_j, $j = 1, \ldots, J$, is a symmetric operator, $D(\overline{A}^{1/2}) \subset \cap_{j=1}^{J} D(B_j)$ and there exist constants a_j and b_j such that

$$\|B_j f\| \leqslant a_j \|\overline{A}^{1/2} f\| + b_j \|f\|, \quad f \in D(\overline{A}^{1/2}).$$

Moreover $|\alpha| < \left(\sum_{j=1}^{J} a_j \|\lambda_j/\sqrt{\omega_{\rm GSB}}\|\right)^{-1}$.

(GSB4) $\omega_{\rm GSB}$ satisfies that (1) $\omega_{\rm GSB}(\cdot)$ is continuous, (2) $\lim_{|k|\to\infty} \omega_{\rm GSB}(k) = \infty$, (3) there exist constants $C > 0$ and $\gamma > 0$ such that

$$|\omega_{\rm GSB}(k) - \omega_{\rm GSB}(k')| \leqslant C|k - k'|^\gamma (1 + \omega_{\rm GSB}(k) + \omega_{\rm GSB}(k')).$$

(GSB5) λ_j, $j = 1, \ldots, J$, is continuous.

Assume (GSB1)–(GSB3). Then it can be shown that $H_{\rm GSB}$ is self-adjoint on $D(H_{\rm GSB,0}) = D(A \otimes 1) \cap D(1 \otimes d\Gamma(\omega_{\rm GSB}))$ and bounded from below. Moreover it is essentially self-adjoint on any core of $H_{\rm GSB,0}$. We introduce assumptions.

(IR) $\lambda_j/\omega_{\rm GSB} \in L^2(\mathbb{R}^d)$, $j = 1, \ldots, J$.
(GSB6) $\omega_{\rm GSB}$ satisfies (B1) with ω replaced by $\omega_{\rm GSB}$.
(GSB7) $\lambda_j \in C^2(\mathbb{R}^d \backslash K)$, $j = 1, \ldots, J$, where K satisfies (**B1**).

Proposition 4.1. *Assume* (GSB1)–(GSB5), (IR) *and* (Gap). *Then there exists a constant* $\alpha_* > 0$ *such that for* α *with* $|\alpha| < \alpha_*$, $H_{\rm GSB}$ *has a ground state* $\varphi_{\rm g}$ *such that* $\|(1 \otimes N^{1/2})\varphi_{\rm g}\| < \infty$.

Proof. See [4, Theorem 1.3, 8, Appendix]. □

Let $f \in C_0^2(\mathbb{R}^d \setminus K)$ and $\Psi, \Phi \in D(H_{\rm GSB})$. We have

$$[a(f), H_{\rm GSB,I}]_W^{D(H_{\rm GSB})}(\Psi, \Phi) = \int_{\mathbb{R}^d} f(k)(\Psi, T_{\rm GSB}(k)\Phi)\, dk,$$

where $T_{\rm GSB}(k) := \sum_{j=1}^{J} \lambda_j(k)(B_j \otimes 1)$.

Theorem 4.2. *Suppose* (GSB1)–(GSB3), (IR), (GSB 6) *and* (GSB 7). *Then it follows that*

$$P_{H_{\rm GSB}} \mathcal{F}_{\rm GSB} \subset D(1 \otimes N^{1/2}) \tag{4.1}$$

and

$$\|(1 \otimes N^{1/2})\varphi_{\rm g}\|^2 = \alpha^2 \int_{\mathbb{R}^d} \|(H_{\rm GSB} - E(H_{\rm GSB}) + \omega_{\rm GSB}(k))^{-1} T_{\rm GSB}(k)\varphi_{\rm g}\|^2\, dk. \tag{4.2}$$

In addition, suppose (Gap). *Then there exists* α_{**} *such that for* α *with* $|\alpha| < \alpha_{**}$,

$$\mathrm{m}(H_{\rm GSB}) \leqslant \mathrm{m}(A). \tag{4.3}$$

Proof. We shall check assumptions (B1)–(B4) and (3) of Theorem 2.9 with the following identifications:

$$\mathcal{F} = \mathcal{F}_{\rm GSB}, \quad H_0 = H_{\rm GSB,0}, \quad H_{\rm I} = H_{\rm GSB,I}, \quad \omega = \omega_{\rm GSB}, \quad D = 1,$$
$$T_{j=1}(k) = T_{\rm GSB}(k).$$

(B1) and (B2) have been already checked. We have

$$\int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\rm GSB}-E(H_{\rm GSB})+\omega_{\rm GSB}(k))}T_{\rm GSB}(k)\varphi_{\rm g})\, dk$$
$$= \sum_{j=1}^J (\Psi, e^{-is(H_{\rm GSB}-E(H_{\rm GSB}))}(B_j\otimes 1)\varphi_{\rm g}) \int_{\mathbb{R}^d} f(k)\lambda_j(k)e^{-is\omega_{\rm GSB}(k)}\, dk. \quad (4.4)$$

Since $f\lambda_j \in C_0^2(\mathbb{R}^d\setminus K)$, we see that by Lemma 2.5, $\left|\int_{\mathbb{R}^d} f(k)\lambda_j(k)e^{-is\omega_{\rm GSB}(k)}\, dk\right| \in L^1([0,\infty), ds)$, which implies, together with (4.4), that (B3) follows. We have

$$\int_{\mathbb{R}^d} \|T_{\rm GSB}(k)\varphi_{\rm g}\|^2\, dk \leqslant J\sum_{j=1}^J\left(\int_{\mathbb{R}^d}|\lambda_j(k)|^2\, dk\right)\|(B_j\otimes 1)\varphi_{\rm g}\|^2 < \infty$$

and

$$\int_{\mathbb{R}^d} \|(H_{\rm GSB}-E(H_{\rm GSB})+\omega_{\rm GSB}(k))^{-1}T_{\rm GSB}(k)\varphi_{\rm g}\|^2\, dk$$
$$\leqslant J\sum_{j=1}^J\left(\int_{\mathbb{R}^d}\frac{|\lambda_j(k)|^2}{\omega_{\rm GSB}(k)^2}\, dk\right)\|(B_j\otimes 1)\varphi_{\rm g}\|^2 < \infty.$$

Thus (B4) and (3) of Theorem 2.9 follow. Hence (4.1) and (4.2) are proven. We check (3.14) in Theorem 3.6 to show (4.3). Note that with some constants c_1 and c_2 independent of α, we have

$$\|(B_j\otimes 1)\varphi_{\rm g}\| \leqslant (a_j\,(c_1E(H_{\rm GSB})+c_2)^{1/2}+b_j)\|\varphi_{\rm g}\|. \quad (4.5)$$

Thus

$$\lim_{\alpha\to 0}\sup_{\varphi_{\rm g}\in P_{H_{\rm GSB}}\mathcal{F}_{\rm GSB}} \frac{\alpha^2\int_{\mathbb{R}^d}\|(H_{\rm GSB}-E(H_{\rm GSB})+\omega_{\rm GSB}(k))^{-1}T_{\rm GSB}(k)\varphi_{\rm g}\|^2\, dk}{\|\varphi_{\rm g}\|^2}$$
$$\leqslant \lim_{\alpha\to 0}\alpha^2 J\sum_{j=1}^J (a_j\,(c_1E(H_{\rm GSB})+c_2)^{1/2}+b_j)^2\|\lambda_j/\omega_{\rm GSB}\|^2 = 0.$$

Then (4.3) follows from Theorem 3.6. □

Corollary 4.3. *Assume* (GSB1)–(GSB4), (GSB6), (GSB7), (IR) *and* (Gap). *Then there exists* α_{***} *such that for* α *with* $|\alpha| < \alpha_{***}$, $H_{\rm GSB}$ *has a ground state and* $\mathrm{m}(H_{\rm GSB}) \leqslant \mathrm{m}(A)$. *In particular in the case of* $\mathrm{m}(A)=1$, $H_{\rm GSB}$ *has a unique ground state.*

Proof. It follows from Proposition 4.1 and Theorem 4.2. □

4.2. The Pauli–Fierz model

The Pauli–Fierz model describes a minimal interaction between electrons with spin 1/2 and a quantized radiation field quantized in the Coulomb gauge. The asymptotic field for $H_{\rm PF}$ is studied in e.g., [14,26]. The Hilbert space for state vectors of the Pauli–Fierz Hamiltonian is given by

$$\mathcal{F}_{\rm PF} := L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^3 \times \{1,2\})).$$

Formally the annihilation operator and the creation operator of $\mathcal{F}_{\rm b}(L^2(\mathbb{R}^3\times\{1,2\}))$ is denoted by $a^\sharp(f,j) = \int f(k) a^\sharp(k,j)\,dk$. The Pauli–Fierz Hamiltonian with ultraviolet cutoff $\hat{\varphi}$ is defined by

$$H_{\rm PF} := \frac{1}{2m}(p\otimes 1 - eA_{\hat{\varphi}})^2 + V\otimes 1 + 1\otimes H_{\rm f} - \frac{e}{2m}(\sigma\otimes 1)\cdot B_{\hat{\varphi}},$$

where $m>0$ and $e\in\mathbb{R}$ denote the mass of an electron and the charge of an electron, respectively. We regard e as a coupling constant. p denotes the momentum operator of an electron, i.e., $p = (p_1, p_2, p_3) = (-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_3})$, and V is an external potential. We identify $\mathcal{F}_{\rm PF}$ as

$$\mathcal{F}_{\rm PF} \cong \mathbb{C}^2 \otimes \int_{\mathbb{R}^3}^{\oplus} \mathcal{F}_{\rm b}(L^2(\mathbb{R}^3\times\{1,2\}))\,dx, \tag{4.6}$$

where $\int_{\mathbb{R}^3}^{\oplus}\cdots dx$ denotes a constant fiber direct integral [35]. $A_{\hat{\varphi}}$ and $B_{\hat{\varphi}}$ denote a quantized radiation field and a quantized magnetic field with ultraviolet cutoff $\hat{\varphi}$, respectively, which are given by, under identification (4.6),

$$A_{\hat{\varphi}} := 1\otimes\int_{\mathbb{R}^3}^{\oplus} A_{\hat{\varphi}}(x)\,dx, \qquad B_{\hat{\varphi}} := 1\otimes\int_{\mathbb{R}^3}^{\oplus} B_{\hat{\varphi}}(x)\,dx$$

with

$$A_{\hat{\varphi}}(x) := \sum_{j=1,2}\int \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}} e(k,j)\left\{e^{-ikx}a^\dagger(k,j) + e^{ikx}a(k,j)\right\}dk$$

and

$$B_{\hat{\varphi}}(x) := \sum_{j=1,2}\int \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}}(-ik\times e(k,j))\left\{e^{-ikx}a^\dagger(k,j) - e^{ikx}a(k,j)\right\}dk.$$

Here $\omega_{\rm PF}(k) := |k|$ and $\hat{\varphi}$ denotes an ultraviolet cutoff function. $H_{\rm f} := d\Gamma([\omega_{\rm PF}])$ is the second quantization of the multiplication operator $[\omega_{\rm PF}] : L^2(\mathbb{R}^3\times\{1,2\}) \to L^2(\mathbb{R}^3\times\{1,2\})$ such that $([\omega_{\rm PF}]f)(k,j) = \omega_{\rm PF}(k)f(k,j)$. Vector $e(k,j) \in \mathbb{R}^3$, $j = 1,2$, denotes a polarization vector satisfying $e(k,1)\cdot e(k,2) = 0$, $e(k,1)\times e(k,2) = k/|k|$ and $|e(k,j)| = 1$, $j = 1,2$. Finally $\sigma := (\sigma_1,\sigma_2,\sigma_3)$ denotes 2×2 Pauli matrices satisfying the anticommutation relations, $\{\sigma_i,\sigma_j\} = 2\delta_{ij}$ for $i,j = 1,2,3$, where $\{A,B\} := AB+BA$. Assumptions (PF1)–(PF3) are as follows.

(PF1) (1) $\sqrt{\omega_{\rm PF}}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega_{\rm PF}}, \hat{\varphi}/\omega_{\rm PF} \in L^2(\mathbb{R}^3)$ and $\hat{\varphi}(k) = \hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$. (2) V is Δ-bounded with a relative bound strictly less than one.

(PF2) (1) $\hat{\varphi} \in C^\infty(\mathbb{R}^3)$. (2) $e(\cdot,j) \in C^\infty(\mathbb{R}^3\setminus\mathcal{Q})$, $j = 1,2$, with some measurable set $\mathcal{Q}$ with its Lebesgue measure zero.

(PF3) The ground state energy of self-adjoint operator $h_{\rm p} := -\frac{1}{2m}\Delta + V$ acting in $L^2(\mathbb{R}^3)$ is discrete.

Let $H_{\rm PF,0}$ be $H_{\rm PF}$ with $e = 0$, i.e., $H_{\rm PF,0} := H_{\rm p}\otimes 1 + 1\otimes H_{\rm f}$, where

$$H_{\rm p} := \begin{pmatrix} h_{\rm p} & 0 \\ 0 & h_{\rm p}\end{pmatrix}$$

acting in $L^2(\mathbb{R}^3;\mathbb{C}^2)\cong L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$. In what follows, simply we write $T\otimes 1$ for $\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\otimes 1$ unless confusions arise. We note that $H_{\rm PF,0}$ is self-adjoint on $D(H_{\rm PF,0}) = D(\Delta\otimes 1)\cap D(1\otimes H_{\rm f})$. Note that $(p\otimes 1)\cdot A_{\hat{\varphi}} = A_{\hat{\varphi}}\cdot(p\otimes 1)$ on $D(H_{\rm PF,0})$. We set

$$H_{\rm PF} = H_{\rm PF,0} + eH_{\rm PF,I},$$

where

$$H_{\rm PF,I} := -\frac{1}{m}(p\otimes 1)\cdot A_{\hat{\varphi}} + \frac{e}{2m}A_{\hat{\varphi}}\cdot A_{\hat{\varphi}} - \frac{1}{2m}(\sigma\otimes 1)\cdot B_{\hat{\varphi}}.$$

Assume (PF1). In [25,27] it is shown that $H_{\rm PF}$ is self-adjoint on $D(H_{\rm PF,0})$ and bounded from below. Moreover it is essentially self-adjoint on any core of $H_{\rm PF,0}$.

Proposition 4.4. *Suppose* (PF1) *and* (PF3). *Then there exists a constant* $e_* \leqslant \infty$ *such that for* e *with* $|e|\leqslant e_*$, $H_{\rm PF}$ *has a ground state such that* $\varphi_{\rm g} \in D(1\otimes N^{1/2})$.

Proof. See e.g., [10,11,17,23,31,32]. □

Remark 4.5. Spinless Pauli–Fierz Hamiltonians are defined by

$$H_{\rm PF}^{\rm spinless} := \frac{1}{2m}(p\otimes 1 - eA_{\hat{\varphi}})^2 + V\otimes 1 + 1\otimes H_{\rm f},$$

which acts in $\mathcal{F} = L^2(\mathbb{R}^3) \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^3 \times \{1,2\}))$. It can be proven that $H_{\rm PF}^{\rm spinless}$ has a ground state $\varphi_{\rm g}$ such that $\varphi_{\rm g} \in D(1 \otimes N^{1/2})$, and it is unique [24]. Then it follows that $P_{H_{\rm PF}^{\rm spinless}} \mathcal{F} \subset D(1 \otimes N^{1/2})$.

We have

$$[1 \otimes a(f,j), H_{\rm PF,I}]_W^{D(H_{\rm PF})}(\Psi, \Phi) = \int f(k)(\Psi, T_{{\rm PF}\,j}(k)\Psi)\, dk,$$

where $T_{{\rm PF}\,j}(k) := T_{{\rm PF}\,j}^{(1)}(k) + T_{{\rm PF}\,j}^{(2)}(k)$ with

$$T_{{\rm PF}\,j}^{(1)}(k) := -\frac{1}{m}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}} e^{-ikx} e(k,j) \cdot (p \otimes 1 - eA_{\hat{\varphi}}),$$
$$T_{{\rm PF}\,j}^{(2)}(k) := -\frac{1}{2m}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}} e^{-ikx} (-ik \times e(k,j)) \cdot (\sigma \otimes 1).$$

Let $H_{\rm PF}^0$ be $H_{\rm PF}$ with $V = 0$. Then the binding energy is defined by

$$E_{\rm bin} := E(H_{\rm PF}^0) - E(H_{\rm PF}).$$

Proposition 4.6. *Suppose* (PF1) *and* (PF3). *Then* $E_{\rm bin} \geqslant -E(Hp)$.

Proof. See [17,22]. □

Assumption (V) is as follows.

(V) Potential $V = V_+ - V_-$ $(V_+(x) = \max\{0, V(x)\},\ V_-(x) = \min\{0, V(x)\})$ satisfies that (1) $\lim_{|x|\to\infty} V_-(x) = V_\infty < \infty$, (2) $|x|^2 V_- \in L^\infty_{\rm loc}(\mathbb{R}^3)$, (3) $E(H_{\rm p}) < -V_\infty$.

Lemma 4.7. *Suppose* (PF1), (PF3) *and* (V). *Then for a sufficiently small* $\varepsilon > 0$ *there exists a constant* $c(\varepsilon)$ *independent of* e *such that*

$$\sup_{\varphi_{\rm g} \in P_H \mathcal{F}} \frac{\|(|x| \otimes 1)\varphi_{\rm g}\|}{\|\varphi_{\rm g}\|} < \frac{c(\varepsilon)}{E_{\rm bin} - V_\infty - \varepsilon}.$$

Proof. It can be proven in the similar manner as [17,22]. □

Remark 4.8. Proposition 4.6 and (3) of (V) imply that $E_{\rm bin} - V_\infty > 0$. Furthermore combining Lemma 4.7 and (3) of (V) imply we have

$$\sup_{\varphi_{\rm g}\in P_H\mathcal{F}} \frac{\|(|x|\otimes 1)\varphi_{\rm g}\|}{\|\varphi_{\rm g}\|} < \frac{c(\varepsilon)}{-E(H_{\rm p}) - V_\infty - \varepsilon} := c_{\rm exp}, \tag{4.7}$$

where we note that $c_{\rm exp} > 0$ is independent of e.

Theorem 4.9. *Assume* (PF1), (PF2), *and* (V). *Then it follows that*

$$P_{H_{\rm PF}}\mathcal{F}_{\rm PF} \subset D(1\otimes N^{1/2}) \tag{4.8}$$

and

$$\|(1\otimes N^{1/2})\varphi_{\rm g}\|^2 = e^2 \sum_{j=1,2} \int_{\mathbb{R}^3} \|(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))^{-1} T_{{\rm PF}\,j}(k)\varphi_{\rm g}\|^2\, dk. \tag{4.9}$$

In addition, assume (PF3), *then there exists a constant e_{**} such that for e with $|e| < e_{**}$,*

$$\mathrm{m}(H_{\rm PF}) \leqslant \mathrm{m}(H_{\rm p}). \tag{4.10}$$

To prove Theorem 4.9 it is sufficient to check (B1)–(B4), (3) of Theorem 2.9 and (3.14) with the following identifications:

$$\mathcal{F} = \mathcal{F}_{\rm PF}, \quad H_0 = H_{\rm PF,0}, \quad H_{\rm I} = H_{\rm PF,I}, \quad \omega = \omega_{\rm PF}, \quad D = 2, \quad T_j(k) = T_{{\rm PF}\,j}(k).$$

Let $K := \bigcup_{n=1}^3 \{(k_1,k_2,k_3)\in\mathbb{R}^3 | k_n = 0\}$ and $\widetilde{K} := K \cup \mathcal{Q} \cup \{0\}$.

Lemma 4.10. *Assume* (PF1) *and* (PF2). *Then for $f \in C_0^2(\mathbb{R}^3\setminus\widetilde{K})$ and $\Psi \in D(H_{\rm PF})$,*

$$\left| \int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))} T_{{\rm PF}\,j}^{(l)}(k)\varphi_{\rm g})\, dk \right| \in L^1([0,\infty), ds), \quad l = 1,2.$$

Proof. We see that

$$\begin{aligned}
&\int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))} T_{{\rm PF}\,j}^{(1)}(k)\varphi_{\rm g})\, dk \\
&\quad = -\frac{1}{m} \sum_{\mu=1,2,3} ((p\otimes 1 - eA_{\hat{\varphi}})_\mu e^{-is(H_{\rm PF} - E(H_{\rm PF}))}\Psi, K_\mu^{(1)}(s,x,j)\varphi_{\rm g}),
\end{aligned}$$

where $K^{(1)}(s,x,j) = \frac{1}{s}(K_1^{(1)}(s,x,j) + x_\mu K_2^{(1)}(s,x,j))$ with

$$K_1^{(1)}(s,x,j) := -i\int_{\mathbb{R}^3} e^{-i(s\omega_{\rm PF}(k)+kx)}\frac{\partial}{\partial k_\mu}\left(\frac{\omega_{\rm PF}(k)}{k_\mu}\frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}}f(k)e(k,j)\right)dk,$$

$$K_2^{(1)}(s,x,j) := \int_{\mathbb{R}^3} e^{-i(s\omega_{\rm PF}(k)+kx)}\frac{\partial}{\partial k_\mu}\left(\frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}}f(k)e(k,j)\right)dk.$$

From the fact that $\hat\varphi\in C^\infty(\mathbb{R}^3)$ and $f\in C_0^2(\mathbb{R}^3\setminus\widetilde{K})$, it follows that for $v=1,2,3$,

$$\frac{\partial}{\partial k_\mu}\left(\frac{\omega_{\rm PF}(k)}{k_\mu}\frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}}f(k)e_v(k,j)\right)\in C_0^\infty(\mathbb{R}^3\setminus\{0\}),$$

$$\frac{\partial}{\partial k_\mu}\left(\frac{\hat\varphi(k)}{\sqrt{2\omega_{\rm PF}(k)}}f(k)e_v(k,j)\right)\in C_0^\infty(\mathbb{R}^3\setminus\{0\}).$$

Thus by [36, Theorem XI.19 (c)] there exist constants c_1 and c_2 such that

$$\sup_x |K_{l,\mu}^{(1)}(s,x,j)| \leqslant \frac{c_l}{1+s},\qquad l=1,2,\qquad \mu=1,2,3. \tag{4.11}$$

By this we have

$$\|K_\mu^{(1)}(s,x,j)\varphi_{\rm g}\| \leqslant \frac{1}{s(s+1)}\left\{c_1\|\varphi_{\rm g}\| + c_2\|(|x|\otimes 1)\varphi_{\rm g}\|\right\}. \tag{4.12}$$

Since $\|(p\otimes 1 - eA_{\hat\varphi})_\mu\Psi\| \leqslant c_1'\|(H_{\rm PF}-E(H_{\rm PF}))\Psi\| + c_2'\|\Psi\|$ with some constants c_1' and c_2', we conclude that

$$\begin{aligned}&\left|-\frac{1}{m}\sum_{\mu=1,2,3}((p\otimes 1 - eA_{\hat\varphi})_\mu e^{-is(H_{\rm PF}-E(H_{\rm PF}))}\Psi, K_\mu^{(1)}(s,x,j)\varphi_{\rm g})\right|\\ &\quad\leqslant \frac{3}{m}(c_1'\|(H_{\rm PF}-E(H_{\rm PF}))\Psi\| + c_2'\|\Psi\|)(c_1\|\varphi_{\rm g}\| + c_2\|(|x|\otimes 1)\varphi_{\rm g}\|)\frac{1}{s(1+s)}.\end{aligned} \tag{4.13}$$

Similarly we can estimate

$$\begin{aligned}&\int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))}{T_{\rm PF}}_j^{(2)}(k)\varphi_{\rm g})\,dk\\ &\quad= -\frac{1}{2m}\sum_{\mu=1,2,3}((\sigma_\mu\otimes 1)e^{-is(H_{\rm PF}-E(H_{\rm PF}))}\Psi, K_\mu^{(2)}(s,x,j)\varphi_{\rm g}),\end{aligned}$$

where $K^{(2)}(s,x,j) = \frac{1}{s}(K_1^{(2)}(s,x,j) + x_\mu K_2^{(2)}(s,x,j))$ with

$$K_1^{(2)}(s,x,j) := -i \int_{\mathbb{R}^3} e^{-i(s\omega_{\rm PF}(k)+kx)} \frac{\partial}{\partial k_\mu}\left(\frac{\omega_{\rm PF}(k)}{k_\mu} \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}} f(k)(-ik\times e(k,j))\right) dk,$$

$$K_2^{(2)}(s,x,j) := \int_{\mathbb{R}^3} e^{-i(s\omega_{\rm PF}(k)+kx)} \frac{\partial}{\partial k_\mu}\left(\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}} f(k)(-ik\times e(k,j))\right) dk.$$

We can see that there exist constants $\widetilde{c}_1$ and $\widetilde{c}_2$ such that

$$\sup_x |K_{l,\mu}^{(2)}(s,x,j)| \leqslant \frac{\widetilde{c}_l}{1+s}, \quad l=1,2, \quad \mu=1,2,3.$$

Then we conclude that

$$\left| -\frac{1}{2m}\sum_{\mu=1,2,3} ((\sigma_\mu\otimes 1)e^{-is(H_{\rm PF}-E(H_{\rm PF}))}\Psi, K_\mu^{(2)}(s,x,j)\varphi_{\rm g}) \right| \leqslant \frac{3}{2m}\|\Psi\| \left\{\widetilde{c}_1\|\varphi_{\rm g}\| + \widetilde{c}_2\|(|x|\otimes 1)\varphi_{\rm g}\|\right\} \frac{1}{s(1+s)}. \tag{4.14}$$

Hence the lemma follows from (4.13) and (4.14). □

Lemma 4.11. *Suppose* (1) *of* (PF1). *Then* $\|T_{\rm PF}{}_j^{(l)}(\cdot)\varphi_{\rm g}\| \in L^2(\mathbb{R}^3)$, $l=1,2$.

Proof. It follows that

$$\|T_{\rm PF}{}_j^{(1)}(k)\varphi_{\rm g}\| \leqslant \sum_{\mu=1}^3 \frac{1}{m}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}\|(p\otimes 1 - eA_{\hat{\varphi}})_\mu\varphi_{\rm g}\|,$$

$$\|T_{\rm PF}{}_j^{(2)}(k)\varphi_{\rm g}\| \leqslant \frac{3}{2m}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{\rm PF}(k)}}|k|\|\varphi_{\rm g}\|.$$

Since $\sqrt{\omega_{\rm PF}}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega_{\rm PF}} \in L^2(\mathbb{R}^3)$, the lemma follows. □

Lemma 4.12. *Assume* (PF1). *Then*

$$\int_{\mathbb{R}^3} \|(H_{\rm PF} - E(H_{\rm PF}) + \omega_{\rm PF}(k))^{-1} T_{\rm PF}{}_j^{(l)}(k)\varphi_{\rm g}\|^2\, dk < \infty, \quad l=1,2.$$

Proof. It is seen that $(x_\mu \otimes 1)\varphi_g \in H_{PF}$ with

$$\frac{i}{m}(p \otimes 1 - eA_{\hat{\varphi}})_\mu \varphi_g = [x_\mu \otimes 1, H_{PF}]\varphi_g = (H_{PF} - E(H_{PF}))(x_\mu \otimes 1)\varphi_g.$$

Then

$$T_{PF}{}_j^{(1)}(k)\varphi_g = -\frac{1}{m}\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}}e^{-ikx}e(k,j)\cdot(-im)(H_{PF} - E(H_{PF}))(x \otimes 1)\varphi_g.$$

Hence we have

$$\begin{aligned}&(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}T_{PF}{}_j^{(1)}(k)\varphi_g \\ &\quad = \frac{i\hat{\varphi}(k)e(k,j)}{\sqrt{2\omega_{PF}(k)}}(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}(H_{PF}(k) - E(H_{PF}))e^{-ikx}(x \otimes 1)\varphi_g,\end{aligned}$$

where we used that e^{ikx} maps $D(H_{PF,0})$ onto itself and on $D(H_{PF})$,

$$H_{PF}(k) := e^{-ikx}H_{PF}e^{ikx} = H_{PF} + \frac{1}{m}(p \otimes 1 - eA_{\hat{\varphi}})\cdot k + \frac{1}{2m}|k|^2.$$

Thus we have

$$\begin{aligned}&\|(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}(H_{PF}(k) - E(H_{PF}))e^{-ikx}(x_\mu \otimes 1)\varphi_g\| \\ &\quad \leqslant \|(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}(H_{PF} - E(H_{PF}))e^{-ikx}(x_\mu \otimes 1)\varphi_g\| \qquad (4.15) \\ &\qquad + \|(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}\frac{1}{m}(p \otimes 1 - eA_{\hat{\varphi}})\cdot ke^{-ikx}(x_\mu \otimes 1)\varphi_g\| \qquad (4.16) \\ &\qquad + \|(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}\frac{1}{2m}|k|^2e^{-ikx}(x_\mu \otimes 1)\varphi_g\|. \qquad (4.17)\end{aligned}$$

It obeys that

$$|(4.15)| \leqslant \|(|x| \otimes 1)\varphi_g\| \qquad (4.18)$$

and

$$|(4.17)| \leqslant \frac{1}{2m}|k|\|(|x| \otimes 1)\varphi_g\|. \qquad (4.19)$$

Note that by $\|(p\otimes 1-eA_{\hat{\varphi}})_\mu\Psi\|\leqslant c_1'\|(H_{\rm PF}-E(H_{\rm PF}))\Psi\|+c_2'\|\Psi\|$,

$$\|(p\otimes 1-eA_{\hat{\varphi}})_\mu(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}\varphi_{\rm g}\|\leqslant c_1'\|\varphi_{\rm g}\|+\frac{c_2'}{\omega_{\rm PF}(k)}\|\varphi_{\rm g}\|.$$

Then

$$|(4.16)|\leqslant\frac{3}{m}(c_1'|k|+c_2')\|(|x|\otimes 1)\varphi_{\rm g}\|. \tag{4.20}$$

Together with (4.18)–(4.20), we have

$$\begin{aligned}&\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}T_{{\rm PF}\,j}^{(1)}(k)\varphi_{\rm g}\|\\ &\quad\leqslant 3\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}\left(1+\frac{|k|}{2m}+\frac{3}{m}(c_1'|k|+c_2')\right)\|(|x|\otimes 1)\varphi_{\rm g}\|.\end{aligned} \tag{4.21}$$

Since $\sqrt{\omega_{\rm PF}}\hat{\varphi},\ \hat{\varphi}/\sqrt{\omega_{\rm PF}}\in L^2(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}T_{{\rm PF}\,j}^{(1)}(k)\varphi_{\rm g}\|^2\,dk<\infty \tag{4.22}$$

follows. Moreover we have

$$\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}T_{{\rm PF}\,j}^{(2)}(k)\varphi_{\rm g}\|\leqslant\frac{3}{2m}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}\|\varphi_{\rm g}\|. \tag{4.23}$$

Hence

$$\int_{\mathbb{R}^3}\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}T_{{\rm PF}\,j}^{(2)}(k)\varphi_{\rm g}\|^2\,dk<\infty \tag{4.24}$$

follows. Thus by (4.22) and (4.24), we get the desired results. □

Proof of Theorem 4.9. Lemmas 4.10–4.12 correspond to assumptions (B3), (B4) and (3) of Theorem 2.9, respectively. Then (4.8) and (4.9) follow from Theorem 2.9. By (4.21), (4.23) and (4.7), we have

$$\lim_{e\to 0}\sup_{\varphi_{\rm g}\in P_{H_{\rm PF}}\mathcal{F}_{\rm PF}} e^2\frac{\sum_{j=1,2}\int_{\mathbb{R}^3}\|(H_{\rm PF}-E(H_{\rm PF})+\omega_{\rm PF}(k))^{-1}T_{{\rm PF}\,j}(k)\varphi_{\rm g}\|^2\,dk}{\|\varphi_{\rm g}\|^2}$$

$$\leqslant \lim_{e\to 0} 6e^2 c_{\rm exp} \int \left\{ 3\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}} \left(1+\frac{|k|}{2m}+\frac{3}{m}(c_1'|k|+c_2')\right)\right\}^2 dk$$
$$+\lim_{e\to 0} 6e^2 \int \left\{ \frac{3}{2m}\frac{|\hat{\varphi}(k)|}{\sqrt{2\omega_{\rm PF}(k)}}\right\}^2 dk = 0.$$

Thus (4.10) follows from Theorem 3.6. □

Remark 4.13. Although, in [28], formula (4.9) has been used to show $\mathrm{m}(H_{\rm p})\leqslant 2$, there is no exact proof to derive this formula in it. See Section 1.3.

In [28] it has been also proven that $2\leqslant \mathrm{m}(H_{\rm PF})$ under some conditions on V. We state a theorem.

Theorem 4.14. *In addition to* (PF1)–(PF3) *and* (V), *we assume* $\mathrm{m}(H_{\rm p}) = 2$ *and* $V(x) = V(-x)$. *Then there exists a constant* e_{***} *such that for* e *with* $|e| < e_{***}$, $\mathrm{m}(H_{\rm PF}) = 2$.

Proof. $\mathrm{m}(H_{\rm PF})\leqslant 2$ follows from Theorem 4.9 and $2\leqslant \mathrm{m}(H_{\rm PF})$ from [28]. □

Example 4.15. Suppose that $V_+ \in L^1_{\rm loc}(\mathbb{R}^3)$ and V_- is infinitesimally small with respect to Δ. Then, by a Feynman–Kac formula, it is shown that $e^{-t(h_{\rm p}-E(h_{\rm p}))}$ is positivity improving in $L^2(\mathbb{R}^3)$. Hence $h_{\rm p}$ has a unique ground state in $L^2(\mathbb{R}^3)$. Then $\mathrm{m}(H_{\rm p}) = 2$.

4.3. The Coulomb–Dirac systems

We can apply the method stated in this paper to a wide class of interaction Hamiltonians in quantum field models. Hamiltonian $H_{\rm CD}$ of the Coulomb–Dirac system is defined as an operator acting in

$$\mathcal{F}_{\rm CD} = \mathcal{F}_{\rm f}(\oplus^4 L^2(\mathbb{R}^3)) \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^3 \times \{1,2\})),$$

where $\mathcal{F}_{\rm f}(\oplus^4 L^2(\mathbb{R}^3))$ denotes a fermion Fock space over $\oplus^4 L^2(\mathbb{R}^3)$. The Coulomb–Dirac system describes an interaction of positrons and relativistic electrons through photons in the Coulomb gauge. Operator $H_{\rm CD}$ is of the form

$$H_{\rm CD} = H_{\rm fermion}\otimes 1 + 1\otimes d\Gamma(\omega_{\rm CD}) + eH_{\rm rad} + e^2 H_{\rm Coulomb},$$

where $H_{\rm fermion}$ denotes a free Hamiltonian of $\mathcal{F}_{\rm f}(\oplus^4 L^2(\mathbb{R}^3))$, $\omega_{\rm CD}$ the multiplication operator by $\omega_{\rm CD}(k) = |k|$, and $H_{\rm rad}$, $H_{\rm Coulomb}$ interaction terms. $H_{\rm CD}$ has been investigated in [12], where the self-adjointness and the existence of a ground state are proven under some conditions. It is known that $\mathrm{m}(H_{\rm fermion}) = 1$. Then using the method in this paper we can also show

$$\mathrm{m}(H_{\mathrm{CD}}) \leqslant \mathrm{m}(H_{\mathrm{fermion}}) = 1,$$

i.e., the ground state of H_{CD} is unique for a sufficiently small e. We omit details.

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