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Classifying spaces with virtually cyclic stabilisers for certain infinite cyclic extensions

Martin Fluch [1]

School of Mathematics, University of Southampton, Southampton, SO17 1BJ, United Kingdom

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ABSTRACT

Let $G := B \rtimes \mathbb{Z}$ be an infinite cyclic extension of a group B where the action of $\mathbb{Z}$ on the set of conjugacy classes of non-trivial elements of B is free. This class of groups includes certain strictly descending HNN-extensions with abelian or free base groups, certain wreath products by $\mathbb{Z}$ and the soluble Baumslag–Solitar groups $BS(1, m)$ with $|m| \geq 2$. We construct a model for $\underline{\underline{E}}G$, the classifying space of G for the family of virtually cyclic subgroups of G, and give bounds for the minimal dimension of $\underline{\underline{E}}G$. Finally we determine the geometric dimension $\underline{\underline{\mathrm{gd}}}G$ when G is a soluble Baumslag–Solitar group.

1. Introduction

Let G be a (discrete) group and let $\mathfrak{F}$ be a *family* of subgroups of G. That is $\mathfrak{F}$ is non-empty and closed under conjugation and taking subgroups. A *classifying space* of G for the family $\mathfrak{F}$ is a G-CW-complex X such that

1. the fixed point space X^H is contractible if $H \in \mathfrak{F}$;
2. the fixed point space X^H is empty if H is any subgroup of G that does not belong to $\mathfrak{F}$.

We also say that X is a model for $E_{\mathfrak{F}}G$. The quotient space X/G is called a model for $B_{\mathfrak{F}}G$.

Classifying spaces exist for any family $\mathfrak{F}$ and are unique up to equivariant homotopy [8]. However in order to do some computation with classifying spaces it is important to know nice representatives in the G-homotopy class of models for $E_{\mathfrak{F}}G$. Generally a model for a classifying space is considered nice if it satisfies some finiteness conditions, for example being finite dimensional or being of finite type (only finitely many equivariant cells in each dimension). For more details see for example [9].

In the case that $\mathfrak{F} = \{1\}$ is the trivial family of subgroups, the universal cover of an Eilenberg–Mac Lane space $K(G, 1)$ is a model for $EG := E_{\mathfrak{F}}G$. A classifying space for the family $\mathfrak{F} = \mathfrak{F}_{\mathrm{fin}}(G)$ of finite subgroups of G is also known as the universal G-space for proper actions and denoted by $\underline{E}G$. For these families many nice models are known. For more details we refer to the comprehensive survey article on classifying spaces by Lück [10].

The object of study in our text is the classifying space for the family $\mathfrak{F} = \mathfrak{F}_{\mathrm{vc}}(G)$ of virtually cyclic subgroups which is denoted by $\underline{\underline{E}}G$. Recently, classifying spaces for this family of subgroups caught the interest of the mathematical community due to its appearance as the geometric object in the Farrell–Jones conjecture in algebraic K- and L-theory (this conjecture is originally stated in [4]). In contrast to models for EG and $\underline{E}G$, the classifying spaces for virtually cyclic subgroups are not well understood. Classes of groups that are understood are word hyperbolic groups [5], virtually polycyclic groups [12], relative hyperbolic groups [7] and CAT(0)-groups [11,3]. Furthermore, there exist general constructions for finite extensions [9] and direct limits of groups [12]. Some more specific constructions can also be found in [2,14].

E-mail address: martin.fluch@gmail.com.

[1] Department of Mathematics, University of Bielefeld, Postbox 10 01 31, 33501 Bielefeld, Germany.

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Among the classes of groups for which there are no known nice models for $\underline{\underline{E}}G$ are soluble groups or, more broadly, elementary amenable groups. Even in the case that G is metabelian there is no known general constructions for a nice model for $\underline{\underline{E}}G$. In a recent paper [6] it has been shown that one cannot expect any finite type model for $\underline{\underline{E}}G$ when G is an elementary amenable group. In general it is conjectured that there exists no finite model for $\underline{\underline{E}}G$ when G is not virtually cyclic [5].

It turns out that virtually cyclic extensions are the major source of obstruction to a general construction of nice models for $\underline{\underline{E}}G$, see for example Theorem 5.1 in [15]. As mentioned above, there exists a general construction for finite extensions. However there is no known general construction for an infinite cyclic extension, not to mention the case of a virtually cyclic extension.

In this article we will consider infinite cyclic extensions of an arbitrary group B. These are precisely the semidirect products $G = B \rtimes \mathbb{Z}$. We will show how to construct a model for $\underline{\underline{E}}G$ when a model for $\underline{\underline{E}}B$ is given, provided that the following condition on the extension is satisfied: $\mathbb{Z}$ acts freely by conjugation on the set of conjugacy classes of non-trivial elements of B. This construction will be such that a finite dimensional model for $\underline{\underline{E}}B$ will give a finite dimensional model for $\underline{\underline{E}}G$. The resulting models will be far away from being of finite type. One major ingredient in our paper is an adaptation of a construction of Juan-Pineda and Leary in [5].

The main result of this paper is Theorem 15, where we give bounds on the minimal dimension a model for $\underline{\underline{E}}G$ can have. The class of groups to which this theorem can be applied to includes certain strictly descending HNN-extensions with abelian or free base groups, certain wreath products by $\mathbb{Z}$, and the soluble Baumslag–Solitar groups $BS(1, m)$ with $|m| \geq 2$. Furthermore, in the latter case we will give an explicit construction of a model for $\underline{\underline{E}}G$ of minimal dimension. This brings us to our second main result, Theorem 20, which gives a precise answer for the possible minimal dimensions of a model for $\underline{\underline{E}}G$ when G is a soluble Baumslag–Solitar group.

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2. Technical preparations

Let B be a group and let $\varphi \in \mathrm{Aut}(B)$. Recall that a model for the semidirect product

$$G := B \rtimes \mathbb{Z}$$

where $\mathbb{Z}$ acts on B via the automorphism φ is the set $B \times \mathbb{Z}$ with the multiplication given by

$$(x, r) \cdot (y, s) := (x\varphi^r(y), r + s).$$

The identity is $(1, 0)$ and the inverse of any element (x, r) is given by $(x, r)^{-1} = (\varphi^{-r}(x^{-1}), -r)$. The group B is embedded via $x \mapsto (x, 0)$ as a normal subgroup of G and we consider $\mathbb{Z}$ embedded as a subgroup of G via $r \mapsto (1, r)$.

Lemma 1. *Assume that B is torsion free and does not contain a subgroup isomorphic to $\mathbb{Z}^2$. Then $\mathbb{Z}$ acts freely by conjugation on the set of conjugacy classes of the non-trivial elements of B if and only if G does not contain a subgroup isomorphic to $\mathbb{Z}^2$.*

Proof. "$\Rightarrow$": Suppose that H is a subgroup of G which is isomorphic to $\mathbb{Z}^2$. Since H cannot be contained in B it follows that there exists $(x, r) \in H \setminus B$. On the other hand H cannot have a trivial intersection with B and thus there exists a non-trivial $(y, 0) \in B \cap H$. Then the commutator

$$[(x, r), (y, 0)] = (x\varphi^r(y)x^{-1}y^{-1}, 0)$$

must be trivial which is the case if and only if $x\varphi^r(y)x^{-1}y^{-1} = 1$. This implies that $\varphi^r(y)$ and y belong to the same conjugacy class in B. Since $r \neq 0$ and $y \neq 1$ this implies that $\mathbb{Z}$ does not act freely on the set of conjugacy classes of non-trivial elements of B.

"$\Leftarrow$": Suppose that $\mathbb{Z}$ does not act freely on the set of conjugacy classes of non-trivial elements of B. Then there exists $1 \neq y \in B$ and $0 \neq r \in \mathbb{Z}$ such that $\varphi^r(y) = x^{-1}yx$ for some $x \in B$. This implies that the non-trivial elements (x, r) and $(y, 0)$ commute. In general (x, r) has infinite order and since B is assumed to be torsion free it follows that the order of $(y, 0)$ is also infinite. Therefore (x, r) and $(y, 0)$ generate a subgroup of G which is isomorphic to $\mathbb{Z}^2$. □

The statement in the next lemma is only non-trivial if B has torsion.

Lemma 2. *Assume that $\mathbb{Z}$ acts freely via conjugation on the set of conjugacy classes of non-trivial elements of B. If H is a virtually cyclic subgroup of G which is not a subgroup of B then H is infinite cyclic.*

Proof. First we note that all $x \in G \setminus B$ have infinite order. Hence the subgroup $\tau(H)$ of H which is generated by all the elements of H which have finite order is a subgroup of B.

By assumption there exists $(x, r) \in H$ with $r \neq 0$. Then the infinite cyclic subgroup of H generated by this element has a trivial intersection with B and therefore also has a trivial intersection with $\tau(H)$. Since H is virtually cyclic this can happen only if $\tau(H)$ is finite.

Assume towards a contradiction that there exists a non-trivial $(y, 0) \in \tau(H)$. Then $(z_k, 0) := (y, 0)^{(x,r)^k}$, $k \in \mathbb{N}$, is a sequence of elements in $\tau(H)$ such that for each $k \in \mathbb{N}$ the element z_k is conjugate in B to $\varphi^{-rk}(y)$. This claim is verified by induction. The case $k = 0$ is trivial. Thus assume that $(z_k, 0) \in \tau(H)$ and that there exists $u \in B$ such that $z_k = u^{-1}\varphi^{-rk}(y)u$. Since $\tau(H)$ is a characteristic subgroup of H and $(x, r) \in H$ it follows that $(z_{k+1}, 0) = (z_k, 0)^{(x,r)} \in \tau(H)$. Furthermore we have that

$$(z_{k+1}, 0) = (z_k, 0)^{(x,r)} = (\varphi^{-r}(x^{-1}z_k x), 0)$$

and hence $z_{k+1} = \varphi^{-r}(x^{-1}u^{-1}\varphi^{-rk}(y)ux) = v^{-1}\varphi^{-r(k+1)}(y)v$ with $v := \varphi^{-r}(ux)$. That is that z_{k+1} is conjugate in B to $\varphi^{-r(k+1)}(y)$. By assumption $\mathbb{Z}$ acts freely via conjugation on the conjugacy classes of non-trivial elements of B and hence all z_k belong to different conjugacy classes. In particular they are all pairwise different. Thus $\{(z_k, 0) : k \in \mathbb{N}\}$ forms an infinite subset of $\tau(H)$. But this is a contradiction as we have shown above that $\tau(H)$ is finite! Therefore $\tau(H)$ must be trivial. It follows that the virtually cyclic group H does not have any torsion and thus it must be infinite cyclic. □

Lemma 3. *Assume that $\mathbb{Z}$ acts freely via conjugation on the set of conjugacy classes of non-trivial elements of B. Then for any $(x, r) \in G \setminus B$ and $y \in B$ we have*

$$(x, r)^y = (x, r) \iff y = 1.$$

Proof. $(x, r)^y = (x, r)$ is equivalent to $\varphi(y) = xyx^{-1}$, which is by assumption on the action of $\mathbb{Z}$ on B equivalent to $y = 1$. □

Lemma 4. *Under the assumptions of the previous lemma, if H is an infinite cyclic subgroup of G that is not a subgroup of B, and $y \in B$, then $|H \cap H^y| = \infty$ if and only if $y = 1$.*

Proof. The "if" statement is trivial. Therefore assume that $y \neq 1$ and let (x, r) be a generator of H. Then $r \neq 0$ and

$$(z, r) := (x, r)^y \neq (x, r)$$

is a generator of H^y where the inequality is due to Lemma 3. Suppose, for a contradiction, that $|H \cap H^y| = \infty$. Then there must exist $k, l \in \mathbb{Z} \setminus \{0\}$ such that $(x, r)^k = (z, r)^l$. In particular this implies that $k = l$. But then we get

$$(z, r)^l = (z, r)^k = \left((x, r)^y\right)^k = \left((x, r)^k\right)^y \neq (x, r)^k,$$

where the last inequality is again due to Lemma 3, and so we achieve our desired contradiction. Hence we must have $|H \cap H^y| \neq \infty$. □

As in [12, p. 502] we define an equivalence relation "$\sim$" on the set $\mathfrak{F}_{\mathrm{vc}}(G) \setminus \mathfrak{F}_{\mathrm{fin}}(G)$ by

$$H \sim K :\Longleftrightarrow |H \cap K| = \infty.$$

We denote by $[H]$ the equivalence class of the group H. If K is not finite then $K \leq H$ implies that $K \sim H$. Furthermore the equivalence relation satisfies $H \sim K$ if and only $H^g \sim K^g$. Therefore the action of G by conjugation on the set $\mathfrak{F}_{\mathrm{vc}}(G) \setminus \mathfrak{F}_{\mathrm{fin}}(G)$ gives an action of G on the set of equivalence classes. If $[H]$ is an equivalence class, then we denote by $G_{[H]}$ the stabiliser of $[H]$.

Given a subgroup H of G, the *commensurator* $\mathrm{Comm}_G(H)$ of H in G is defined as the subgroup

$$\mathrm{Comm}_G(H) := \{g \in G : |H : H \cap H^g| \text{ and } |H^g : H \cap H^g| \text{ are finite}\}.$$

This subgroup is also known as the *virtual normaliser* $VN_G(H)$ of the subgroup H in G. In general it contains the normaliser $N_G(H)$ of H in G as its subgroup. In the case that H is a virtually cyclic subgroup of G which is not finite we have

$$\mathrm{Comm}_G(H) = \{g \in G : |H \cap H^g| = \infty\}.$$

In particular we have that $\mathrm{Comm}_G(H) = G_{[H]}$ in this case.

Lemma 5. *Assume that $\mathbb{Z}$ acts freely by conjugation on the set of non-trivial conjugacy classes of non-trivial elements of B. Then $\mathrm{Comm}_G(H)$ is infinite cyclic for any virtually cyclic subgroup H of G that is not a subgroup of B.*

Proof. Any such virtually cyclic subgroup H of G is infinite cyclic by Lemma 2. Therefore $G_{[H]} = \mathrm{Comm}_G(H)$. Suppose that $G_{[H]}$ is not infinite cyclic. Then the canonical projection $\pi\colon B \rtimes Z \to \mathbb{Z}$ cannot map $G_{[H]}$ isomorphically onto its image. Hence there exists a non-trivial $y \in G_{[H]} \cap \ker(\pi) = G_{[H]} \cap B$. Since H is infinite cyclic we get $|H \cap H^y| \neq \infty$ by Lemma 4 which is equivalent to $[H] \neq [H^y]$, and this is a contradiction to the assumption that $y \in G_{[H]}$. Therefore $G_{[H]} = \mathrm{Comm}_G(H)$ must be infinite cyclic. □

Proposition 6. *Let G be an arbitrary group and let $\mathfrak{F}$ and $\mathfrak{G}$ be families of subgroups of G such that*

$$\mathfrak{F}_{\mathrm{fin}}(G) \subset \mathfrak{F} \subset \mathfrak{G} \subset \mathfrak{F}_{\mathrm{vc}}(G).$$

Assume that the commensurator $\mathrm{Comm}_G(H) \in \mathfrak{G}$ for any $H \in \mathfrak{G} \setminus \mathfrak{F}$, then every $H \in \mathfrak{G} \setminus \mathfrak{F}$ is contained in a unique maximal element $H_{\max} \in \mathfrak{G}$ and $N_G(H_{\max}) = H_{\max}$.

Proof. Since H is an infinite virtually cyclic subgroup of G it follows that $G_{[H]} = \mathrm{Comm}_G(H)$ and thus $G_{[H]} \in \mathfrak{G}$ by assumption.

Trivially we have that $H \le G_{[H]}$. If $K \in \mathfrak{G}$ with $H \le K$, then $H \sim K$ since H is not finite, and for any $k \in K$ we get $[H^k] = [K^k] = [K] = [H]$. Therefore any $k \in K$ stabilises $[H]$. This implies $K \le G_{[H]}$ and thus $G_{[H]}$ is maximal and unique in $\mathfrak{G} \setminus \mathfrak{F}$, that is $H_{\max} = G_{[H]}$.

Finally, $H_{\max} \le N_G(H_{\max}) \le \mathrm{Comm}_G(H_{\max}) = G_{[H_{\max}]} = H_{\max}$ and hence $H_{\max} = N_G(H_{\max})$. □

Together with Lemma 2, we get then the following result:

Corollary 7. *Let $G = B \rtimes \mathbb{Z}$ and assume that $\mathbb{Z}$ acts freely by conjugation on the set of conjugacy classes of non-trivial elements of B. Then every $H \in \mathfrak{F}_{\mathrm{vc}}(G) \setminus \mathfrak{F}_{\mathrm{vc}}(B)$ is contained in a unique maximal element $H_{\max} \in \mathfrak{F}_{\mathrm{vc}}(G) \setminus \mathfrak{F}_{\mathrm{vc}}(B)$ and $N_G(H_{\max}) = H_{\max}$. Moreover, any $H \in \mathfrak{F}_{\mathrm{vc}}(G) \setminus \mathfrak{F}_{\mathrm{vc}}(B)$ has a trivial intersection with B.* □

3. Juan-Pineda and Leary's construction

Let G be an arbitrary group and assume that $\mathfrak{F}$ and $\mathfrak{G}$ are two families of subgroups of G which satisfy the conditions of Proposition 6.

Let H be a virtually cyclic group. Then by [5, p. 137], there exists a unique maximal normal finite subgroup N of H and precisely one of the cases occurs: H is finite, H/N is infinite cyclic (we say H is *orientable*) or H/N is infinite dihedral (we say H is *non-orientable*).

The following result is a natural generalisation of Proposition 9 and Corollary 10 in [5]. Juan-Pineda and Leary's proof can be used unchanged to verify these statements.

Proposition 8 (*Juan-Pineda and Leary*)**.** *Assume that every $H \in \mathfrak{G} \setminus \mathfrak{F}$ is contained in a unique maximal element $H_{\max} \in \mathfrak{G}$ and $N_G(H_{\max}) = H_{\max}$. Moreover, assume that any $H \in \mathfrak{G} \setminus \mathfrak{F}$ has finite intersections with the elements of $\mathfrak{F}$. Let $\mathcal{C}$ be a complete set of representatives of conjugacy classes of the maximal elements of $\mathfrak{G} \setminus \mathfrak{F}$. Denote by $\mathcal{C}_o$ the set of orientable elements of $\mathcal{C}$ and denote by $\mathcal{C}_n$ the set of non-orientable elements of $\mathcal{C}$. Then a model for $E_{\mathfrak{G}}G$ can be obtained from model for $E_{\mathfrak{F}}G$ by attaching*

1. *orbits of 0-cells indexed by $\mathcal{C}$;*
2. *orbits of 1-cells indexed by $\mathcal{C}_o \cup \{1, 2\} \times \mathcal{C}_n$;*
3. *orbits of 2-cells indexed by $\mathcal{C}$.*

Furthermore, a model for $B_{\mathfrak{G}}G$ can be obtained from a model for $B_{\mathfrak{F}}G$ by attaching 2-cells indexed by $\mathcal{C}_o$. □

Juan-Pineda and Leary's construction is essentially the following. For each maximal $H \in \mathfrak{G} \setminus \mathfrak{F}$ there exists a 1-dimensional model E_H for $\underline{E}H$ which is homeomorphic to the real line. The model E_H has one orbit of 1-cells and either one or two orbits of 0-cells depending on if H is orientable or non-orientable. Let X be a model for $E_{\mathfrak{F}}G$. Then a model for $E_{\mathfrak{G}}G$ is obtained from X by attaching pieces of the form

$$X_H := E_H \times [0, 1] \big/ \sim$$

to X for each maximal element $H \in \mathfrak{G} \setminus \mathfrak{F}$ using suitable equivariant maps $\eta\colon E_H \times \{0\} \to X$ and where the equivalence relation "$\sim$" identifies all elements $(x, 1) \in X_H$. Juan-Pineda and Leary have shown how to implement this construction such that we get a G-CW-complex with the desired properties.

Note that in the case $\mathfrak{G} = \mathfrak{F}_{\mathrm{vc}}(G)$ and $\mathfrak{F} = \mathfrak{F}_{\mathrm{fin}}(G)$, we recover the original statements in [5]. However, we apply it to the case that $G = B \rtimes \mathbb{Z}$, $\mathfrak{F} = \mathfrak{F}_{\mathrm{vc}}(B)$ and $\mathfrak{G} = \mathfrak{F}_{\mathrm{vc}}(G)$. If $\mathbb{Z}$ acts freely by conjugation on the set of conjugacy classes of non-trivial elements of B, then Corollary 7 tells us that we can use Proposition 8 in order to construct a model for $\underline{\underline{E}}G$ from a model for $E_{\mathfrak{F}_{\mathrm{vc}}(B)}G$. However, in order to obtain this way a nice model for $\underline{\underline{E}}G$ we need to have a nice model for $E_{\mathfrak{F}_{\mathrm{vc}}(B)}G$ to start with. In the next section we will give a general construction for such a model if a nice model for $\underline{\underline{E}}B$ is given.

4. Constructing a model for $E_{\mathfrak{F}}G$ from a model for $E_{\mathfrak{F}}B$

We carry out the construction in a setting that is more general than in Section 3. Let $G = B \rtimes \mathbb{Z}$ be an arbitrary infinite cyclic extension, where $\mathbb{Z}$ acts on B via an automorphism $\varphi \in \mathrm{Aut}(B)$. Let $\mathfrak{F}$ be a family of subgroups of B. We assume that $\mathfrak{F}$ is invariant under the automorphism φ, that is, $\varphi^k(H) \in \mathfrak{F}$ for every $H \in \mathfrak{F}$ and $k \in \mathbb{Z}$. This implies that $H \in \mathfrak{F}$ if and only if $\varphi(H) \in \mathfrak{F}$ for any subgroup H of B. Furthermore this implies that $\mathfrak{F}$ is not just a family of subgroups of B but also a family of subgroups of G.

We begin our construction with the assumption that we are given a model X for $E_{\mathfrak{F}}B$. For each $k \in \mathbb{Z}$ let X_k be a copy of X seen as a set. We define a B-action

$$\Phi_k\colon B \times X_k \to X_k$$

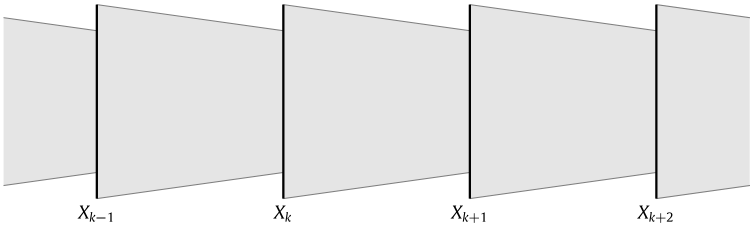

Fig. 1. A schematic picture of the B-CW-complex Y.

on X_k by $\Phi_k(g,x) := \varphi^{-k}(g)x$. Note that each X_k is a model for $E_{\mathfrak{F}}B$ since $\mathfrak{F}$ is assumed to be invariant under the automorphism φ.

Since X_0 and X_1 are models for $E_{\mathfrak{F}}B$ there exists a B-map $f\colon X_0 \to X_1$. In other words f is a continuous map $f\colon X \to X$ which satisfies $f(gx) = \varphi^{-1}(g)f(x)$ for every $x \in X$ and $g \in B$. By the equivariant Cellular Approximation Theorem [8, p. 32] we may assume without loss of generality that f is cellular. Denote by X_∞ the disjoint union of B-spaces

$$X_\infty := \coprod_{k\in\mathbb{Z}} (X_k \times [0,1])$$

and let Y be the quotient space

$$Y := X_\infty / \sim$$

under the equivalence relation generated by $(x,1) \sim (f(x),0)$ whenever $x \in X_k$ and $f(x) \in X_{k+1}$ for some $k \in \mathbb{Z}$. Since f is a cellular B-map it follows that Y is a B-CW-complex. Essentially, it is a mapping telescope which extends to infinity in both directions, see Fig. 1. Note that if X is an n-dimensional B-CW-complex, then Y is $(n+1)$ dimensional B-CW-complex.

Lemma 9. *The B-CW-complex Y is a model for $E_{\mathfrak{F}}B$.*

Proof. Let H be a subgroup of B such that $H \notin \mathfrak{F}$. Let $k \in \mathbb{Z}$ and $x \in X_k$. Since $\mathfrak{F}$ is assumed to be invariant under the automorphism φ we have $\varphi^{-k}(H) \notin \mathfrak{F}$. Therefore there exists a $h \in H$ such that $\varphi^{-k}(h)x \neq x$. But then

$$\Phi_k(h,x) = \varphi^{-k}(h)x \neq x,$$

which implies that $x \notin X_k^H$. Thus $X_k^H = \emptyset$ for every $k \in \mathbb{Z}$ and it follows that $Y^H = \emptyset$.

On the other hand, consider the case that $H \in \mathfrak{F}$. Since the family $\mathfrak{F}$ is assumed to be invariant under the automorphism φ it follows that $\varphi^k(H) \in \mathfrak{F}$ for every $k \in \mathbb{Z}$. Then $X_k^H = X^{\varphi^k(H)}$ is contractible for every $k \in \mathbb{Z}$. It follows that the subcomplex Y^H is the infinite mapping telescope of the contractible spaces X_k^H. Since any map from an n-sphere into Y^H is contained in a finite subtelescope whose deformation retracts to its contractible right-hand end space it follows that Y^H is weakly contractible. Since Y^H has the structure of a CW-complex this implies that Y^H is contractible. □

For every $(x,t) \in X_k \times [0,1]$ and $(g,r) \in G$ set

$$\Psi((g,r),(x,t)) := (\Phi_{k+r}(g,x), t) \in X_{k+r} \times [0,1].$$

Straightforward calculation shows that this induces a well defined action

$$\Psi\colon G \times Y \to Y$$

of G on Y, which extends the already existing B-action on Y. If $(g,r) \in G \setminus B$, then $r \neq 0$ and therefore clearly $\Psi((g,r),x) \neq x$ for any $x \in Y$. Then together with Lemma 9 this implies that Y is an $(n+1)$-dimensional model for $E_{\mathfrak{F}}G$. Altogether we have then shown the following result.

Proposition 10. *Let $G = B \rtimes \mathbb{Z}$ be an arbitrary infinite cyclic extension where $\mathbb{Z}$ acts on B via an automorphism $\varphi \in \mathrm{Aut}(B)$. Let $\mathfrak{F}$ be a family of subgroups of B which is invariant under the automorphism φ. If there exists an n-dimensional model for $E_{\mathfrak{F}}B$ then there exists an $(n+1)$-dimensional model for $E_{\mathfrak{F}}G$.* □

5. Examples

Strictly descending HNN-extensions are a natural source for candidates for infinite cyclic extensions $G = B \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts freely by conjugation on the set of conjugacy classes of the non-trivial elements of B.

The general setup is the following. Let B_0 be a group and $\varphi\colon B_0 \to B_0$ a monomorphism. Recall that the descending HNN-extension determined by this data is the group G given by the presentation

$$G := \langle B_0, t \mid t^{-1}xt = \varphi(x) \text{ for all } x \in B_0 \rangle$$

and this group is usually denoted by B_0*_φ in the literature. The group B_0 is called the base group of the HNN-extension. The HNN-extension is called *strictly descending* if the monomorphism φ is not an isomorphism. We consider B_0 in the obvious way as a subgroup of G.

Conjugation by $t \in G$ defines an automorphism of G which agrees on B_0 with φ, which we will therefore denote by the same symbol. In other words, the monomorphism $\varphi\colon B_0 \to B_0$ extends to the whole group G if we set

$$\varphi\colon G \to G, x \mapsto \varphi(x) := t^{-1}xt.$$

For each $k \in \mathbb{Z}$ we set $B_k := \varphi^k(B_0)$. We obtain this way a descending sequence

$$\cdots \supset B_{-2} \supset B_{-1} \supset B_0 \supset B_1 \supset B_2 \supset \cdots$$

of subgroups of G. This sequence of subgroups is strictly descending if and only if the HNN-extension is strictly descending. We denote the directed union of all these B_k by B. The automorphism φ restricts to an automorphism of B which is therefore a normal subgroup of G. It is standard fact that we can write G as the semidirect product $G = B \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on B via the automorphism φ restricted to B.

Lemma 11. *Assume that for every non-trivial $x \in B_0$ there exists a $k \in \mathbb{N}$ such that $x \notin \varphi^k(B_0)$. Given $x \in B_0$ denote by $[x]$ the set of all elements in B_0 which are conjugate in B_0 to x. Assume that for each $x \in B_0$ we are given a finite subset $[x]' \subset [x]$, which only depends on the conjugacy class $[x]$ of x in B_0, such that $\varphi([x]') \subset [\varphi(x)]'$ for every $x \in B_0$. Then $\mathbb{Z}$ acts freely on the set of conjugacy classes of non-trivial elements of B.*

Proof. We suppose that $\mathbb{Z}$ does not act freely on the set of conjugacy classes of non-trivial elements of B. Then there exists $x \in B$ and $n \geq 1$ such that $\varphi^n(x)$ is conjugate in B to x. Without any loss of generality we may assume that $x \in B_0$ (otherwise replace x by $\varphi^k(x)$ for a suitable $k \in \mathbb{N}$). Furthermore, without any loss of generality we may assume that $x \in [x]'$. Finally we may assume without any loss of generality that $\varphi^n(x)$ is actually conjugate in B_0 to x (otherwise, again, replace x by $\varphi^k(x)$ for a suitable $k \in \mathbb{N}$).

Now $\varphi^{rn}(x) \in [x]'$ for any $r \geq 1$. Since $[x]'$ is finite this implies that $\varphi^{rn}(x) = \varphi^{sn}(x)$ for some $s > r$. Therefore $\varphi^{(s-r)n}(x) = x$ and since $(s-r)n > 0$ it follows that $x \in \varphi^k(B_0)$ for any $k \in \mathbb{N}$. However, this is a contradiction on the hypothesis that $\mathbb{Z}$ does not act freely on the set of conjugacy classes of non-trivial elements of B. Therefore the opposite must be true. $\square$

Note that the requirement that for every non-trivial element $x \in B_0$ there exists a $k \in \mathbb{N}$ such that $x \notin \varphi^k(B_0)$ implies that the descending HNN-extension $G = B_0*_\varphi$ is actually strictly descending. Furthermore, we can conclude from it that the intersection of the groups B_k, $k \in \mathbb{Z}$, is trivial.

Example 12. Let B_0 be an abelian group and $\varphi\colon B_0 \to B_0$ a monomorphism such that for every non-trivial $x \in B_0$ there exists a $k \in \mathbb{N}$ such that $x \notin \varphi^k(B_0)$. Since B_0 is abelian, each conjugacy class $[x]$ of elements in B_0 contains precisely one element. Therefore Lemma 11 states that $\mathbb{Z}$ acts freely by conjugation on the set of non-trivial elements of B. In particular we can use Proposition 8 to obtain a model for $\underline{\underline{E}}G$ from a model for $E_{\mathfrak{F}_{vc}(B)}G$.

Let B_0 be a free group. An element $x \in B_0$ is called *cyclically reduced* if it cannot be written as $x = u^{-1}yu$ for some non-trivial $u, y \in B_0$. It follows from [13, Section 1.4] that every element $x \in B_0$ is conjugate to a cyclically reduced element x' and that there are only finitely many cyclically reduced elements in B_0 which are conjugate to x. Therefore

$$[x]' := \{x' \in [x] : x' \text{ is cyclically reduced}\}$$

is a finite subset of $[x]$ for every $x \in B_0$. Then the following two assumptions on the monomorphism $\varphi\colon B_0 \to B_0$ are necessary in order to apply Lemma 11:

1. For every non-trivial $x \in B_0$ there exists a $k \in \mathbb{N}$ such that $x \notin \varphi^k(B_0)$;
2. If x is a cyclically reduced element in B_0, then so is $\varphi(x)$.

Example 13. Let X be an arbitrary non-empty set and let $B_0 := F(X)$ be the free group on the basis X. Let $\{\alpha_x\}_{x\in X}$ be a collection of integers such that $|\alpha_x| \geq 2$ for every $x \in X$. Consider the endomorphism $\varphi\colon B_0 \to B_0$ that maps any basis element x to x^{α_x}. Then φ is a monomorphism which satisfies the assumptions (1) and (2) above. Lemma 11 tells us then that we can use Proposition 8 to construct a model for $\underline{\underline{E}}G$ from a model for $E_{\mathfrak{F}_{vc}(B)}G$.

Example 14. Another example of a strictly descending HNN-extension (in disguise) is the standard wreath product $A \wr \mathbb{Z}$ of an arbitrary group A by $\mathbb{Z}$ which is defined as follows. Let A_k be a copy of A for each $k \in \mathbb{Z}$. Let B be the coproduct of all these A_k and let $\mathbb{Z}$ act on B via φ which maps A_k identically onto A_{k+1} for all $k \in \mathbb{Z}$. Then

$$A \wr \mathbb{Z} := B \rtimes \mathbb{Z}.$$

Since each A_k is normal in B the above definition of φ forces the action of $\mathbb{Z}$ on the set of conjugacy classes of non-trivial elements of B to be free. Therefore we can apply Proposition 8 in this case, too.

6. Dimensions

Given a family $\mathfrak{F}$ of subgroups of G, a model for $E_{\mathfrak{F}}G$ is only defined uniquely up to G-homotopy. Consider a model for $E_{\mathfrak{F}}G$. One particular invariant of the group G is called the *geometric dimension* of G with respect to the family $\mathfrak{F}$, and this is defined as being the least possible dimension of a model for $E_{\mathfrak{F}}G$. It is denoted by $\operatorname{gd}_{\mathfrak{F}} G$ and may be infinite. In the case that $\mathfrak{F} = \{1\}$ we recover the classical geometric dimension of the group G. In the case that $\mathfrak{F} = \mathfrak{F}_{\mathrm{vc}}(G)$ we denote the geometric dimension by $\underline{\underline{\operatorname{gd}}}\, G$.

Theorem 15. *Let $G = B \rtimes \mathbb{Z}$ and assume that $\mathbb{Z}$ acts freely via conjugation on the conjugacy classes of non-trivial elements of B. Then*

$$\underline{\underline{\operatorname{gd}}}\, B \leq \underline{\underline{\operatorname{gd}}}\, G \leq \underline{\underline{\operatorname{gd}}}\, B + 1.$$

Proof. Since (in general) a model for $\underline{\underline{E}}G$ is always a model for $\underline{\underline{E}}B$ via restriction, we have that the second inequality is the only non-trivial one. If X is an n-dimensional model for $\underline{\underline{E}}B$, then the join construction in Section 4 gives an $(n + 1)$-dimensional model for $E_{\mathfrak{F}_{\mathrm{vc}}(B)}G$.

Since $\mathbb{Z}$ acts freely via conjugation on the conjugacy classes of non-trivial elements of B it follows that B cannot be virtually cyclic. Thus $n + 1 \geq 2$ and attaching cells of dimension at most 2 does not increase the dimension of the resulting space. Hence Proposition 8 yields an $(n + 1)$-dimensional model for $\underline{\underline{E}}G$ and this concludes the proof. □

Corollary 16. *Let $G = B_0 *_{\varphi}$ is a descending HNN-extension as in Section 5. If $G = B \rtimes \mathbb{Z}$ satisfies the conditions of the previous theorem then*

$$\underline{\underline{\operatorname{gd}}}\, B_0 \leq \underline{\underline{\operatorname{gd}}}\, G \leq \underline{\underline{\operatorname{gd}}}\, B_0 + 2.$$

Proof. As exploited previously, since B_0 is a subgroup of G, the second inequality is the only non-trivial part of the statement. The group B is the countable direct union of the conjugates of B_0 in G. Therefore an n-dimensional model for $\underline{\underline{E}}B_0$ gives rise to an $(n + 1)$-dimensional model for $\underline{\underline{E}}B$ by a construction of Lück and Weiermann [12, Section 4]. Now the claim follows from the previous theorem. □

Example 17. Let $G = B_0 *_{\varphi}$ be a descending HNN-extension with B_0 a free group. If B_0 has rank 1, then G is a soluble Baumslag–Solitar group and this case is treated below in Theorem 20. Thus we may assume that B_0 has rank at least 2. Free groups are torsion free and act freely on a tree which is therefore a 1-dimensional model for $\underline{E}B_0$. Free groups are word hyperbolic and therefore Proposition 9 in [5] states the existence of a 2-dimensional model for $\underline{\underline{E}}B_0$. On the other hand by Remark 16 in [5] there cannot exist a model for $\underline{\underline{E}}B_0$ less than 2. Therefore $\underline{\underline{\operatorname{gd}}}\, B_0 = 2$. Now the direct union B of all conjugates of B_0 in G is locally free and therefore does not contain a subgroup of isomorphic to $\mathbb{Z}^2$. Then Lemma 1 states that we can apply Corollary 16 if and only if G does not contain a subgroup isomorphic to $\mathbb{Z}^2$. Therefore we get in this case the estimation $2 \leq \underline{\underline{\operatorname{gd}}}\, G \leq 4$.

Example 18. Consider the wreath product $G = A \wr \mathbb{Z}$ where A is a countable locally finite group. Then

$$B := \coprod_{k \in \mathbb{Z}} A$$

is also a countable locally finite group. Then B is a countable colimit of its finite subgroups B_λ and Theorem 4.3 in [12] gives the estimate $\underline{\underline{\operatorname{gd}}}\, B \leq \sup\{\underline{\underline{\operatorname{gd}}}\, B_\lambda\} + 1$. Since $\underline{\underline{\operatorname{gd}}}\, B_\lambda = 0$ for every λ and we get $\underline{\underline{\operatorname{gd}}}\, B \leq 1$. On the other hand B is not virtually cyclic and therefore $\underline{\underline{\operatorname{gd}}}\, B \neq 0$, that is $\underline{\underline{\operatorname{gd}}}\, B = 1$. We have seen that G does satisfy the requirements of Theorem 15. Therefore we get the estimate $1 \leq \underline{\underline{\operatorname{gd}}}(A \wr \mathbb{Z}) \leq 2$. Note that $A \wr \mathbb{Z}$ is not locally virtually cyclic. Therefore we can furthermore exclude the possibility $\underline{\underline{\operatorname{gd}}}(A \wr \mathbb{Z}) = 1$ using the next proposition. Thus we have altogether

$$\underline{\underline{\operatorname{gd}}}(A \wr \mathbb{Z}) = 2.$$

Note that the smallest concrete example of a group of this type is the lamplighter group $L = \mathbb{Z}_2 \wr \mathbb{Z}$ where $\mathbb{Z}_2$ is the cyclic group of the integers modulo 2.

Proposition 19. *Let G be a group with $\underline{\underline{\operatorname{gd}}}\, G = 1$. Then G is a locally virtually cyclic and not finitely generated.*

Proof. The assumption $\underline{\underline{\operatorname{gd}}}\, G = 1$ implies that G has a tree T as a model for $\underline{\underline{E}}G$. If G is finitely generated then Corollary 3 to Proposition 26 in [16, p. 64] states that $T^G \neq \emptyset$ which implies that G is virtually cyclic. But this means $\underline{\underline{\operatorname{gd}}}\, G = 0$ which contradicts the assumption that $\underline{\underline{\operatorname{gd}}}\, G = 1$. Therefore G is not finitely generated.

Let H be a finitely generated subgroup of G. Since H is a subgroup of G we have $\underline{\underline{\operatorname{gd}}}\, H \leq \underline{\underline{\operatorname{gd}}}\, G$. However, since H is finitely generated the case $\underline{\underline{\operatorname{gd}}}\, H \neq 1$ cannot occur. Therefore $\underline{\underline{\operatorname{gd}}}\, H = 0$ which implies that H is virtually cyclic. Hence G is locally virtually cyclic. □

We conclude this article with a complete answer to the geometric dimension of the soluble Baumslag–Solitar groups with respect to the family of virtually cyclic subgroups. These groups belong to a class of two-generator and one-relator groups introduced by Baumslag and Solitar in [1]. Their class consists of the groups

$$BS(p, q) = \langle x, t \mid t^{-1}x^p t = x^q \rangle$$

where p and q are non-zero integers. The soluble Baumslag–Solitar groups are precisely those groups which are isomorphic to $BS(1, m)$ for some $m \neq 0$. These groups can also be written as

$$BS(1, m) = \mathbb{Z}[1/m] \rtimes \mathbb{Z},$$

where $\mathbb{Z}[1/m]$ is the subgroup of the rational numbers $\mathbb{Q}$ generated by all powers of $1/m$ and where $\mathbb{Z}$ acts on $\mathbb{Z}[1/m]$ by multiplication by m. The group $BS(1, 1)$ is $\mathbb{Z}^2$ and $BS(1, -1)$ is the Klein bottle group $\mathbb{Z} \rtimes \mathbb{Z}$. If $|m| \geq 2$, then $BS(1, m)$ belongs to the case described in Example 12, as well as to the case described in Example 13.

Theorem 20. *Let $G = BS(1, m)$ be a soluble Baumslag–Solitar group. Then*

$$\underline{\underline{\mathrm{gd}}}\, G = \begin{cases} 3 & \text{if } |m| = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. The case $|m| = 1$ has been answered in Example 3 and Remark 15 in [5].

Thus we assume that $|m| \geq 2$. The group $\mathbb{Z}[1/m]$ is countable and locally virtually cyclic. Therefore $\underline{\underline{\mathrm{gd}}}\, \mathbb{Z}[1/m] \leq 1$ by [12, Theorem 4.3]. The action of $\mathbb{Z}$ on the conjugacy classes of non-trivial elements of $\mathbb{Z}[1/m]$ is free and thus we can apply Theorem 15 to obtain the estimate $\underline{\underline{\mathrm{gd}}}\, G \leq 2$. However, $\underline{\underline{\mathrm{gd}}}\, G$ cannot be zero as G is not virtually cyclic. Moreover G is finitely generated and therefore $\underline{\underline{\mathrm{gd}}}\, G \neq 1$ by Proposition 19. Thus $\underline{\underline{\mathrm{gd}}}\, G = 2$ is the only remaining possibility. $\square$

We conclude this article with a model for $\underline{\underline{E}}G$ for the soluble Baumslag–Solitar groups $G = BS(1, m)$, $|m| \neq 1$, which is nicer than the one obtained from the general construction in Section 4 together with Proposition 8. The group G can be realised as the fundamental group of a graph (G, Y) of groups in the sense of [16] where Y is a loop and where the vertex and edge groups are all infinite cyclic. Let T be the Bass–Serre tree associated with this graph of groups. It follows that T is not only a model for $\underline{\underline{E}}\mathbb{Z}[1/m]$, but also a model for $E_{\mathfrak{F}_{\mathrm{vc}}(\mathbb{Z}[1/m])}G$. We can apply Proposition 8 and obtain a model X for $\underline{\underline{E}}G$ by attaching orbits of 0-, 1- and 2-cells to T indexed by the infinitely many conjugacy classes of maximal infinite cyclic subgroups of G which are not contained in the subgroup $\mathbb{Z}[1/m]$. Furthermore, since $Y = T/G$, we obtain a model for $\underline{\underline{B}}G$ by attaching infinitely many 2-cells along the loop Y. That is, a model for $\underline{\underline{B}}G$ is given by the quotient space $(D^2 \times \mathbb{Z})/\sim$ where the equivalence relation is given by $(x, k) \sim (x, l)$ for all $x \in \partial D^2 = S^1$ and all $k, l \in \mathbb{Z}$.

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