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Quantum hyperbolic invariants of 3-manifolds with $PSL(2,\mathbb{C})$-characters

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Abstract

We construct *quantum hyperbolic invariants* (QHI) for triples (W, L, ρ), where W is a compact closed oriented 3-manifold, ρ is a flat principal bundle over W with structural group $PSL(2,\mathbb{C})$, and L is a non-empty link in W. These invariants are based on the Faddeev–Kashaev's *quantum dilogarithms*, interpreted as matrix-valued functions of suitably decorated hyperbolic ideal tetrahedra. They are explicitly computed as state sums over the decorated hyperbolic ideal tetrahedra of the *idealization* of any fixed $\mathcal{D}$*-triangulation*; the $\mathcal{D}$-triangulations are simplicial 1-cocycle descriptions of (W, ρ) in which the link is realized as a Hamiltonian subcomplex. We also discuss how to set the Volume Conjecture for the coloured Jones invariants $J_N(L)$ of hyperbolic knots L in S^3 in the framework of the general QHI theory.

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1. Introduction

In the two papers [18,22], Kashaev proposed a new infinite family $\{K_N\}$, $N > 1$ being any odd positive integer, of conjectural complex-valued topological invariants for pairs (W, L), where L is a link in a compact closed oriented 3-manifold W. These invariants should be computed as *state sums* $K_N(\mathcal{T})$ supported by some kind of heavily decorated triangulation $\mathcal{T}$ for (W, L). The main

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ingredients of the state sums were the Faddeev–Kashaev's matrix version of the *quantum dilogarithms* at the Nth root of unity $\zeta = \exp(2\pi i/N)$, suitably associated to the decorated tetrahedra of $\mathcal{T}$. The nature of these decorated triangulations was mysterious, but it was clear that they fulfilled non-trivial global constraints which made their existence not evident a priori. Beside this neglected *existence* and *meaning* problem, a main question left unsettled was the *invariance* of the value of $K_N(\mathcal{T})$ when $\mathcal{T}$ varies. However, Kashaev proved the invariance of $K_N(\mathcal{T})$ under certain 'moves' on $\mathcal{T}$, which reflect fundamental identities verified by the quantum dilogarithms. He also showed in [19] that $K_N(S^3, L)$ is indeed a well-defined invariant, by reducing its state sum formula to one based on planar $(1,1)$-tangle presentations of L (as for the Alexander polynomial) and involving a constant *Kashaev's R-matrix*.

On another hand, Faddeev–Kashaev [17], Bazhanov–Reshetikhin [4] and Kashaev [20] had already computed the semi-classical limit of (various versions of) the quantum dilogarithms and their five term 'pentagon' identities in terms of the classical Euler and Rogers dilogarithm functions, which are known to be related to the computation of the volume of spherical or hyperbolic simplices. This definitely suggested the possibility of a deep intriguing relationship between hyperbolic geometry and the invariants $K_N(S^3, L)$. In this direction, the so-called Kashaev's *Volume Conjecture* [21] predicts that when L is a hyperbolic link in S^3, one can recover the hyperbolic volume of $S^3 \setminus L$ from the asymptotic behaviour of $K_N(S^3, L)$, when $N \to \infty$. More recently, Murakami–Murakami [25] proved that the Kashaev's R-matrix can be enhanced into a Yang–Baxter operator which allows one to define the coloured Jones polynomial $J_N(L)$ for links L in S^3 (evaluated at $\zeta = \exp(2i\pi/N)$ and normalized by $J_N(\text{unknot}) = 1$), so that $K_N(S^3, L) = J_N^N(L)$. This gave a new formulation of the Volume Conjecture, discussed in [25,33], in terms of those celebrated invariants of links.

The new formulation of $J_N(L)$ using quantum dilogarithms was an important achievement, but it also had the negative consequence of putting aside the initial purely 3-dimensional and more geometric set-up for links in an arbitrary compact closed oriented 3-manifold W, willingly forgetting the complicated and somewhat mysterious decorated triangulations.

In our opinion, this set-up deserved to be understood and developed as a full quantum field theory, also in the perspective of finding an appropriate geometric framework for a well-motivated more general version of the Volume Conjecture. The present paper, which is the first of a series, establishes some fundamental facts of our program on this matter. The main result is the construction of so-called *quantum hyperbolic invariants* (QHI) for compact closed oriented 3-manifolds endowed with an embedded non-empty link and a flat principal bundle with structural group $PSL(2,\mathbb{C})$. The QHI generalize the Kashaev's conjectural topological invariants.

1.1. Description of the paper

We are mainly concerned with pairs (W, ρ) where W is a compact closed oriented 3-manifold and ρ is a flat principal bundle over W with structural group $PSL(2,\mathbb{C})$. By using the hauptvermutung, depending on the context, we will freely assume that W is endowed with a (necessarily unique) PL or smooth structure, and use differentiable or PL homeomorphisms. The pairs (W, ρ) are considered up to orientation preserving homeomorphisms of W and flat bundle isomorphisms of ρ. Equivalently, ρ is identified with a conjugacy class of representations of the fundamental group of W in $PSL(2,\mathbb{C})$, i.e. with a $PSL(2,\mathbb{C})$-character of W. Compact oriented hyperbolic 3-manifolds with their hyperbolic holonomies furnish a main example of pairs (W,ρ). There are other natural examples (W,ρ_α) associated to ordinary cohomology classes $\alpha \in H^1(W;\mathbb{C})$ (see Section 2.2).

In Section 2 we introduce special combinatorial descriptions of (W,ρ) called $\mathscr{D}$-*triangulations*. These are "decorated" triangulations T of W, where the decoration consists of a system b of edge orientations of a special kind (called *branching*), and of a 'generic' $PSL(2,\mathbb{C})$-valued 1-cocycle z on (T,b). This genericity condition allows us to define a simple explicit procedure of *idealization* which converts any $\mathscr{D}$-triangulation $\mathscr{T}$ into a suitably structured family $\mathscr{T}_\mathscr{I}$ of oriented hyperbolic ideal tetrahedra, called an $\mathscr{I}$-*triangulation* for (W,ρ). Each hyperbolic tetrahedron of $\mathscr{T}_\mathscr{I}$ has the vertices ordered by the branching b, and its geometry is encoded by the cross-ratio moduli in $\mathbb{C}\setminus\{0,1\}$ associated to its edges. The $\mathscr{I}$-triangulations have remarkable global properties. In particular their moduli satisfy, at every edge, the usual compatibility condition needed when one tries to construct hyperbolic 3-manifolds by gluing ideal tetrahedra. This means that given an $\mathscr{I}$-triangulation we can construct pairs $(\tilde{\rho},s)$, where $\tilde{\rho}$ is a representative of the character ρ and s is a piecewise-straight section of the flat bundle $\widetilde{W}\times_{\tilde{\rho}}\bar{\mathbb{H}}^3 \to W$, with structural group $PSL(2,\mathbb{C})$ and total space the quotient of $\widetilde{W}\times\bar{\mathbb{H}}^3$ by the diagonal action of $\pi_1(W)$ and $\tilde{\rho}$.

We also define the notions of $\mathscr{D}$- and $\mathscr{I}$-*transits* between $\mathscr{D}$- and $\mathscr{I}$-triangulations of (W,ρ). These are supported by the usual elementary moves on triangulations of 3-manifolds, but they also include the transits of the respective extra-structures. We prove the remarkable fact that, via the idealization, the $\mathscr{D}$-transits dominate the $\mathscr{I}$-transits.

In Section 3, we consider for any odd positive integer $N>1$ certain *basic state sums* $\mathfrak{L}_N(\mathscr{T}_\mathscr{I})\in\mathbb{C}$ supported by the idealization $\mathscr{T}_\mathscr{I}$ of any $\mathscr{D}$-triangulation $\mathscr{T}$ for (W,ρ). The main ingredients of these state sums are the Faddeev–Kashaev (non-symmetric) matrix quantum dilogarithms, viewed as matrix-valued functions depending on the moduli of branched hyperbolic ideal tetrahedra. At this point some comments are in order.

These matrix quantum dilogarithms (quantum dilogarithms for short) were originally derived in [18,22] as matrices of $6j$-*symbols* for the cyclic representation theory of a Borel quantum subalgebra $\mathscr{B}_\zeta$ of $U_\zeta(sl(2,\mathbb{C}))$, where $\zeta=\exp(2i\pi/N)$. Such matrices describe the associativity of the tensor product in this category. Here are two key facts. First, the isomorphism classes of irreducible cyclic representations of $\mathscr{B}_\zeta$ are parametrized by the elements with non-zero upper diagonal term in the Borel subgroup B of $PSL(2,\mathbb{C})$ of upper triangular matrices. Moreover, the specific 'Clebsch–Gordan' decomposition rule into irreducibles of cyclic tensor products of such representations relies on a (generic) B-valued 1-cocycle-like property. These facts may be seen at hand, or alternatively they can be deduced from the theory of quantum coadjoint action of De Concini–Kac–Procesi [12], applied to the group B. We recall them in Appendix A, as well as the properties of the quantum dilogarithms that we need; for full details we refer to [1].

Thus, when associating irreducible cyclic representations of $\mathscr{B}_\zeta$ to the edges of a branched tetrahedron (Δ,b), generic B-valued 1-cocycles on Δ seem to play a fundamental role to associate quantum dilogarithms to it. For this reason, we early considered the QHI only for B-valued characters of W (see [2]). However, we eventually realized that the quantum dilogarithms do in fact only depend on particular ratios of parameters expressed in terms of the cocycle values, which may naturally be interpreted as moduli for idealized tetrahedra. Also, the basic identities they satisfy are only related to certain $\mathscr{I}$-transits. As the idealization works for arbitrary $PSL(2,\mathbb{C})$-characters on W, this and the symmetrization procedure explained below finally leads to the present general formulation of the theory. The quantum dilogarithms do not appear in this way as directly related to the whole cyclic representation theory of $U_\zeta(sl(2,\mathbb{C}))$. Of course, it would be most useful to compute/compare explicitely the matrices of $6j$-symbols for this theory. We expect that the theory of quantum coadjoint action leads to generalizations of the QHI for other semisimple Lie groups than $PSL(2,\mathbb{C})$. [1]

The value of the basic state sums $\mathfrak{L}_N(\mathcal{T}_{\mathcal{I}})$ is not invariant with respect to the change of branching, and it is invariant only for some specific instance of $\mathcal{I}$-transit. So, in order to construct invariants for (W,ρ) based on the quantum dilogarithms, these should be modified in such a way that the corresponding modified state sums are (at least) branching invariant and invariant with respect to *all* instances of $\mathcal{I}$-transits. We do this via a specific procedure of (partial) symmetrization of the quantum dilogarithms.

In Section 4 we show that this local symmetrization leads to fix an arbitrary non-empty link L in W, considered up to ambient isotopy, in order to fix one coherent globalization. So we incorporate this *link-fixing* in all the discussion: we consider triples (W,L,ρ) up to orientation preserving homeomorphisms of (W,L) and flat bundle isomorphisms of ρ, and we provide the appropriate notion of $\mathcal{D}$-triangulation for a triple (W,L,ρ). This is a $\mathcal{D}$-triangulation (T,b,z) for (W,ρ) in which the link L is realized as a Hamiltonian subcomplex H (i.e. H contains all the vertices of T). We also refine the $\mathcal{D}$-transits to preserve this Hamiltonian property of H.

The globalization of the symmetrization of the quantum dilogarithms is governed, for all odd positive integer $N>1$, by any fixed *integral charge* c on (T,H). An integral charge is a $\mathbb{Z}$-valued function of the edges of the (abstract) tetrahedra of T that satisfies suitable non-trivial global conditions, and which eventually encodes H, hence the link L. In fact, for any fixed N, we rather use the reduction mod(N) of 'half' the charge, i.e. $c'(e)=(p+1)c(e)\,\mathrm{mod}(N)$, where $N=2p+1$. This is a main point where it is important that N is odd.

The integral charges are a subtle ingredient of our construction. Their structure is very close to the one of the "flattenings" used by Neumann in his work on Cheeger–Chern–Simons classes of hyperbolic manifolds [26–28]. The main results concerning the existence and the structure of the integral charges are obtained by adapting some fundamental results of Neumann.

All this gives the notion of *charged* $\mathcal{D}$-triangulation $(\mathcal{T},c)=(T,H,b,z,c)$ for a triple (W,L,ρ); we stress that their existence is not an evident fact. The $\mathcal{D}$- and $\mathcal{I}$-transits are extended to transits of charged triangulations. This is the final set-up for defining the QHI: the idealization $(\mathcal{T}_{\mathcal{I}},c)$ of any charged $\mathcal{D}$-triangulation supports modified state sums $H_N(\mathcal{T}_{\mathcal{I}},c)\in\mathbb{C}$ based on the symmetrized quantum dilogarithms. Up to a sign and an Nth root of unity multiplicative factor, $H_N(\mathcal{T}_{\mathcal{I}},c)$ is invariant with respect to the choice of branching and for all instances of charged $\mathcal{I}$-transits.

In Section 4.2 we state the two main results of the present paper, proved in Sections 4.3 and 4.4, respectively: the existence of charged $\mathcal{D}$-triangulations for any triple (W,L,ρ), and the fact that the value of the state sums $H_N(\mathcal{T}_{\mathcal{I}},c)$ does not depend on the choice of $(\mathcal{T}_{\mathcal{I}},c)$ up to sign and Nth root of unity factors. This proof of invariance consists in reducing the full invariance to the transit invariance mentioned above. We eventually get the QHI $H_N(W,L,\rho)$, and $K_N(W,L,\rho)=H_N(W,L,\rho)^{2N}$ is a well-defined complex-valued invariant for every odd integer $N>1$.

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[1] The referee informed the authors that Kashaev and Reshetikhin recently constructed new invariants for complements of tangles in S^3 based on this theory, after a preliminary version of the present paper was put on the web in January 2001. See Kashaev and Reshetikhin, *Invariants of tangles with flat connections in their complements*, *I*: *Invariants and holonomy R-matrices*, *II*: *Holonomy R-matrices related to quantized envelopping algebras at roots of* 1, arXiv:math.AT/0202211.

In Section 4.5 we discuss some complements about the QHI. In particular, we prove a *duality* property related to the change of the orientation of W.

We had presented in [2] the construction of QHI for flat B-bundles on W, where B is the Borel subgroup of $PSL(2,\mathbb{C})$ of upper triangular matrices. In that case we adopted a slightly different symmetrization procedure. The resulting state sums differ from $H_N(\mathcal{T}_{\mathcal{I}},c)$, which work for arbitrary $PSL(2,\mathbb{C})$-bundles, by a scalar factor depending on the charged $\mathcal{D}$-triangulation $(\mathcal{T},c)$, not only on its idealization $(\mathcal{T}_{\mathcal{I}},c)$ (see Remark 4.31). The topological invariants $K_N(W,L)$ conjectured by Kashaev correspond to the particular case of these B-QHI, when ρ is the *trivial* flat bundle.

In Section 5 we discuss how to set the Volume Conjecture for the Jones invariants $J_N(L)$ of hyperbolic links L in S^3 in the framework of the general QHI theory.

An appropriate conceptual framework for both the QHI and the dilogarithmic invariant defined in [3] stems from the theory of *scissors congruence classes* (see [14], [27] and the references therein for details on this theory). It is elaborated in [2] for flat B-bundles, and in general in [3].

Let us conclude by saying that another idea on the background of our work, that is at least a meaningful heuristic support, is to look at it as part of an "exact solution" of the Euclidean analytic continuation of $(2+1)$ quantum gravity with negative cosmological constant, that was outlined in [34]. This should be a gauge theory with gauge group $SL(2,\mathbb{C})$ and an action of Chern–Simons type. Hyperbolic 3-manifolds are the empty "classical solutions". The Volume Conjecture discussed in Section 5 essentially agrees with the expected "semi-classical limits" of the partition functions of this theory (see [34, p. 77]).

2. $\mathcal{D}$-triangulations for a pair (W, ρ)

We first recall few generalities before defining the $\mathcal{D}$-triangulations.

2.1. Generalities on triangulations and spines

For the fundations of this theory, including the existence of spines, the reconstruction of manifolds from them and the complete calculus of triangulation/spine-moves, we refer to [10,24,30]. Other references are [6,7]. One finds also a clear treatment of this material in [32] (note that sometimes the terminologies do not agree). We shall refer to the topological space underlying a cell complex as its *polyhedron*.

Consider a tetrahedron Δ with its usual triangulation with 4 vertices, and let C be the interior of the 2-skeleton of the dual cell decomposition. A *simple* polyhedron P is a 2-dimensional compact polyhedron such that each point of P has a neighbourhood which can be embedded into an open subset of C. A simple polyhedron P has a natural stratification given by its singularities; P is *standard* (in [32] one uses the term *cellular*) if all the strata of this stratification are open cells of the appropriate dimension $\leqslant 2$. Depending on the dimension, we call the strata of a standard polyhedron P *vertices*, *edges* and *regions*.

Every compact 3-manifold Y (which for simplicity we assume connected) with non-empty boundary has *standard spines* [10], that is standard polyhedra P together with an embedding in $Int(Y)$ such that Y is a regular neighbourhood of P. Moreover, Y can be reconstructed from any of its standard spines. The standard polyhedra underlying standard spines of oriented 3-manifolds are characterized by the property of carrying a suitable "screw-orientation" along the edges [6]; a compact oriented 3-manifold Y can be reconstructed from any of its oriented standard spines. From now on we assume that Y is oriented, and we shall only consider oriented standard spines of it. Since we shall always work with combinatorial data encoded by triangulations/spines, which define the corresponding manifold only up to PL-homeomorphisms, we shall systematically forget the underlying embeddings.

A *singular* triangulation of a polyhedron Q is a triangulation in a weak sense, namely, self-adjacencies and multiple adjacencies of 3-simplices along 2-faces are allowed. For any Y as above, let us denote by $Q(Y)$ the space obtained by collapsing each connected component of ∂Y to a point. A (topological) *ideal triangulation* of Y is a singular triangulation T of $Q(Y)$ such that the vertices of T are precisely the points of $Q(Y)$ corresponding to the components of ∂Y.

For any ideal triangulation T of Y, the 2-skeleton of the *dual* cell decomposition of $Q(Y)$ is a standard spine $P(T)$ of Y. This procedure can be reversed, so that we can associate to each standard spine P of Y an ideal triangulation $T(P)$ of Y such that $P(T(P))=P$. Thus standard spines and ideal triangulations are dual equivalent viewpoints which we will freely intermingle. By removing small open neigbourhoods of the vertices of $Q(Y)$, any ideal triangulation leads to a cell decomposition of Y by *truncated tetrahedra*, which restricts to a singular triangulation of ∂Y.

Any ideal triangulation T of Y can be considered as a finite family $\{\Delta_i\}$ of *oriented abstract* tetrahedra, each being endowed with the standard triangulation with 4 vertices and the orientation induced by the one of Y, together with identifications of pairs of distinct (abstract) 2-faces. We will often distinguish between edges and 2-faces *in* T, that is considered after the identifications, and *abstract* edges and 2-faces, that is considered as simplices of the abstract Δ_i's. We view each Δ_i as positively embedded as a straight tetrahedron in $\mathbb{R}^3$ endowed with the orientation specified by the standard basis (the 'right-hand screw rule').

Consider now a compact closed oriented 3-manifold W. For any $r_0 \geqslant 1$, let $Y = W_{r_0} = W \setminus r_0D^3$ be the manifold obtained by removing r_0 disjoint open balls from W. By definition $Q(Y) = W$ and any ideal triangulation of Y is a singular triangulation of W with r_0 vertices; moreover, it is easily seen that all singular triangulations of W come in this way from ideal triangulations. We shall adopt the following terminology. A singular triangulation of W is simply called a *triangulation*. Ordinary triangulations (where neither self-adjacencies nor multi-adjacencies are allowed) are said to be *regular*.

The main advantage in using singular triangulations (resp. standard spines) instead of regular triangulations consists in the fact that there exists a *finite* set of moves which are sufficient in order to connect, by means of finite sequences of these moves, any two singular triangulations (resp. standard spines) of the same manifold. Let us recall some elementary moves on the triangulations (resp. simple spines) of a polyhedron $Q(Y)$ that we shall use throughout the paper; see Figs. 1 and 2.

The $2 \to 3$ *move*: Replace the triangulation T of a portion of $Q(Y)$ made by the union of 2 tetrahedra with a common 2-face f by the triangulation made by 3 tetrahedra with a new common edge which connect the two vertices opposite to f. Dually this move is obtained by sliding a portion of some region of $P(T)$ along an edge e, until it bumps into another region.

The bubble move: Replace a face of a triangulation T of $Q(Y)$ by the union of two tetrahedra glued along three faces. Dually this move is done by gluing a closed 2-disk D via its boundary ∂D on the standard spine $P(T)$, with exactly two transverse intersection points of ∂D along some edge

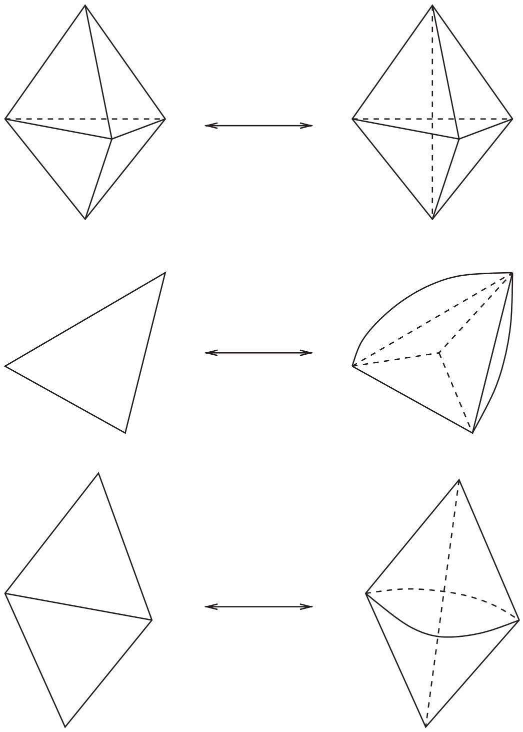

Fig. 1. The moves between singular triangulations.

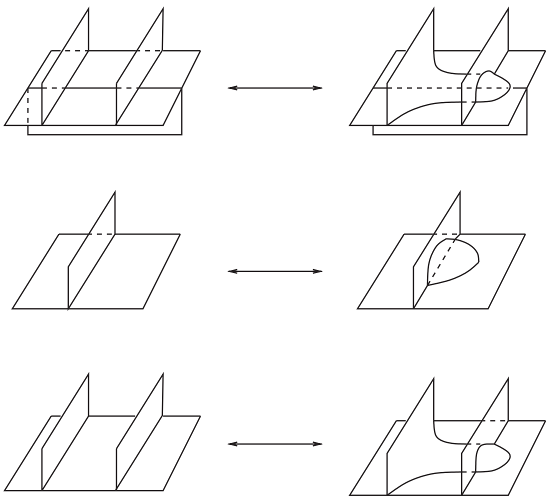

Fig. 2. The moves on standard spines.

of $P(T)$. The new triangulation thus obtained is dual to a spine of $Y \setminus D^3$, where D^3 is an open ball in the interior of Y.

The $0 \to 2$ *move*: Replace two adjacent faces of a triangulation T of $Q(Y)$ by the union of two tetrahedra glued along two faces, so that the other faces match the two former ones. The dual of this move is the same as for the $2 \to 3$ move, except that now we slide portions of regions *away* from the edges of $P(T)$.

Standard spines of the same compact oriented 3-manifold Y with boundary and with at least two vertices (which, of course, is a painless requirement) may always be connected by means of a finite sequence of the (dual) $2 \to 3$ move and its inverse. In order to handle triangulations of closed oriented 3-manifolds we also need a move which allows us to vary the number of vertices. The simplest way is to use the bubble move. Note that a bubble move followed by a $2 \to 3$ move with an adjacent tetrahedron gives a $1 \to 4$ move: this simply consists in subdividing a tetrahedron Δ by the cone over its 2-skeleton, with centre at an interior point of Δ.

Although the $2 \to 3$ move and the bubble move generate a complete calculus for triangulations and standard spines, it is useful to introduce the $0 \to 2$ move, or *lune* move. The inverse of the lune move is not always admissible because one could lose the standardness property of spines when using it. We say that a move which increases (resp. decreases) the number of tetrahedra is *positive* (resp. *negative*). In some situations it may be useful to use only positive moves. For that we need the following technical result due to Makovetskii [23]:

Proposition 2.1. *Let P and P' be standard spines of Y. There exists a spine P'' of Y such that P'' can be obtained from both P and P' via finite sequences of positive* $0 \to 2$ *and* $2 \to 3$ *moves.*

In this paper we shall use a restricted class of triangulations.

Definition 2.2. A *quasi-regular* triangulation T of a compact closed 3-manifold W is a triangulation where all edges have distinct vertices. A move $T \to T'$ is *quasi-regular* if both T and T' are quasi-regular.

Of course any regular triangulation of W is quasi-regular. We will also need the 2-dimensional version of the above facts. Given a compact closed surface S, there is a natural notion of ideal triangulation T of $S_{r_0} = S \setminus r_0 D^2$ (for arbitrary r_0) which corresponds to the notion of (singular) triangulation of S with r_0 vertices. The 1-skeleton P of the dual cell decomposition of T has only trivalent vertices and is a standard spine of S_{r_0}. In Fig. 3 we show 2-dimensional moves on triangulations and their dual standard spines: the $2 \to 2$ "flip" move, which is the 2-dimensional analogue of the $2 \to 3$ move, the 2-dimensional bubble move, and the $1 \to 3$ move, which is the 2-dimensional analogue of the above $1 \to 4$ move. Similarly to the 3-dimensional case, the $1 \to 3$ move is a composition of a bubble move and a $2 \to 2$ move. It is known that any two arbitrary triangulations of S with the same number of vertices can be connected by a finite sequence of $2 \to 2$ moves; hence, to connect arbitrary triangulations of S we only need a further move which increases by one the number of vertices. Finally, we still have the notion of quasi-regular triangulations of a surface S.

Let W be a compact closed oriented 3-manifold, T be a quasi-regular triangulation of W, and v_0 be a vertex of T. The link $S = Link(v_0, T)$ with its natural triangulation T_{v_0} can be identified with one of the spherical connected component of the boundary of $Y = W_{r_0}$, triangulated, as we said before, by the restriction of the natural cell decomposition of Y via the truncated tetrahedra of T. The cone over S with centre v_0 is $Star(v_0, T)$, the star of v_0 in T, so its natural triangulation is the cone over the triangulation T_{v_0} of S.

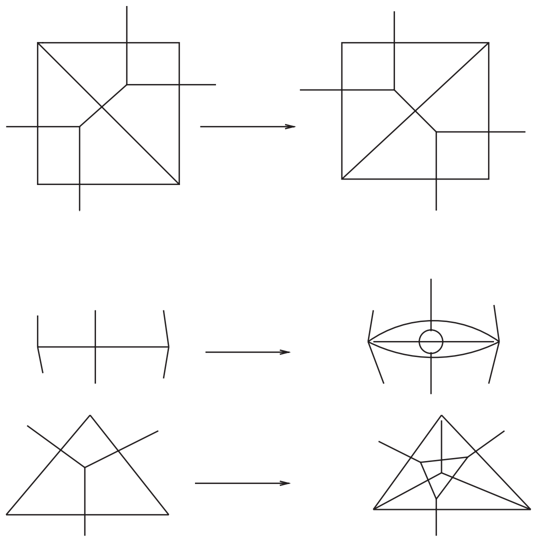

Fig. 3. 2-dimensional moves.

Note that the trace on ∂Y of a $2 \to 3$ move in T consists of three $2 \to 2$ moves and a couple of $1 \to 3$ moves. By quasi-regularity of T, this implies that any $2 \to 2$ or $1 \to 3$ move on T_{v_0} can be induced by suitable $2 \to 3$ moves around v_0.

2.2. Generalities on flat principal $PSL(2,\mathbb{C})$-bundles of closed 3-manifolds

Let W be a compact closed oriented 3-manifold, and ρ be a flat principal bundle over W with structural group $PSL(2,\mathbb{C})$. We consider the pair (W,ρ) up to oriented homeomorphisms of W and flat bundle isomorphisms of ρ. Equivalently, ρ is identified with a conjugacy class of representations of the fundamental group of W in $PSL(2,\mathbb{C})$, i.e. with a $PSL(2,\mathbb{C})$-character of W.

Let T be a triangulation of W with oriented edges. Denote by $Z^1(T;PSL(2,\mathbb{C}))$ the set of $PSL(2,\mathbb{C})$-valued simplicial 1-cocycles on T. In particular, for such a cocycle z we have $z(-e) = z(e)^{-1}$. A 0-cochain is a $PSL(2,\mathbb{C})$-valued function defined on the vertices of T. We denote by $[z]$ the equivalence class of $z \in Z^1(T;PSL(2,\mathbb{C}))$ up to cellular coboundaries: two 1-cocycles z and z' are equivalent if there exists a 0-cochain λ such that for any oriented edge e of T with ordered endpoints v_0, v_1, we have $z'(e) = \lambda(v_0)^{-1} z(e) \lambda(v_1)$. We denote this quotient set by $H^1(T;PSL(2,\mathbb{C}))$. The common refinements (subdivisions) of any two triangulations T and T' induce isomorphisms $H^1(T;PSL(2,\mathbb{C})) \cong H^1(T';PSL(2,\mathbb{C}))$. So $H^1(T;PSL(2,\mathbb{C}))$ can be identified with the set of isomorphism classes of flat principal $PSL(2,\mathbb{C})$-bundles on W, which itself may be described as the reduction of the sheaf cohomology set $H^1(W;\mathscr{C}^\infty(PSL(2,\mathbb{C})))$ to $H^1(W;PSL(2,\mathbb{C}))$ (i.e. where $PSL(2,\mathbb{C})$ is endowed with the discrete topology).

Compact-oriented hyperbolic 3-manifolds with their holonomy furnish a main example of pairs (W,ρ). There are other natural examples (W,ρ_α) coming from the ordinary simplicial cohomology of W, as follows. Let us denote by B the Borel subgroup of $SL(2,\mathbb{C})$ of upper triangular matrices.

There are two distinguished abelian subgroups of B:

(1) the *Cartan* subgroup $C = C(B)$ of diagonal matrices; it is isomorphic to the multiplicative group $\mathbb{C}^*$ via the map which sends $A = (a_{ij}) \in C$ to a_{11};
(2) the *parabolic* subgroup $Par(B)$ of matrices with double eigenvalue 1; it is isomorphic to the additive group $\mathbb{C}$ via the map which sends $A = (a_{ij}) \in Par(B)$ to $x = a_{12}$.

Denote by G any such abelian subgroup of B. There is a natural map $H^1(T;G) \to H^1(T;B)$ induced by the inclusion, and $H^1(T;G)$ is endowed with the usual abelian group structure. Note that $H^1(T;Par(B)) = H^1(T;\mathbb{C})$ is the ordinary (singular or de Rham) first cohomology group of W. Hence the inclusion $B \subset SL(2,\mathbb{C})$ allows us to associate to each 1-cohomology class $\alpha \in H^1(W,G)$ a pair (W,ρ_α). In particular, we can consider the trivial flat bundle ρ_0 on W.

For our purposes, we need to specialize the kind of triangulations, edge orientations and $PSL(2,\mathbb{C})$-valued simplicial 1-cocycles representing flat $PSL(2,\mathbb{C})$-bundles.

2.3. Branchings

Let us first specialize the kind of edge orientations. We do it for ideal triangulations of an arbitrary compact oriented 3-manifold Y with boundary. Let P be a standard spine of Y, and consider the dual ideal triangulation $T = T(P)$. Recall the notion of abstract tetrahedron of T.

Definition 2.3. A *branching* b of T is a choice of orientation for each edge of T such that on each abstract tetrahedron Δ of T it is associated to a total ordering v_0, v_1, v_2, v_3 of the (abstract) vertices by the rule: each edge is oriented by the arrow emanating from the smallest endpoint.

Note that for each $j = 0,\dots,3$ there are exactly j b-oriented edges incoming at the vertex v_j; hence there are only one source and one sink of the branching. This is equivalent to saying that for any 2-face f of Δ the boundary of f is not coherently oriented. In dual terms, a branching is a choice of orientation for each region of P such that for each edge of P we have the same induced orientation only twice. In particular, the edges of P have an induced prevailing orientation.

Branchings, mostly in terms of spines, have been widely studied and applied in [7–9]. A branching of P gives it the extra-structure of an embedded and oriented (hence normally oriented) *branched surface* in $Int(Y)$. Moreover, a branched spine P carries a suitable positively transverse *combing* of Y (i.e. a non-vanishing vector field).

Given a branching b on a oriented tetrahedron Δ (realized in $\mathbb{R}^3$ as stipulated in Section 2.1), denote by $E(\Delta)$ the set of b-oriented edges of Δ, and by e' the edge opposite to e. We put $e_0 = [v_0, v_1]$, $e_1 = [v_1, v_2]$ and $e_2 = [v_0, v_2] = -[v_2, v_0]$. This fixed ordering of the edges of the 2-face opposite to the vertex v_3 will be used all along the paper. The ordered triple of edges

$$(e_0 = [v_0, v_1], e_2 = [v_0, v_2], e'_1 = [v_0, v_3]) \tag{1}$$

departing from v_0 defines a *b-orientation* of Δ. This orientation may or may not agree with the orientation of Y. In the first case we say that Δ is of index $*_b = 1$, and it is of index $*_b = -1$ otherwise. The 2-faces of Δ can be named by their opposite vertices. We orient them by working as above on the boundary of each 2-face f: there is a b-ordering of the vertices of f, and an orientation of f which induces on ∂f the prevailing orientation among the three b-oriented edges.

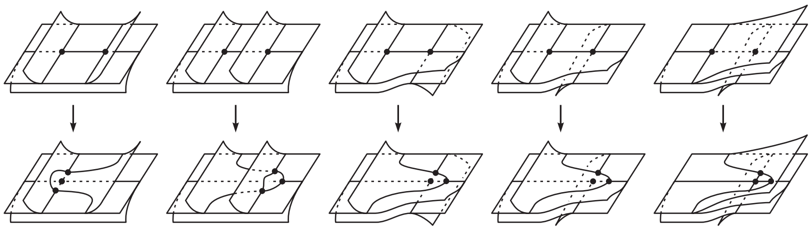

Fig. 4. $2 \to 3$ sliding moves.

This 2-face orientation corresponds dually to the orientation on the edges of P mentioned above. These considerations apply to each abstract tetrahedron of any branched triangulation (T, b) of Y.

2.3.1. Branching's existence and transit

In general, a given ideal triangulation T of Y may not admit any branching.

Definition 2.4. Given *any* choice g of edge-orientations on T and any move $T \to T'$, a transit $(T, g) \to (T', g')$ is given by any choice g' of edge-orientations on T' which agrees with g on the common edges of T and T'. This makes a *branching transit* if both g and g' are branchings. We will use the same terminology for moves on branched standard spines.

Concerning the existence of branched triangulations there is the following result:

Proposition 2.5 (Benedetti and Petronio [7, Theorem 3.4.9]). *For any system g of edge-orientations on T there exists a finite sequence of positive $2 \to 3$ transits such that the final (T', g') is actually branched.*

On another hand, any quasi-regular triangulation T of a closed 3-manifold W admits branchings of a special type, defined by fixing any total ordering of its vertices and by stipulating that the edge $[v_i, v_j]$ is positively oriented iff $j > i$. These branchings are called total ordering branchings. Any quasi-regular move which preserves the number of vertices also preserves the total orderings on the set of vertices, hence it obviously induces *total ordering branching transits*. If it increases the number of vertices, one can extend to the new vertex, in several different ways, the old total orderings of the set of vertices. Again, any of these ways induces a total odering branching transit. If (T, b) is an arbitrary branched triangulation of Y (i.e. T is not necessarily quasi-regular nor b is of total ordering type) and $T \to T'$ is either a positive $2 \to 3$, $0 \to 2$ or bubble move, then it can be completed, sometimes in different ways, to a branched transit $(T, b) \to (T', b')$. Any of these ways is a possible transit. On the contrary, it is easily seen that a negative $3 \to 2$ or $2 \to 0$ move may not be "branchable" at all.

For the sake of clarity, we show in Figs. 4–6 the whole set of $2 \to 3$ and $0 \to 2$ (dual) branched transits on standard spines, up to evident symmetries. Note that the middle sliding move in Fig. 4 corresponds dually to the branched triangulation move shown in Fig. 8. Following [7], one can distinguish two families of branched transits: the *sliding moves*, which actually preserve the positively

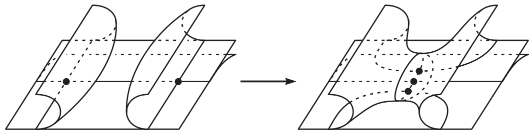

Fig. 5. $2 \to 3$ bumping move.

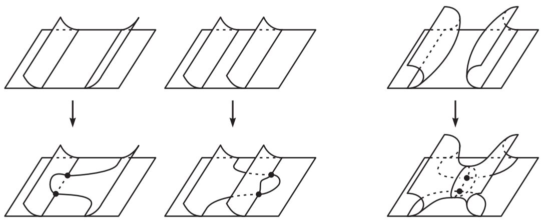

Fig. 6. Branched lune moves.

transverse combing mentioned after Definition 2.4, and the *bumping moves*, which eventually change it. We shall not exploit this difference in the present paper; see however Remark 4.30.

Finally, we note that for the proof of the main Theorem 4.14 it is enough to use only total ordering branchings, but we need to consider general branchings to extend the construction of the QHI to other situations (see the first point in Section 4.5, and the discussion on cusped manifolds in Section 5).

2.4. $\mathcal{D}$-triangulations for (W, ρ) and their idealization

We will now select certain generic $PSL(2,\mathbb{C})$-valued 1-cocycles on branched quasi-regular triangulations (T,b) of W, so as to define the $\mathcal{D}$-triangulations.

Definition 2.6. A *$\mathcal{D}$-triangulation* for the pair (W,ρ) consists of a triple $\mathcal{T} = (T,b,z)$ where: T is a quasi-regular triangulation of W; b is a branching of T; z is a $PSL(2,\mathbb{C})$-valued 1-cocycle on (T,b) representing ρ, such that (T,b,z) is *idealizable* (see Definition 2.7).

The name '$\mathcal{D}$-triangulation' refers to the fact that they are "decorated" by the branching and the cocycle (and, later in Definition 4.1, "distinguished" by an Hamiltonian link). If z is $PSL(2,\mathbb{C})$-valued 1-cocycle on (Δ,b), we write $z_j = z(e_j)$ and $z'_j = z(e'_j)$. Then, one reads for instance the cocycle condition on the 2-face opposite to v_3 as $z_0 z_1 z_2^{-1} = 1$. This holds for each abstract tetrahedron of any branched triangulation (T,b) of W and for (the restrictions of) any $PSL(2,\mathbb{C})$-valued 1-cocycle z on (T,b).

Consider the half space model of the hyperbolic space $\mathbb{H}^3$. We orient it as an open set of $\mathbb{R}^3$. The natural boundary $\partial\bar{\mathbb{H}}^3 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ of $\mathbb{H}^3$ is oriented by its complex structure. We realize $PSL(2,\mathbb{C})$ as the group of orientation preserving isometries of $\mathbb{H}^3$, with the corresponding conformal action on $\mathbb{CP}^1$.

Definition 2.7. Let (Δ, b, z) be a branched tetrahedron endowed with a $PSL(2,\mathbb{C})$-valued 1-cocycle z. It is *idealizable* iff

$$u_0 = 0,\ u_1 = z_0(0),\ u_2 = z_0 z_1(0),\ u_3 = z_0 z_1 z_0'(0)$$

are 4 distinct points in $\mathbb{C} \subset \mathbb{CP}^1 = \partial\bar{\mathbb{H}}^3$. These 4 points span a (possibly degenerate) hyperbolic ideal tetrahedron with ordered vertices. A triangulation (T, b, z) is *idealizable* iff all its abstract tetrahedra (Δ_i, b_i, z_i) are idealizable.

If (Δ, b, z) is idealizable, for all $j = 0, 1, 2$ one can associate to e_j and e_j' the same *cross-ratio* modulus $w_j \in \mathbb{C} \setminus \{0, 1\}$ of the hyperbolic ideal tetrahedron spanned by (u_0, u_1, u_2, u_3); we refer to [5, Chapter 5] for details on the meaning and the role of cross-ratio moduli in hyperbolic geometry. We have (indices $\mathrm{mod}(\mathbb{Z}/3\mathbb{Z})$):

$$w_{j+1} = 1/(1 - w_j) \tag{2}$$

and

$$w_0 = (u_2 - u_1)u_3/u_2(u_3 - u_1).$$

Let us write $p_0 = u_1(u_3 - u_2)$, $p_1 = (u_2 - u_1)u_3$, and $p_2 = -u_2(u_3 - u_1)$. Then

$$w_j = -p_{j+1}/p_{j+2}. \tag{3}$$

Set $w = (w_0, w_1, w_2)$ and call it a *modular triple*. The ideal tetrahedron spanned by (u_0, u_1, u_2, u_3) is non-degenerate iff the imaginary parts of the w_j's are not equal to zero; in such a case they share the same sign $*_w = \pm 1$.

Definition 2.8. We call (Δ, b, w) the *idealization* of the idealizable (Δ, b, z), and identify it with the branched tetrahedron in $\bar{\mathbb{H}}^3$ spanned by (u_0, u_1, u_2, u_3). For any $\mathscr{D}$-triangulation $\mathscr{T} = (T, b, z)$ of (W, ρ), its *idealization* $\mathscr{T}_{\mathscr{I}} = (T, b, w)$ is given by the family $\{(\Delta_i, b_i, w_i)\}$ of idealizations of the (Δ_i, b_i, z_i)'s. We say that $\mathscr{T}_{\mathscr{I}}$ is an $\mathscr{I}$-triangulation for (W, ρ). It is non-degenerate if each $\{(\Delta_i, b_i, w_i)\}$ is non-degenerate.

Remark 2.9. (1) We could incorporate the non-degeneracy assumption into the notion of idealizable tetrahedron. All the constructions of the present paper would run in the same way. The non-degenerate assumption simplifies the exposition and also certain proofs concerning the classical dilogarithmic invariant of (W, ρ) considered in [3].

(2) Since $PSL(2,\mathbb{C})$ acts on $\mathbb{CP}^1$ via Moebius transformations $z_j : x \mapsto (a_j x + b_j)/(c_j x + d_j)$, it is immediate to formulate for any given quasi-regular triangulation T a simple system of algebraic equalities on the entries of the z_j's, whose zero set describes non-idealizable cocycles.

(3) In [2] we have used so-called *full* B-valued 1-cocycles z to construct the QHI for B-characters. 'Full' means that for any edge e the upper diagonal entry $x(e)$ of $z(e)$ is non-zero. It is easy to verify that a B-valued 1-cocycle is full iff it is idealizable. The idealization we proposed for such cocycles was in fact a specialization of the present general procedure. We can simply write the moduli for the idealization of a $\mathscr{D}$-tetrahedron with a full B-valued 1-cocycle as $w_j = -q_{j+1}/q_{j+2}$, where $q_j = x(e_j)x(e_j')$ for $j = 0, 1$, and $q_2 = -x(e_2)x(e_2')$ (beware that $p_i \neq q_i$).

(4) It follows from the cocycle condition or from relation (2) that $p_0 + p_1 + p_2 = 0$ (and also that $q_0 + q_1 + q_2 = 0$—see remark (3)).

The following lemma is immediate:

Lemma 2.10. *For any $PSL(2,\mathbb{C})$-character ρ, any quasi-regular branched triangulation (T,b) of W can be completed to a $\mathcal{D}$-triangulation (T,b,z) for the pair (W,ρ).*

In fact, given any $PSL(2,\mathbb{C})$-valued 1-cocycle, one can perturb it by the coboundaries of generic 0-cochains which are injective on the vertices of T, so that we get idealizable 1-cocycles.

2.4.1. Tetrahedral symmetries

The idealization has a good behaviour with respect to a change of branching (the 'tetrahedral symmetries'). Indeed, we have:

Lemma 2.11. *Denote by S_4 the permutation group on four elements. A permutation $p \in S_4$ of the vertices of an idealizable tetrahedron (Δ,b,z) gives another idealizable tetrahedron (Δ,b',z'). The permutation turns the idealization (Δ,b,w) into an isometric (Δ,b',w'), where for each edge e of Δ we have $w'(e)=w(e)^{\varepsilon(p)}$ and $\varepsilon(p)$ is the signature of p.*

Proof. Consider for instance the transposition $(0,1)$. It turns the (ordered) set of points $0, z_0(0)$, $z_0z_1(0)$, $z_0z_1z_0'(0)$ into $0,(z_0)^{-1}(0),z_1(0),z_1z_0'(0)$. By applying on this second set the hyperbolic isometry z_0, one gets the first set after the transposition of 0 and $z_0(0)$. Things go similarly for any other permutation. Then the lemma follows immediately, due to the behaviour of cross-ratios with respect to vertex permutations. □

2.4.2. Hyperbolic edge compatibility

We are now concerned with an important global property of the idealized triangulations $\mathcal{T}_{\mathcal{I}}$. Before to state it, let us stress that when dealing with modular triples one has to be careful with the orientations. Recall that every $\mathcal{I}$-tetrahedron (Δ,b,w) is oriented by definition; in the case of an $\mathcal{I}$-triangulation this is given by the orientation of W. There is also the b-orientation encoded by the sign $*=*_b$. The idealization 'physically' realizes the vertices of Δ on $\partial\bar{\mathbb{H}}^3$, with the ordering induced by b. When the spanned ideal tetrahedron is non-degenerate, the b-orientation may or may not agree with the one induced by the fixed orientation of $\mathbb{H}^3$, which is encoded by the sign $*_w$ of the modular triple. Let $\mathcal{T}_{\mathcal{I}}=(T,b,w)$ be an $\mathcal{I}$-triangulation. The preceding discussion shows that the contribution of each (Δ_i,b_i,w_i) to any computation with the moduli is given by the $w(e)^*$'s, where e is any edge of Δ_i and $*=*_{b_i}$. The next Lemma 2.12 is a first concretization of this fact (see also the notion of $\mathcal{I}$-transit below). Denote by $E(T)$ the set of edges of T, by $E_\Delta(T)$ the whole set of edges of the associated abstract tetrahedra $\{\Delta_i\}$, and by $\varepsilon_T : E_\Delta(T) \to E(T)$ the natural identification map.

Lemma 2.12. *For any edge $e \in E(T)$ we have $\prod_{a\in\varepsilon_T^{-1}(e)} w(a)^*=1$, where $*=\pm 1$ according to the b-orientation of the tetrahedron Δ_i that contains a.*

Proof. Looking at $Star(e,T)$ we see that up to a sign two consecutive moduli partially compensate along the common face of the corresponding tetrahedra. For instance, in Fig. 7 the left (resp. right)

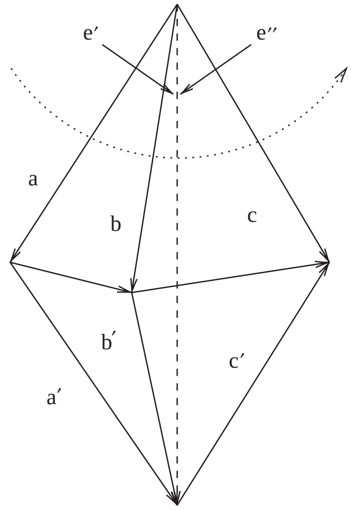

Fig. 7. The compatibility relation around an edge.

tetrahedron is negatively (resp. positively) b-oriented; we have

$$w(e')^{-1}w(e'') = (ab')(-bc')/(ba')(b'c) = -ac'/a'c.$$

This and Lemma 2.11 show that the same holds true when the two tetrahedra are simultaneously positively or negatively b-oriented. Continuing this way around e, we end up with $\prod_{a\in\varepsilon_T^{-1}(e)} w(a)^* = \pm 1$. Each -1 contribution comes from a tetrahedron where the b-orientations of the two faces containing e are opposite (that is, when the corresponding a is e_2 or e_2'). Since W is orientable, a short closed loop about e may only meet an even number of such tetrahedra. This gives the result. □

This lemma means that around each edge the signed moduli verify the usual compatibility condition needed when one tries to construct hyperbolic 3-manifolds by gluing hyperbolic ideal tetrahedra. So the $\mathcal{I}$-triangulations have the following geometric interpretation. Given an $\mathcal{I}$-triangulation $\mathcal{T}_{\mathcal{I}} = (T, b, w)$ of (W, ρ), lift T to a cellulation $\widetilde{T}$ of the universal cover $\widetilde{W}$, and fix a base point $\tilde{x}_0$ in the 0-skeleton of $\widetilde{T}$; denote by x_0 the projection of $\tilde{x}_0$ onto W. Then, for any tetrahedron in $\widetilde{T}$ that contains $\tilde{x}_0$, use the moduli of the corresponding $(\Delta_i, b_i, w_i) \in \mathcal{T}_{\mathcal{I}}$ to define an hyperbolic ideal tetrahedron. Do this by respecting the gluings in $\widetilde{T}$. Starting from the vertices adjacent to $\tilde{x}_0$ and continuing in this way, we construct an image in $\bar{\mathbb{H}}^3$ of a complete lift of T in $\widetilde{T}$, having one tetrahedron in each $\pi_1(W)$-orbit. The key point is that Lemma 2.12 implies that for any two paths of tetrahedra in $\widetilde{T}$ having a same starting point, we get the same end point. This construction extends to a piecewise-linear map $D : \widetilde{W} \to \bar{\mathbb{H}}^3$, equivariant with respect to the action of $\pi_1(W)$ and $PSL(2, \mathbb{C})$. So we eventually find: a representation $\tilde{\rho} : \pi_1(W, x_0) \to PSL(2, \mathbb{C})$ with character ρ and satisfying $D(\gamma(x)) = \tilde{\rho}(\gamma)D(x)$ for each $\gamma \in PSL(2, \mathbb{C})$; a piecewise-straight continuous section of the flat bundle $\widetilde{W} \times_{\tilde{\rho}} \bar{\mathbb{H}}^3 \to W$, with structural group $PSL(2, \mathbb{C})$ and total space the quotient of $\widetilde{W} \times \bar{\mathbb{H}}^3$ by the diagonal action of $\pi_1(W)$ and $\tilde{\rho}$. The map D behaves formally as a developing map for a $(PSL(2, \mathbb{C}), \mathbb{H}^3)$-structure on W (see e.g. [5, Chapter B] for this notion).

2.4.3. $\mathscr{D}$- and $\mathscr{I}$-transits

We consider now moves on $\mathscr{D}$-triangulations $\mathscr{T}=(T,b,z)$ and $\mathscr{I}$-triangulations $\mathscr{T}_{\mathscr{I}}=(T,b,w)$ for the pair (W,ρ), called $\mathscr{D}$- and $\mathscr{I}$-*transits*, respectively. They are supported by the bare triangulation moves mentioned in Section 2.1, but they also include the transits of the respective extra-structures. First of all we require that they are quasi-regular moves. We stress that this is not an automatic fact; on the contrary this leads to one main technical complication in the proofs. Then we require that $(T_0,b_0)\leftrightarrow(T_1,b_1)$ is a branching transit in the sense of Definition 2.4.

Definition 2.13. Let (T_0,b_0), (T_1,b_1) be branched quasi-regular triangulations and $z_k\in Z^1(T_k;PSL(2,\mathbb{C}))$, $k=0,1$. We have a *cocycle transit* $(T_0,z_0)\leftrightarrow(T_1,z_1)$ if z_0 and z_1 agree on the common edges of T_0 and T_1. This makes an *idealizable cocycle transit* if both z_0 and z_1 are idealizable 1-cocycles, and in this case we say that $(T_0,b_0,z_0)\leftrightarrow(T_1,b_1,z_1)$ is a $\mathscr{D}$-transit.

It is not hard to see that z_0 and z_1 as above represent the same flat bundle ρ. Note that for $2\to3$ and $0\to2$ moves, given z_k there is only one (resp. at most one) z_{k+1} with this property. We stress that in some special cases a $2\to3$ transit of an idealizable cocycle can actually not preserve the idealizability, but generically this does not hold. For positive bubble moves there is always an infinite set of possible (idealizable) cocycle transits. The following lemma shows that the $\mathscr{D}$-transits are generic.

Lemma 2.14. *Let (T,b) be a branched quasi-regular triangulation of W. Suppose that $(T,b)=(T_1,b_1)\to\cdots\to(T_s,b_s)=(T',b')$ is a finite sequence of quasi-regular $2\leftrightarrow3$ branching transits. Then for each T_i there exists a dense open set U_i of $PSL(2,\mathbb{C})$-valued 1-cocycles, in the quotient topology of $PSL(2,\mathbb{C})^{r_1(T_i)}$ as a space of matrices ($r_1(T_i)$ being the number of edges of T_i), such that for every $z_i\in U_i$, (T_i,b_i,z_i) is a $\mathscr{D}$-triangulation, and the transit $T_i\to T_{i+1}$ maps U_i into U_{i+1}. Moreover each class $\alpha\in H^1(W;PSL(2,\mathbb{C}))\cong H^1(T_i;PSL(2,\mathbb{C}))$ can be represented by cocycles in U_i.*

Proof. Each $2\leftrightarrow3$, $0\leftrightarrow2$ or negative bubble transit $(T_i,b_i,z_i)\to(T_{i+1},b_{i+1},z_{i+1})$ defines an algebraic surjective map from $Z^1(T_i;PSL(2,\mathbb{C}))$ to $Z^1(T_{i+1};PSL(2,\mathbb{C}))$. Since all edges of T_{i+1} have distinct vertices, there are no trivial (two term) cocycle relations on T_{i+1}. Hence the set of idealizable cocycles for which a transit fails to be idealizable is contained in a proper algebraic subvariety of $Z^1(T_i;PSL(2,\mathbb{C}))$. Working by induction on s we get the conclusion. $\square$

Let us now consider the transits for the idealized triangulations. Consider the convex hull of five distinct points $u_0,u_1,u_2,u_3,u_4\in\partial\bar{H}^3$, with the two possible triangulations Q_0 Q_1 made of the oriented hyperbolic ideal tetrahedra Δ^i obtained by omitting u_i, $i=0,\dots,4$. An edge e of $Q_i\cap Q_{i+1}$ belongs to one tetrahedron of Q_i iff it belongs to two tetrahedra of Q_{i+1}. Then, the modulus of e in Q_i is the product of the two moduli of e in Q_{i+1}. Also, the product of the moduli on the central edge of Q_1 is equal to 1.

Let $T\to T'$ be a $2\to3$ move. Consider the two (resp. three) abstract tetrahedra of T (resp. T') involved in the move. They determine subsets $\widetilde{E}(T)$ of $E_\Delta(T)$ and $\widetilde{E}(T')$ of $E_\Delta(T')$. Put $\widehat{E}(T)=E_\Delta(T)\setminus\widetilde{E}(T)$ and $\widehat{E}(T')=E_\Delta(T')\setminus\widetilde{E}(T')$. Clearly one can identify $\widehat{E}(T)$ and $\widehat{E}(T')$. The above configurations Q_0 and Q_1 and the considerations made before Lemma 2.12 lead to the following definition:

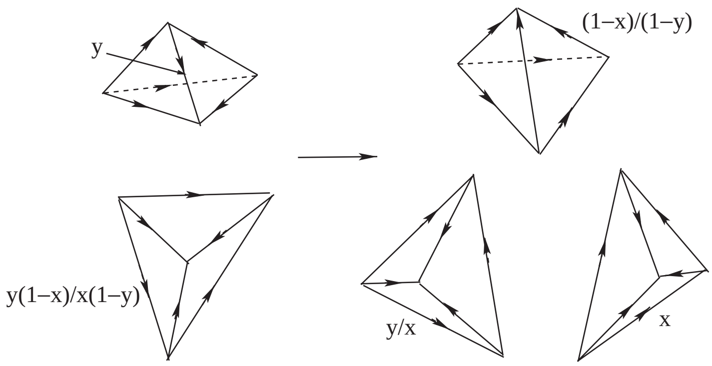

Fig. 8. A $2 \leftrightarrow 3$ ideal transit.

Definition 2.15. A $2 \to 3$ *$\mathscr{I}$-transit* $(T,b,w) \to (T',b',w')$ of $\mathscr{I}$-triangulations for a pair (W,ρ) is such that

(1) w and w' agree on $\widehat{E}(T) = \widehat{E}(T')$;
(2) for each common edge $e \in \varepsilon_T(\widetilde{E}(T)) \cap \varepsilon_{T'}(\widetilde{E}(T'))$ we have

$$\prod_{a \in \varepsilon_T^{-1}(e)} w(a)^* = \prod_{a' \in \varepsilon_{T'}^{-1}(e)} w'(a')^*, \tag{4}$$

where $* = \pm 1$ according to the b-orientation of the abstract tetrahedron containing a (resp. a'). We have a $0 \to 2$ (resp. *bubble*) *$\mathscr{I}$-transit* if the above first condition is satisfied, and we replace the second by:
(2′) for each edge $e \in \varepsilon_{T'}(\widetilde{E}(T'))$ we have

$$\prod_{a' \in \varepsilon_{T'}^{-1}(e)} w'(a')^* = 1. \tag{5}$$

$\mathscr{I}$-transits for negative $3 \to 2$ moves are defined in exactly the same way, and for negative $2 \to 0$ and bubble moves w' is defined by simply forgetting the moduli of the two disappearing tetrahedra. The condition (1) above implies that the product of the $w'(a')^*$'s around the new edge is equal to 1. A $2 \leftrightarrow 3$ $\mathscr{I}$-transit is shown in Fig. 8; we only indicate the first component 'w_0' of each modular triple. In general, the relations (4) may imply that w or w' equals 0 or 1 on some edges. In that case, the $2 \leftrightarrow 3$ $\mathscr{I}$-transit fails. In particular, in Fig. 8 we assume that $x \neq y$.

Note that for $2 \leftrightarrow 3$ $\mathscr{I}$-transits w' is uniquely determined by w. On the contrary, there is one degree of freedom for (positive) $0 \to 2$ and bubble $\mathscr{I}$-transits. Relation (5) simply means that such transits give *the same* modular triples to the two new tetrahedra, for their b-orientations are opposite.

The next proposition states the remarkable fact that $\mathscr{D}$- and $\mathscr{I}$-transits together with the idealization make commutative diagrams, that is the $\mathscr{D}$-transits *dominate* the $\mathscr{I}$-transits.

Proposition 2.16. *Consider a fixed pair (W,ρ), and denote by $\mathscr{I}$ the idealization map $\mathscr{T} \to \mathscr{T}_{\mathscr{I}}$ on its $\mathscr{D}$-triangulations. For any $\mathscr{D}$-transit $\mathfrak{d}$ there exists an $\mathscr{I}$-transit $\mathfrak{i}$ (resp. for any $\mathfrak{i}$ there exists $\mathfrak{d}$) such that $\mathfrak{i} \circ \mathscr{I} = \mathscr{I} \circ \mathfrak{d}$.*

For $2 \leftrightarrow 3$ transits there is also an uniqueness statement.

Proof. By using the tetrahedral symmetries of Lemma 2.11, it is enough to show the proposition for one branching transit configuration (for instance the one of Fig. 8). The idealization map defines embeddings of the $\mathscr{D}$-tetrahedra of this configuration as branched (b-oriented) ideal tetrahedra in $\mathbb{H}^3$. Since orientation preserving isometries do not alter the moduli, the union of these tetrahedra may be viewed as the convex hull of five distinct ordered points on $\mathbb{CP}^1$, such that the ordering induces the branching on each tetrahedron. Then the verification follows immediately from the definition of the idealization. □

Note that the possible failures of $2 \to 3$ transits of idealizable cocycles that we mentioned after Definition 2.13 exactly correspond to the failures of $2 \to 3$ $\mathscr{I}$-transits (for instance when $x = y$ in Fig. 8).

3. Quantum dilogarithms and basic state sums for pairs (W, ρ)

Let $N = 2p+1 > 1$ be a fixed odd positive integer, and put $\zeta = \exp(2i\pi/N)$. The quantum algebraic origin of the Faddeev–Kashaev's matrix quantum dilogarithms is discussed in Appendix A. Here we forget this origin, and, for the reader's convenience, we simply introduce the special functions needed for defining *basic state sums* supported by the $\mathscr{I}$-triangulations $\mathscr{T}_{\mathscr{I}}$ of any pair (W,ρ). The main property of these basic state sums is to be invariant for some *specific* instances of $\mathscr{I}$-transits.

We denote by g the analytic function defined for any complex number x with $|x| < 1$ by

$$g(x) := \prod_{j=1}^{N-1} (1 - x\zeta^j)^{j/N}$$

and set $h(x) := x^{-p} g(x)/g(1)$ when x is non-zero (one computes that $|g(1)| = N^{1/2}$). We shall still write g for its analytic continuation to the complex plane with cuts from the points $x = \exp(i\varepsilon)\zeta^k$, $k = 0, \ldots, N-1$, $\varepsilon \in \mathbb{R}$, to infinity. Hereafter we will implicitly assume that ε is such that the cuts are away from the points where g is evaluated (things will not depend on this choice).

Consider the curve $\Gamma = \{x^N + y^N = z^N\} \subset \mathbb{C}P^2$ (homogeneous coordinates), and the rational functions on Γ given for any $n \in \mathbb{N}$ by

$$\omega(x, y, z \mid n) = \prod_{j=1}^{n} \frac{(y/z)}{1 - (x/z)\zeta^j}. \tag{6}$$

These functions are periodic in their integer argument, with period N. Denote by δ the N-periodic Kronecker symbol, i.e. $\delta(n) = 1$ if $n \equiv 0 \bmod (N)$, and $\delta(n) = 0$ otherwise. Set $[x] = N^{-1}(1 - x^N)/(1 - x)$.

The elementary building blocks of the basic state sums are the $N^2 \times N^2$-matrix valued quantum dilogarithms and their inverses, whose matrix entries are the rational functions defined on the curve Γ by

$$R(x, y, z)^{\gamma,\delta}_{\alpha,\beta} = h(z/x)\zeta^{\alpha\delta + \alpha^2/2}\omega(x, y, z \mid \gamma - \alpha)\delta(\gamma + \delta - \beta),$$

$$\bar{R}(x, y, z)^{\alpha,\beta}_{\gamma,\delta} = \frac{[x/z]}{h(z/x)}\, \zeta^{-\alpha\delta - \alpha^2/2}\, \frac{\delta(\gamma + \delta - \beta)}{\omega(\frac{x}{\zeta}, y, z \mid \gamma - \alpha)}.$$

We can interpret these matrices as functions of $\mathscr{I}$-tetrahedra as follows. Let (Δ, b, w) be an $\mathscr{I}$-tetrahedron. The 1-skeleton of the cell decomposition of Δ dual to the canonical triangulation with 4 vertices is made of 4 edges incident at an interior point of Δ. As we said in Section 2.3, the orientations of these edges are complementary to the b-orientations of the dual 2-faces of Δ. Two of them are pointing inwards Δ, and the others are pointing outwards. So they form two distinguished pairs. Let us order the two edges of each pair as the corresponding 2-faces of Δ (ordered by the opposite vertices). We can associate to both ordered pairs a copy of $\mathbb{C}^N \otimes \mathbb{C}^N$ (with the standard basis), which we denote respectively by $I_1 \otimes I_2$ (for 'inwards') and $O_1 \otimes O_2$ (for 'outwards').

Write $w_i = -p_{i+1}/p_{i+2}$ (indices mod$(\mathbb{Z}/3\mathbb{Z})$) as in (3). Recall from Remark 2.9 (4) that $p_0 + p_1 + p_2 = 0$. Fix common determinations of the Nth roots of the p_i's, and denote them by p_i'. We define a matrix $\mathfrak{L}_N(\Delta, b, w) : I_1 \otimes I_2 \to O_1 \otimes O_2$ by

$$\mathfrak{L}_N(\Delta, b, w) = \begin{cases} R(p_1', p_0', -p_2') & \text{if } \ * = 1, \\ \bar{R}(p_1', p_0', -p_2') & \text{if } \ * = -1, \end{cases}$$

where $* = \pm 1$ according to the b-orientation of Δ. Since $\mathfrak{L}_N(\Delta, b, w)$ is homogeneous in the p_i''s, it only depends on (b, w).

Let $\mathscr{T}_{\mathscr{I}} = (T, b, w)$ be any $\mathscr{I}$-triangulation for (W, ρ). Let us consider the 1-skeleton C of the cell decomposition dual to T, with the edges oriented as above. By associating to each (Δ_i, b_i, w_i) the corresponding operator $\mathfrak{L}_N(\Delta_i, b_i, w_i)$, one gets an *operator network* whose complete contraction gives a scalar $\mathfrak{L}_N(\mathscr{T}_{\mathscr{I}}) \in \mathbb{C}$ (note that there is no edge with free ends in C). This has an explicit expression as a *state sum* as follows. A *state* is a function defined on the edges of C, with values in $\{0, \ldots, N-1\}$. Any state α determines an entry (a $6j$-symbol) $\mathfrak{L}_N(\Delta_i, b_i, w_i)_\alpha$ of $\mathfrak{L}_N(\Delta_i, b_i, w_i)$, for each (Δ_i, b_i, w_i). Set

$$\mathfrak{L}_N(\mathscr{T}_{\mathscr{I}})_\alpha = \prod_i \mathfrak{L}_N(\Delta_i, b_i, w_i)_\alpha$$

and

$$\mathfrak{L}_N(\mathscr{T}_{\mathscr{I}}) = \sum_\alpha \mathfrak{L}_N(\mathscr{T}_{\mathscr{I}})_\alpha. \tag{7}$$

Given any maximal tree τ in C, we can consider the polyhedron P_τ obtained by cutting T along the faces dual to the edges of $C \setminus \tau$. Fix an ordering of these faces. Then we can write $\mathfrak{L}_N(\mathscr{T}_{\mathscr{I}})$ as the trace of the operator obtained by composing the $\mathfrak{L}_N(\Delta_i, b_i, w_i)$'s along the faces dual to the edges of τ. The domain (resp. target) space of this operator is the tensor product of one copy of $\mathbb{C}^N$ for each face of ∂P_τ whose dual edge points inwards (resp. outwards) P_τ, with the same ordering. We have the following key facts:

(a) Straightforward computations show that $\mathfrak{L}_N(\Delta, b, w)$ does not respect the tetrahedral symmetries, i.e. it is not invariant if we change the branching.
(b) $\mathfrak{L}_N(\mathscr{T}_{\mathscr{I}})$ is invariant *only* for some peculiar instances of $2 \leftrightarrow 3$ $\mathscr{I}$-transits. One among them is shown in Fig. 8. This instance corresponds to the *basic pentagon identity* satisfied by $\mathfrak{L}_N(\Delta, b, w)$ (see (A.4) in Appendix A).

Before we overcome these problems, let us disgress a bit to motivate and explain the approach we will follow.

3.1. Quantum vs. classical dilogarithms

There is a 'classical' analogue of $\mathfrak{L}_N(\mathcal{T}_{\mathcal{I}})$, which we now describe. We refer to [3] for details. Denote by log the standard branch of the logarithm, with arguments in $]-\pi,\pi]$. Put $\mathfrak{D}=\mathbb{C}\setminus\{(-\infty;0)\cup(1;+\infty)\}$. The *Rogers dilogarithm* is the complex analytic function defined over $\mathfrak{D}$ by

$$\mathrm{L}(x)=-\frac{\pi^2}{6}-\frac{1}{2}\int_0^x\left(\frac{\log(t)}{1-t}+\frac{\log(1-t)}{t}\right)\mathrm{d}t, \tag{8}$$

where we integrate first along the path $[0;1/2]$ on the real axis and then along any path in $\mathfrak{D}$ from $1/2$ to x. Here we add $-\pi^2/6$ so that $\mathrm{L}(1)=0$. It is well-known that L verifies the fundamental Schaeffer's identity:

$$\mathrm{L}(x)-\mathrm{L}(y)+\mathrm{L}(y/x)-\mathrm{L}\left(\frac{1-x^{-1}}{1-y^{-1}}\right)+\mathrm{L}\left(\frac{1-x}{1-y}\right)=0, \tag{9}$$

when x, y are real and $0<y<x<1$. In fact, this identity characterizes the Rogers dilogarithm: if $f(0;1)\to\mathbb{R}$ is a 3 times differentiable function satisfying (9) for all $0<y<x<1$, then $f(x)=k\mathrm{L}(x)$ for a suitable constant k. By analytic continuation, relation (9) also holds true for complex parameters x, y with $\mathrm{Im}(y)\neq 0$, providing that x lies inside the triangle formed by 0, 1 and y. Note that for such x, y all the arguments of L in (9) have imaginary parts with the same sign. For every non-degenerate $\mathcal{I}$-tetrahedron (Δ,b,w) set $\mathrm{L}(\Delta,b,w)=\mathrm{L}(w_0)$, and for every $\mathcal{I}$-triangulation $\mathcal{T}_{\mathcal{I}}$ set $\mathrm{L}(\mathcal{T}_{\mathcal{I}})=\sum_i \mathrm{L}(\Delta_i,b_i,w_i)$.

We note that $\mathrm{L}(\Delta,b,w)$ does not respect the tetrahedral symmetries. Also, with the above restriction on the moduli, the Schaeffer's identity implies the invariance of $\mathrm{L}(\mathcal{T}_{\mathcal{I}})$ for the same specific instance of $\mathcal{I}$-transit shown in Fig. 8, and considered in (b) above. On another hand, $\mathfrak{L}_N(\Delta,b,w)$ is a peculiar matrix representation of a specific operator Φ acting on a suitable completion of the $\mathbb{C}$-algebra generated by two elements a, b satisfying $ab=\zeta ba$ (see [1,2]). The operator Φ may be defined by an N-dependent power series whose dominant term for $N\to\infty$ essentially involves dilogarithms. It satisfies a non-commutative version of Relation (9), which induces the basic pentagon identity (A.4) in the particular matrix representation defining $\mathfrak{L}_N(\Delta,b,w)$. The dominant term for $N\to\infty$ of that 'quantum Schaeffer's identity' satisfied by Φ is the exponential of (9) up to a multiplicative constant times N, where L is expressed as its power series expansion for $|x-1/2|<1$ [4,20]. It turns out that this result also holds for the matrix entries of $\mathfrak{L}_N(\Delta,b,w)$. These facts justify the following name: $\mathfrak{L}_N(\Delta,b,w)$ is the N^2-dimensional *non symmetric quantum dilogarithm*, computed on the given $\mathcal{I}$-tetrahedron.

In order to construct invariants for (W,ρ) based on $\mathfrak{L}_N(\Delta,b,w)$, these should be modified so that the corresponding modified state sums are invariant with respect to the *whole* set of instances of $\mathcal{I}$-transits, as well as the choice of branching. This is done as follows. Formally similar problems have been solved in [3] to define a dilogarithmic invariant $\mathrm{R}(W,\rho)$ based on $\mathrm{L}(\Delta,b,w)$.

3.2. Symmetrized quantum dilogarithms

Let (Δ,b,w) be an $\mathcal{I}$-tetrahedron. The notion of integral charges on hyperbolic ideal tetrahedra that we are going to define is strictly related to that of *flattenings*, introduced by Neumann in his work on Cheeger–Chern–Simons classes of hyperbolic manifolds [26–28]. Flattenings also emerge straightforwardly in [3], to get the invariance of $\mathrm{L}(\varDelta,b,w)$ with respect to a change of branching, that we discussed above. In a similar way, the integral charges are going to be used in order to (partially) repare the same non-invariance of the quantum dilogarithms $\mathfrak{L}_N(\varDelta,b,w)$. The main difference between integral charges and flattenings is that the charges do not depend on the moduli; a charge defines a flattening on a non-degenerate $\mathscr{I}$-tetrahedron only if $*_w=-1$.

Definition 3.1. An *integral charge* on $(\varDelta,b,w)$ is a $\mathbb{Z}$-valued map defined on the edges of $\varDelta$ such that $c(e)=c(e')$ for opposite edges e and e', and $c_0+c_1+c_2=1$ (where $c_i=c(e_i)$). We call $c(e)$ the charge of e.

Write $N=2p+1$, and for each edge e of $\varDelta$ set $c'(e)=(p+1)c(e)\,\mathrm{mod}(N)$, viewed as a point in $\{0,\dots,N-1\}$. Recall the notation p_i' for the determinations of the Nth roots of the p_i's.

Definition 3.2. The N^2-dimensional *symmetrized quantum dilogarithm* is the matrix valued function $\mathfrak{R}_N(\varDelta,b,w,c): I_1\otimes I_2\to O_1\otimes O_2$ defined on the set of charged $\mathscr{I}$-tetrahedra $(\varDelta,b,w,c)$ and given by

$$\mathfrak{R}_N(\varDelta,b,w,c)=\begin{cases}((-p_1'/p_2')^{-c_1}(-p_2'/p_0')^{c_0})^pR'(w\,|\,c) & \text{if } *=1,\\ ((-p_1'/p_2')^{-c_1}(-p_2'/p_0')^{c_0})^p\bar{R}'(w\,|\,c) & \text{if } *=-1,\end{cases}\tag{10}$$

where $*=\pm1$ according to the b-orientation of $\varDelta$, and the matrix entries of $R'(w\,|\,c)$ and $\bar{R}'(w\,|\,c)$ are respectively

$$R'(w\,|\,c)^{\gamma,\delta}_{\alpha,\beta}=\zeta^{c_1'(\gamma-\alpha)}R(p_1',p_0',-p_2')^{\gamma-c_0',\delta}_{\alpha,\beta-c_0'},$$

$$\bar{R}'(w\,|\,c)^{\alpha,\beta}_{\gamma,\delta}=\zeta^{c_1'(\gamma-\alpha)}\bar{R}(p_1',p_0',-p_2')^{\alpha,\beta+c_0'}_{\gamma+c_0',\delta}.\tag{11}$$

As for $\mathfrak{L}_N(\varDelta,b,w)$, we see from (6) that $\mathfrak{R}_N(\varDelta,b,w,c)$ only depends on (b,w,c), and not on the choice of the Nth roots p_i' of the p_i's.

Write $v=g(1)/|g(1)|$. Let S and T be the $N\times N$ invertible square matrices with matrix entries

$$T_{m,n}=v\zeta^{m^2/2}\delta(m+n),\quad S_{m,n}=N^{-1/2}\zeta^{mn}.$$

We have

$$S^4=id,\quad S^2=\zeta'(ST)^3$$

for some root of unity ζ'. Hence the matrices S and T define a projective N-dimensional representation Θ of $SL(2,Z)$. The following lemma describes the tetrahedral symmetries of $\mathfrak{R}_N$ in terms of Θ. Recall that the symmetry group on four elements numbered from 0 to 3 is generated by the transpositions (01), (12) and (23).

Lemma 3.3. *Let $(\varDelta,b,w,c)$ be a charged $\mathscr{I}$-tetrahedron with $*_b=+1$. If we change b via the transpositions* (01), (12) *and* (23) *of the vertices we have respectively*

$$\mathfrak{R}_N((01)(\varDelta,b,w,c))\equiv_N \pm T_1^{-1}\mathfrak{R}_N(\varDelta,b,w,c)T_1,$$

$$\mathfrak{R}_N((12)(\varDelta,b,w,c))\equiv_N \pm S_1^{-1}\mathfrak{R}_N(\varDelta,b,w,c)T_2,$$

$$\mathfrak{R}_N((23)(\varDelta,b,w,c))\equiv_N \pm S_2^{-1}\mathfrak{R}_N(\varDelta,b,w,c)S_2,$$

where $\equiv_N$ means equality up to multiplication by Nth roots of unity. Here we write $T_1 = T \otimes 1$, etc.

Proof. Use the relations $w_0w_1w_2=-1$ between the moduli and $c_0+c_1+c_2=1$ between the charges to rewrite the scalar factors in both sides of each equality in terms of the same variables. For instance, for the first equality we have on the left-hand side:

$$((w_0')^{-1})^{-c_2}((w_2')^{-1})^{c_0})^p = ((w_0')^{-c_1+1}((w_0'w_2')^{-c_0}))^p,$$

where $w_0' = -p_1'/p_2'$, $w_1' = -p_2'/p_0'$ and $w_2' = -p_0'/p_1'$. As $w_0'w_1'w_2' = -1$, up to a sign this is equal to $(w_0')^p$ times $((w_0')^{-c_1}((w_1')^{c_0}))^p$. This last scalar is exactly the one appearing on the right-hand side. Then the result follows from Proposition A.4 in Appendix A. We do the same for the other transpositions. $\square$

3.3. Complete pentagon relations

Let us say that an $\mathscr{I}$-triangulation $\mathscr{T} = (T,b,w)$ of (W,ρ) is *roughly charged* if every abstract tetrahedron (Δ_i,b_i,w_i) is equipped with an integral charge c_i. We say 'roughly' because in Section 4 it shall be necessary to specialize to integral charges satisfying global constraints. By replacing in (7) the non-symmetric quantum dilogarithms with the symmetrized ones, we obtain new state sums

$$\mathfrak{R}_N(\mathscr{T}_{\mathscr{I}},c) = \sum_\alpha \prod_i \mathfrak{R}_N(\Delta_i,b_i,w_i,c_i)_\alpha. \tag{12}$$

The next step is to introduce a suitable notion of *charged $\mathscr{I}$-transit*, such that $\mathfrak{R}_N(\mathscr{T}_{\mathscr{I}},c)$ is invariant for all instances of $2 \leftrightarrow 3$ charged $\mathscr{I}$-transit. As the integral charges do not depend on the moduli, also a charged $\mathscr{I}$-transit is obtained by completing a usual $\mathscr{I}$-transit with a moduli-independent *charge transit*. We use the notations of Definition 2.15.

Definition 3.4. We say that there is a *charge transit* $(T,c) \leftrightarrow (T',c')$ if c' equals c on the edges of the abstract tetrahedra of T not involved in the move, and for any other edge e we have the *transit of sum* condition:

$$\sum_{a\in\varepsilon_T^{-1}(e)} c(a) = \sum_{a'\in\varepsilon_{T'}^{-1}(e)} c'(a'). \tag{13}$$

Note that for positive $2 \to 3$ transits this relation implies that the sum of the charges around the new edge after the move is equal to 2.

Proposition 3.5. *For any charged $2 \leftrightarrow 3$ $\mathscr{I}$-transit $(T,b,w,c) \leftrightarrow (T',b',w',c')$ we have*

$$\prod_{\Delta_i\subset T} \mathfrak{R}_N(\Delta_i,b_i,w_i,c_i) \equiv_N \pm \prod_{\Delta_i'\subset T'} \mathfrak{R}_N(\Delta_i',b_i',w_i',c_i').$$

Proof. Denote by $f(\Delta,b,w,c)$ the scalar factor in front of the matrices R' and $\bar{R}'$ in (10). By Proposition A.6 in Appendix A we see that the statement is true if the $\mathscr{I}$-transit is the one shown in

Fig. 8, and if we remove $f(\Delta_j, b_j, w_j, c_j)$ from $\mathfrak{R}_N(\Delta_j, b_j, w_j, c_j)$, for each tetrahedron Δ_j involved in the move. We claim that we also have

$$\prod_{\Delta_j \subset T} f(\Delta_j, b_j, w_j, c_j) \equiv_N \pm \prod_{\Delta'_j \subset T'} f(\Delta_j, b_j, w_j, c_j). \tag{14}$$

Indeed, denote by c^i the integral charge of the tetrahedron opposite to the ith vertex (for the ordering of the vertices induced by the branching), and rewrite the moduli as in Fig. 8. Let log be the standard branch of the logarithm. Up to Nth roots of unity the left-hand side of (14) is

$$\exp\Big(\frac{p}{N}(-c_1^1 \log(y) + c_0^1 \log(1-y))\Big)$$
$$\times \exp\Big(\frac{p}{N}(-c_1^3 \log(y(1-x)/x(1-y)) + c_0^3 \log((y-x)/x(1-y))\Big)$$

and the right-hand side is

$$\exp\Big(\frac{p}{N}(-c_1^0 \log(x) + c_0^0 \log(1-x))\Big) \exp\Big(\frac{p}{N}(-c_1^2 \log(y/x) + c_0^2 \log(1-y/x))\Big)$$
$$\times \exp\Big(\frac{p}{N}(-c_1^4 \log((1-x)/(1-y)) + c_0^4 \log((x-y)/(1-y))\Big).$$

Consider the exponents in these formulas. An elementary computation using Relation (13) shows that they are equal up to $2i\pi/N$. For instance, the coefficient of $\log(y)$ in the left-hand side is $-c_1^1 - c_1^3 = -c_1^2$, whereas in the right-hand side it is $-c_1^2$. Things go similarly for the coefficients of $\log(1-y)$, etc. Hence the statement is true for the $\mathscr{I}$-transit shown in Fig. 8. We get the result for all the instances of $\mathscr{I}$-transit by using Lemma 3.3, together with the fact that the action of the matrices S and T cancel on a common face of two tetrahedra (details on this claim are given in the proof of Lemma 4.15). □

4. Link-fixing and QHI for triples (W, L, ρ)

We first refine the notion of charged $\mathscr{I}$-triangulation so as to make it *stable* for charged $\mathscr{I}$-transits. A naive idea would be to require that the sum of the charges around each edge of T is equal to 2. But simple combinatorial considerations show that such tentative global integral charges do not exist. A way to overcome this difficulty is to fix an arbitrary non-empty link L in W, considered up to ambient isotopy, and to incorporate this link-fixing in all the constructions. This eventually leads to the definition of the QHI for triples (W, L, ρ).

Definition 4.1. A *distinguished triangulation* of (W, L) is a pair (T, H) such that T is a triangulation of W and H is a *Hamiltonian* subcomplex of the 1-skeleton of T which realizes the link L (Hamiltonian means that H contains all the vertices of T).

Definition 4.2. A $\mathscr{D}$-*triangulation* $\mathscr{T} = (T, H, b, z)$ *for a triple* (W, L, ρ) consists of a $\mathscr{D}$-triangulation (T, b, z) for (W, ρ) such that (T, H) is a distinguished triangulation of (W, L).

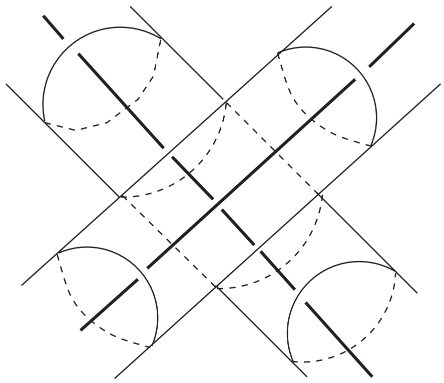

Fig. 9. A tunnel junction over a diagram crossing.

So a $\mathscr{D}$-triangulation for (W, L, ρ) is a distinguished and quasi-regular triangulation of (W, L), decorated by a branching b and a $PSL(2, \mathbb{C})$-valued simplicial 1-cocycle z. We postpone to Section 4.3 the proof of the existence of such $\mathscr{D}$-triangulations for any triple (W, L, ρ).

Example 4.3. *The tunnel construction*. Here is a simple construction of distinguished and quasi-regular triangulations of (S^3, L) derived from link diagrams.

Remove two ordered open 3-balls $B^3_\pm$ from S^3 away from the link L. We get a manifold homeomorphic to $S^2 \times [-1, 1]$ with the embedded 2-sphere $\Sigma = S^2 \times \{0\}$ as a simple spine, and two ordered spherical boundary components $\Sigma_\pm$. Consider a generic projection $\pi(L)$ of $L \subset S^2 \times [-1, 1]$ onto Σ such that every connected component of $\Sigma \setminus \pi(L)$, called a *diagram region*, is an open 2-disk (for instance, this is automatic if L is a knot). Then, as usual, encode L by a link diagram on Σ with support $\pi(L)$, by specifying the under/over crossings with respect to the direction normal to Σ and going from Σ_- towards Σ_+. Dig tunnels on Σ around $\pi(L)$, by respecting the under/over crossings, as in Fig. 9. Glue 2-disk walls inside the tunnels, one between each of the tunnel junctions, such that their boundaries span meridians. So there is one wall for each arc of the link diagram. In this way we get a standard spine P corresponding to a quasi-regular triangulation T of S^3. To obtain a distinguished triangulation (T, H) of (S^3, L) do as follows. There are two distinguished vertices $\pm v$ in T, at the interior of the balls $B^3_\pm$ we have initially removed. The edges of T which are dual to the walls realize L and contain all the vertices of T except $\pm v$. Select one wall D, and remove from L the interior of the edge dual to D. We get an arc with two vertices of T as endpoints. Connect one of these vertices to $+v$ and the other with $-v$, by means of the edges of T dual to the two opposite regions contained in the boundary of the tunnel around the removed edge (see the left side of Fig. 10). Finally connect $+v$ with $-v$ by another edge dual to an adjacent region of P contained in Σ. This construction gives a distinguished and quasi-regular triangulation of (S^3, L). Note that we can define a very particular branching b on T as follows. Fix an orientation of L. Then, the walls are positively oriented in accordance with the orientations of L and S^3, and the other regions of P are positively oriented with respect to the flow transverse to P and traversing $S^2 \times [-1, 1]$ from Σ_- towards Σ_+.

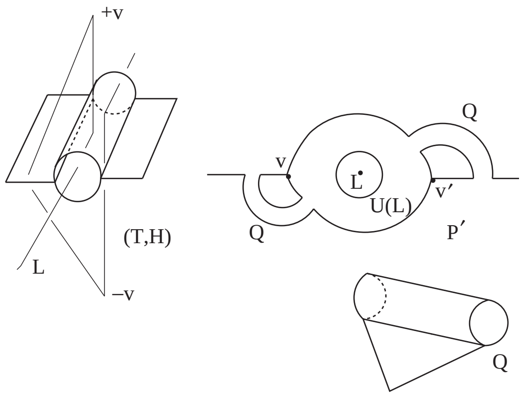

Fig. 10. Final steps of the constructions of (T,H) and T'.

Next, we show how to modify slightly the above construction in order to obtain an ideal triangulation for $Y = S^3 \setminus U(L)$, where $U(L)$ is an open tubular neighbourhood of L. Remove from P all the 2-disk walls. We get a standard spine P'' of $Y' = S^3 \setminus \{U(L) \cup B_+ \cup B_-\}$. Then modify P'' near the (removed) wall D as shown on the right side of Fig. 10. In fact we attach to P'' two copies of the 2-dimensional polyhedron Q, and then we remove 4 open 2-disks at their ends, on P''. The effect is to remove the interior of two 1-handles connecting $\partial U(L)$ with $\Sigma_\pm$, so that the so obtained P' is a standard spine of Y.

Note that for both P and P' there is the same pattern of 4 vertices at each diagram crossing (Fig. 9). It corresponds to an octahedron of T or T' made of 4 tetrahedra. In P, there are 2 more vertices for each wall (hence for each arc in the diagram). In P' there are just 2 further vertices (indicated as v and v' in Fig. 10). The non-tunnel regions of P which are contained in Σ exactly correspond to the original diagram regions. For both constructions, the diagram arc corresponding to the selected wall D plays a peculiar role. Also, the adjacent regions are modified by the respective final steps. One can obviously orient the regions of Q so that the branching b of P extends to a branching of P'.

We have to refine the notion of $\mathcal{D}$-transit in order to incorporate the fixed link L. Roughly speaking, a *$\mathcal{D}$-transit $(T,H,b,z) \to (T',H',b',z')$ of $\mathcal{D}$-triangulations for (W,L,ρ)* consists of a $\mathcal{D}$-transit $(T,b,z) \to (T',b',z')$ of $\mathcal{D}$-triangulations for (W,ρ) such that the two Hamiltonian subcomplexes H and H' which realize L coincide on the tetrahedra not involved by the underlying move. Precisely

(1) Any positive $0 \to 2$ or $2 \to 3$ move $T \to T'$ naturally specializes to a move $(T,H) \to (T',H')$; in fact $H' = H$ is still Hamiltonian. The inverse moves are defined in the same way. In particular, for negative $3 \to 2$ moves we require that the disappearing edge of T belongs to $T \setminus H$;

(2) For positive bubble moves, we assume that an edge e of H lies in the boundary of the involved face f; then e lies in the boundary of a unique 2-face f' of T' containing the new vertex of T'. We define the Hamiltonian subcomplex H' of T' just by replacing e with the other two edges of f'. The inverse move is defined in the same way.

Definition 4.4. The above moves make sense for (non-necessarily quasi-regular) distinguished triangulations of (W,L). We will refer to them as *distinguished moves*.

4.1. Integral charges on (T,H)

Let (T,H) be a distinguished triangulation of (W,L). Let us recall the notations already used in Lemma 2.12. We denote by $E(T)$ the set of edges of T, by $E_\Delta(T)$ the whole set of edges of the associated abstract tetrahedra $\{\Delta_i\}$, and by $\varepsilon_T : E_\Delta(T) \to E(T)$ the natural identification map.

Let s be a simple closed curve in W in general position with respect to T. We say that *s has no back-tracking* if it never departs a tetrahedron of T across the same 2-face by which it entered. Thus each time s passes through a tetrahedron, it selects the edge between the entering and departing faces.

Definition 4.5. An *integral charge* on a distinguished triangulation (T,H) of (W,L) is a map $c : E_\Delta(T) \to \mathbb{Z}$ such that the restriction of c to each abstract tetrahedron Δ of T is an integral charge (see Definition 3.1), and such that the following global properties are satisfied:

(1) for each $e \in E(T) \setminus E(H)$ we have $\sum_{e' \in \varepsilon^{-1}(e)} c(e') = 2$,
  for each $e \in E(H)$ we have $\sum_{e' \in \varepsilon^{-1}(e)} c(e') = 0$.
(2) Let s be any curve which has no back-tracking with respect to T. Each time s enters a tetrahedron of T the map c associates an integer to the selected edge. Let $c(s)$ be the sum of these integers. Then, for each s we have $c(s) \equiv 0 \bmod 2$.

We call $c(e)$ the *charge* of the edge e.

A map $c : E_\Delta(T) \to \mathbb{Z}$ inducing a charge on each tetrahedron of T and satisfying Definition 4.5 (1) defines an element $[c] \in H^1(W; \mathbb{Z}/2\mathbb{Z})$. The meaning of Definition 4.5 (2) is that we prescribe $[c] = 0$. Note that any integral charge c on (T,H) eventually encodes H, hence the link L.

Definition 4.6. A *charged $\mathcal{D}$-triangulation* for a triple (W,L,ρ) consists of a couple $(\mathcal{T},c)$ where $\mathcal{T} = (T,H,b,z)$ is $\mathcal{D}$-triangulation for (W,L,ρ), and c is an integral charge on (T,H).

Theorem 4.7. *For every distinguished triangulation (T,H) of (W,L) there exist integral charges. In particular, every $\mathcal{D}$-triangulation $\mathcal{T}$ of a triple (W,L,ρ) can be charged.*

This theorem is obtained by adapting, almost verbatim, Neumann's proof of the existence of combinatorial flattenings of ideal triangulations of compact 3-manifolds whose boundary is a union of tori (Theorem 2.4.(i) and Lemma 6.1 of [26]). In Neumann's situation there is no link but the manifold has a non-empty boundary; only the first condition of Definition 4.5 (1) is present, and there is a further condition in Definition 4.5 (2) about the behaviour of the charges on the boundary. In our situation, as W is a closed manifold, this further condition is essentially empty. The second condition in Definition 4.5 (1) together with the fact that H is Hamiltonian replace the role of the non-empty boundary in the combinatorial algebraic considerations that lead to the existence of combinatorial flattenings. All the details of this adaptation are contained in [1, Proposition 2.2.5].

Next we describe the structure of the set of integral charges on (T,H), which is an affine space over an integer lattice. Again, this is an adaptation to the present situation of a result of [26]. Let (T,H) be a distinguished triangulation of (W,L), and choose an abtract tetrahedron Δ of T. By definition, there are only two degrees of freedom in choosing the charges of the edges of Δ. Assume

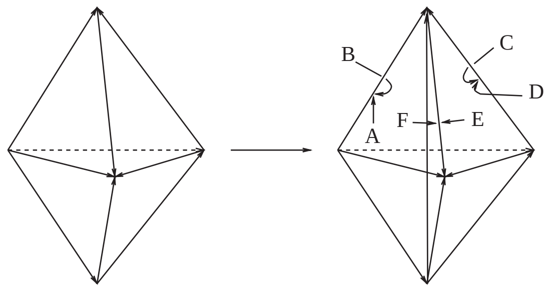

Fig. 11. $2 \to 3$ charge transits are generated by Neumann's vectors $w(e)$.

for simplicity that T is branched, and use the branching to order the edges of Δ as in (1), from e_0 to e_2'. Hence, given a branching on T there is a preferred ordered pair of charges $(c_1^{\Delta}, c_2^{\Delta}) = (c(e_0), c(e_1))$ for each abstract tetrahedron Δ.

Set $d_1^{\Delta} := c_1^{\Delta}$ and $d_2^{\Delta} = -c_2^{\Delta}$. Let r_0 and r_1 be, respectively, the number of vertices and edges of T. An easy computation with the Euler characteristic shows that there are exactly $r_1 - r_0$ tetrahedra in T. If we order the tetrahedra of T in a sequence $\{\Delta^i\}_{i=1,\dots,r_1-r_0}$, one can write down an integral charge on (T, H, b) as a vector $c = c(d) \in \mathbb{Z}^{2(r_1-r_0)}$, with

$$c = (d_1^{\Delta^1}, \dots, d_1^{\Delta^{r_1-r_0}}, d_2^{\Delta^1}, \dots, d_2^{\Delta^{r_1-r_0}})^{\mathrm{t}}.$$

Proposition 4.8 (Baseilhac [1, Corollary 2.2.7]). *The difference between any two integral charges c and c' on (T, H) is a linear combination with integer coefficients of determined vectors $d(e) \in \mathbb{Z}^{2(r_1-r_0)}$ associated to the edges $e \in E(T): c' = c + \sum_e \lambda_e d(e)$.*

The vectors $d(e)$ have the following form. For any abstract tetrahedron Δ^i glued along a specific edge e, define $r_1^{\Delta^i}(e)$ (resp. $r_2^{\Delta^i}(e)$) as the number of occurrences of $d_1^{\Delta^i}$ (resp. $d_2^{\Delta^i}$) in $\varepsilon^{-1}(e) \cap \Delta^i$. Then

$$d(e) = (r_2^{\Delta^1}, \dots, r_2^{\Delta, r_1-r_0}, -r_1^{\Delta^1}, \dots, -r_1^{\Delta^{r_1-r_0}})^t \in \mathbb{Z}^{2(r_1-r_0)}.$$

Example 4.9. Consider the situation depicted in the right of Fig. 11. Denote by Δ^j the tetrahedron opposite to the jth vertex. We have

$$r_1^{\Delta^0}(e) = -1, \quad r_1^{\Delta^2}(e) = 0, \qquad r_1^{\Delta^4}(e) = -1,$$
$$r_2^{\Delta^0}(e) = 1, \qquad r_2^{\Delta^2}(e) = -1, \quad r_2^{\Delta^4}(e) = 1,$$

where e is the central edge. Then $d(e) = (1, -1, 1, 1, 0, 1)^{\mathrm{t}}$.

4.1.1. Charge transit

Charge transits for roughly charged triangulations of (W, ρ) have been described in Definition 3.4. We have to prove that they specialize well to integral charges on (T, H).

Lemma 4.10. *Let $(T_1,H_1) \to (T_2,H_2)$ be any distinguished move between distinguished triangulations of (W,L). Assume that c_1 is an integral charge on (T_1,H_1), and that c_1 transits as a rough charge c_2 on (T_2,H_2). Then c_2 is actually an integral charge on (T_2,H_2).*

Definition 4.11. We have a *charged $\mathscr{D}$-transit* $(\mathscr{T}_1,c_1) \to (\mathscr{T}_2,c_2)$ between charged $\mathscr{D}$-triangulations of a triple (W,L,ρ) if $\mathscr{T}_1 \to \mathscr{T}_2$ is a $\mathscr{D}$-transit and $(T_1,H_1,c_1) \to (T_2,H_2,c_2)$ is a transit of integral charges as in Lemma 4.10.

Lemma 4.12. *Suppose that $(T_1,H_1) \to (T_2,H_2)$ is a $2 \to 3$ move between distinguished triangulations of (W,L). Fix integral charges c_1, c_2 on (T_1,H_1) and (T_2,H_2) respectively, and put*

$$C(e,c_2,T) = \{c_2' = c_2 + \lambda d(e), \lambda \in \mathbb{Z}\},$$

where e is the edge that appears and $w(e)$ is as in Example 4.9. *The integral charges c_2' obtained by varying the charge transit $(T_1,H_1,c_1) \to (T_2,H_2,c_2')$ exactly span $C(e,c_2,T)$.*

Proof of Lemma 4.10. First consider the $2 \to 3$ moves. It follows from Definition 3.4 that we can restrict our attention to $Star(e,T_2)$. Consider the situation of Fig. 11, and denote by Δ^i the tetrahedron opposite to the ith vertex. Let c^i be the integral charge on Δ^i and c^i_{jk} the value of c^i on the edge with vertices v_j and v_k. Relation (13) implies that the sum of the charges around each of the edges of $T_1 \cap T_2$ stays equal. Moreover it gives:

$$c^1_{02} + c^1_{24} + c^1_{40} = (c^4_{02} - c^3_{02}) + (c^0_{24} - c^3_{24}) + (c^2_{04} - c^3_{04}) = c^4_{13} + c^0_{13} + c^2_{13} - (c^3_{02} + c^3_{24} + c^3_{40}),$$

where in the second equality we use the fact that opposite edges of a tetrahedron share the same charge. Since $c^1_{02} + c^1_{24} + c^1_{40} = c^3_{02} + c^3_{24} + c^3_{40} = 1$ we have $c^4_{13} + c^0_{13} + c^2_{13} = 2$. Similar computations show that (13) forces c_2 to induce an integral charge on each abstract tetrahedron of T_2. As H_1 is not altered by a $2 \to 3$ move, we conclude that c_2 verifies Definition 4.5 (1).

Next consider the $0 \to 2$ moves. Any non-branched $0 \to 2$ move $(T_1,H_1) \to (T_2,H_2)$ is a composition of $2 \to 3$ and $3 \to 2$ moves [30]. In particular, the negative moves in this composition do not involve the edges of $E(T_1) \cap E(T_2)$. Also, the integral charges do not depend on branchings. Then our previous conclusion for $2 \to 3$ charge transits (which obviously holds for $3 \to 2$ ones) is still true for $0 \to 2$ charge transits. For such a transit $(T_1,H_1,c_1) \to (T_2,H_2,c_2)$, denote by Δ' and Δ'' the new tetrahedra. It is easy to verify that it is defined by $s_1 := c_2(\varepsilon^{-1}(e) \cap \Delta') + c_2(\varepsilon^{-1}(e) \cap \Delta'') = 0$ for each $e \in E(T_1) \cap E(T_2)$, by $s_2 := c_2(\varepsilon^{-1}(e_c) \cap \Delta') + c_2(\varepsilon^{-1}(e_c) \cap \Delta'') = 2$ on the new interior edge e_c, and by $s_3 := c_2(\varepsilon^{-1}(e') \cap \Delta') + c_2(\varepsilon^{-1}(e'') \cap \Delta'') = 2$ on the edges e' and e'' opposite to e_c in Δ' and Δ'', respectively.

Finally consider the bubble moves. Remark that a distinguished bubble move $(T_1,H_1) \to (T_2,H_2)$ is *abstractly* obtained from the final configuration of a $0 \to 2$ move by gluing two more faces. The resulting face contains the two new edges of H_2. Define a charge transit for a distinguished bubble move via the very same formulas as for a $0 \to 2$ move. This makes sense, because the sum of the charges is equal to $s_1 = 0$ along each of the two new edges of H_2, to $s_2 = 2$ along the other interior edge of $\Delta' \cap \Delta''$, and to $s_3 = 2$ along the former edge of H_1. Hence for bubble charge transits c_2 also satisfies Definition 4.5 (1).

Let us show that c_2 also verifies Definition 4.5 (2). As above it is enough to consider a $2 \to 3$ move $(T_1,H_1) \to (T_2,H_2)$. Denote by e the edge that appears. We have to prove that for any simple

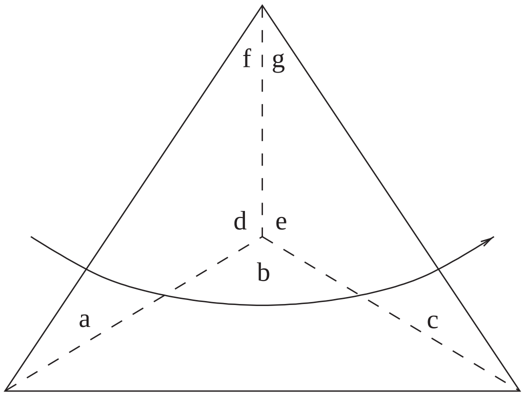

Fig. 12. Proof of 4.5 (2) for c_2.

closed curve s without back-tracking with respect to T_1 and T_2 we have $c_2(s) \equiv 0 \bmod(2)$. Fig. 12 shows an instance of s in a section of the three tetrahedra of T_2 glued along e. In this picture the charges $a, \dots, g$ are attached to the dihedral angles of the tetrahedra. Using the first two conditions in Definition 4.5 for c_2, we see that

$$-a + b - c = (d + f - 1) + (2 - d - e) + (e + g - 1) = f + g.$$

Then $c_2(s) = c_1(s) \equiv 0$. □

Proof of Lemma 4.12. Again consider Fig. 11. The symbols E, D, F, A, C, B denote the charges on the top edges of Δ^0, Δ^2 and Δ^4, respectively. The space of solutions of the linear system (13) of relations which define c_2 from c_1 is one dimensional. Hence there is a single degree of freedom in choosing these charges. Fix a particular choice for them, hence for c_2. If c_2' is defined by decreasing B by 1, we have

$$c_2'(d) - c_2(d) = (1, -1, 1, 1, 0, 1)^{\mathrm{t}} = d(e) \in \mathbb{Z}^{2(r_1 - r_0)}.$$

This shows that the integral charges on T_2 obtained by varying the charge transit may only differ by a $\mathbb{Z}$-multiple of $d(e)$. □

4.2. The QHI state sums

We are ready to state the main results of the present paper.

Theorem 4.13. *For every triple (W, L, ρ) there exist charged $\mathcal{D}$-triangulations $(\mathcal{T}, c)$.*

Fix a triple (W, L, ρ), and let $(\mathcal{T}, c) = ((T, H, b, z), c)$ be any charged $\mathcal{D}$-triangulation of it, with associated charged $\mathcal{I}$-triangulation $(\mathcal{T}_{\mathcal{I}}, c)$. Denote by n_0 the number of vertices of T. Recall the state sums defined in (12).

Theorem 4.14. *For every odd integer $N > 1$, the value of the (normalized) state sum $H_N(\mathscr{T}_{\mathscr{I}}, c) = N^{-n_0}\mathfrak{R}_N(\mathscr{T}_{\mathscr{I}}, c)$ does not depend on the choice of $(\mathscr{T}, c)$, up to sign and multiplication by Nth roots of unity. Hence, up to this ambiguity, it defines a* quantum hyperbolic invariant $H_N(W, L, \rho) \in \mathbb{C}$.

This shows that $K_N(W, L, \rho) = H_N(W, L, \rho)^{2N}$ is a well-defined complex valued invariant of (W, L, ρ). We can prove immediately the invariance of the QHI state sums $H_N(\mathscr{T}_{\mathscr{I}}, c)$ with respect to the choice of branching and the charged $\mathscr{I}$-transits. Recall that Lemma 2.11 describes how vary the moduli when we change the branching of an $\mathscr{I}$-triangulation.

Lemma 4.15. *Suppose that $(\mathscr{T}', c)$ is obtained from $(\mathscr{T}, c)$ be changing the branching. Then*

$$H_N(\mathscr{T}_{\mathscr{I}}, c) \equiv_N \pm H_N(\mathscr{T}'_{\mathscr{I}}, c).$$

Proof. Any change of branching on a fixed triangulation translates on each of its abstract tetrahedra Δ_i as a composition of transpositions of the vertices. By Lemma 3.3 such transpositions induce an equivariant projective action of $SL(2, \mathbb{Z})$ on the carrying spaces $I_1 \otimes I_2$ and $O_1 \otimes O_2$ of $\mathfrak{R}_N(\Delta_i, b_i, w_i, c_i)$, which are associated to pairs of faces of Δ_i. This action is defined via matrices $S^{\pm 1}$ and $T^{\pm 1}$. For each (branched) face, it depends on the b-orientation of Δ_i: the action is turned into its inverse if we change the agreement between the b-orientation of the face and the orientation induced as a boundary of Δ_i. We can see this by simply changing in the formulas of Lemma 3.3 the side where the above matrices act. Since a face is always given opposite boundary orientations by the two adjacent tetrahedra, a change of branching may only alter $\mathfrak{R}_N(\mathscr{T}_{\mathscr{I}}, c)$ by the projective factor, which is a sign or an Nth root of unity. □

Remark 4.16. Note that the branching is a necessary ingredient for defining the state sums. Moreover, the branching invariance results from global considerations, as the individual quantum dilogarithms have been only partially symmetrized. This makes a difference, for instance, with respect to the state sums used for the Turaev–Viro invariants.

Lemma 4.17. *Let $(\mathscr{T}, c) \to (\mathscr{T}', c')$ be any transit of charged $\mathscr{D}$-triangulations for (W, L, ρ). Then*

$$H_N(\mathscr{T}_{\mathscr{I}}, c) \equiv_N \pm H_N(\mathscr{T}'_{\mathscr{I}}, c').$$

Proof. We use the fact that the $\mathscr{D}$-transits dominate $\mathscr{I}$-transits (see Proposition 2.16). For $2 \leftrightarrow 3$ transits, the transit invariance of the QHI state sums has been already proved in Proposition 3.5. For the other transits it is obtained as follows.

Consider the abstract $2 \leftrightarrow 3$ $\mathscr{I}$-transit shown in Fig. 8. Denote by Δ^i the tetrahedron opposite to the ith vertex. Do a further $2 \to 3$ $\mathscr{I}$-transit on Δ^0 and Δ^2. A mirror image of Δ^4 appears, which together with Δ^4 forms the final configuration of a $0 \to 2$ $\mathscr{I}$-transit. Moreover, the other two new $\mathscr{I}$-tetrahedra have exactly the same decorations and gluings than Δ^1 and Δ^3. Hence Proposition 3.5 implies that, after a trivial simplification, such sequences of $\mathscr{I}$-transits (varying the branching and using Lemma 3.3) translate as the following *orthogonality relations* for the $0 \leftrightarrow 2$ $\mathscr{I}$-transits (above for Δ^4):

$$\mathfrak{R}_N(\Delta, b, w, c)\mathfrak{R}_N(\Delta, \bar{b}, w, \bar{c}) \equiv_N \pm id \otimes id.$$

Here $\bar{b}$ and $\bar{c}$ denote the branching and the integral charge mirror to b and c, as given by a $0 \to 2$ branched charged move (the explicit formulas for c are given in the proof of Lemma 4.10). The mirror moduli are the same. By taking the trace over one of the tensor factors in the orthogonality relations, we get the *normalization relations* corresponding to the bubble $\mathscr{I}$-transits. In these relations there is an N in factor; we compensate it by normalizing with N^{-n_0} in $H_N(\mathscr{T}_{\mathscr{I}}, c)$. □

The rest of this section shall be mainly devoted to the proof of Theorems 4.13 and 4.14.

4.3. Existence of $\mathscr{D}$-triangulations for (W, L, ρ)

We prove Theorem 4.13. As the existence of integral charges has been already settled, it remains to show the existence of $\mathscr{D}$-triangulations for any triple (W, L, ρ).

Recall Definitions 4.1 and 4.2. We prove at first the existence of distinguished triangulations for pairs (W, L). Let us describe these triangulations (T, H) in terms of dual spines. Let $M = W \setminus U(L)$, where $U(L)$ is an open tubular neighbourhood of L in W, and S be the union of $t_i \geqslant 1$ parallel copies on ∂M of the meridian m_i of the component L_i of L, $i = 1, \ldots, n$. Set $r = \sum_i t_i$.

Definition 4.18. We say that a spine Q of M is *quasi-standard and adapted* to L of *type* $t=(t_1, \ldots, t_n)$ if:

(i) Q is a simple polyhedron with boundary ∂Q consisting of r circles. These circles bound (unilaterally) r annular regions of Q. The other regions are cells.
(ii) $(Q, \partial Q)$ is properly embedded in $(M, \partial M)$ and transversely intersects ∂M at S (we also say that Q is relative to S).
(iii) Q is a spine of M.

Let Q be a spine of M adapted to L. Filling each boundary component of Q by a 2-disk we get a standard spine $P = P(Q)$ of $W_r = W \setminus rD^3$. The dual triangulation $T(P)$ of W contains L as a Hamiltonian subcomplex. Conversely, starting from any distinguished triangulation (T, H) and removing an open disk in each of the regions dual to an edge of H, we pass from $P = P(T)$ to a quasi-standard spine $Q = Q(P)$ of M adapted to L. So adapted spines and distinguished triangulations (T, H) are equivalent objects.

Lemma 4.19. *Quasi-standard spines of M adapted to L and of arbitrary type, hence distinguished triangulations of (W, L) with an arbitrary number of vertices, do exist.*

Proof. Let $\tilde{P}$ be any standard spine of M. Consider a *normal* retraction $h: M \to \tilde{P}$. Recall that M is the mapping cylinder of h. For each region R of $\tilde{P}$, $h^{-1}(R) = R \times I$; for each edge e, $h^{-1}(e) = e \times \{$a "tripode"$\}$; for each vertex v, $h^{-1}(v) = \{$a "quadripode"$\}$. We can assume that S is in general position with respect to h, so that the mapping cylinder of the restriction of h to S is a simple spine of M relative to S. Possibly after doing some $0 \to 2$ moves, far from the boundary curves, we obtain a quasi-standard spine Q adapted to L. □

We get the stronger existence result we need with the help of more distinguished moves (see Definition 4.4).

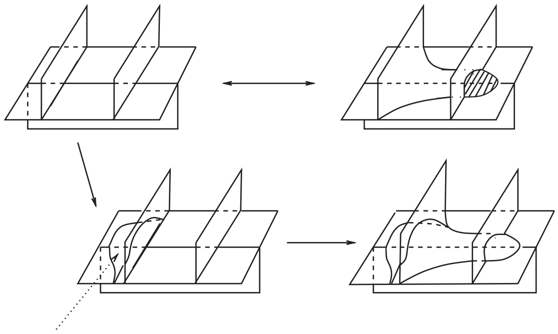

Fig. 13. How to turn a non quasi-regular move into a quasi-regular one by capping off the sector of immersion of the corresponding 3-cell.

Proposition 4.20. *For any pair (W,L) there exist distinguished and quasi-regular triangulations.*

Proof. Let (T,H) be any distinguished triangulation of (W,L). It is not quasi-regular if some edge e of T is a loop, i.e. if the ends of e are identified. In the cellulation $D(T)$ of W dual to T, this means that the spine $P=P(T)$ contains some region $R=R(e)$ which has the same 3-cell C on both sides: the boundary of C is a sphere S immersed at R. Let us say that R is *bad*. We construct a distinguished and quasi-regular triangulation (T',H') by doing some distinguished bubble moves on (T,H) (thus adding new 3-cells to $D(T)$). Then we slide portions of their "capping" disks until they cover the bad regions, thus desingularizing all the boundary 2-spheres.

Let us formalize this argument. Any (dual) bubble move $P\to P'$ is obtained by gluing a closed 2-disk D^2 along its boundary ∂D^2, with two transverse intersection points of ∂D^2 with an edge e of P (see the second move in Fig. 2). Denote by A and B, $A\cup B=\partial D^2$, the two arcs thus defined. The bubble move is distinguished if at least one of A or B lies on a region R_H of P dual to an edge of H. The two new regions of P' dual to edges of H' are D^2 and the region bounded by ∂D^2 and adjacent to R_H. We call D^2 the *capping disk* of the bubble move. Note that a bubble move does not increase the number of bad regions, and that any $2\leftrightarrow 3$ move done by sliding a portion of the capping disk also has this property as long as ∂D^2 is embedded.

Let now $R\in S$ be a bad region (dashed in the top right of Fig. 13), where S is a singular sphere as above. Using distinguished bubble moves we may always assume that each connected component of H has at least two vertices. Since (T,H) is distinguished, there are exactly two regions R_H and R'_H in the cellular decomposition of S which are dual to edges of H. As above, do a bubble move that involves R_H (for instance), and slide a portion of its capping disk D^2 via $2\leftrightarrow 3$ moves along the 1-skeleton of S, until it reaches a vertex of R. This is obviously always possible. The only thing is to keep track of the region initially bounded by ∂D^2 and adjacent to R_H; we cannot remove it, for it is dual to an edge of H. Also, if ∂D^2 was no longer embedded after this sequence of moves, we could find a shorter sequence leading to the same vertex of R. So at each step we still have (dual) distinguished triangulations with no more bad regions. Next expand D^2 over R by doing further $2\leftrightarrow 3$ moves along the edges of ∂R, possibly arranged so that they give $0\leftrightarrow 2$ moves. If R is embedded in S, we can choose such a sequence of moves so that D^2 is embedded at each

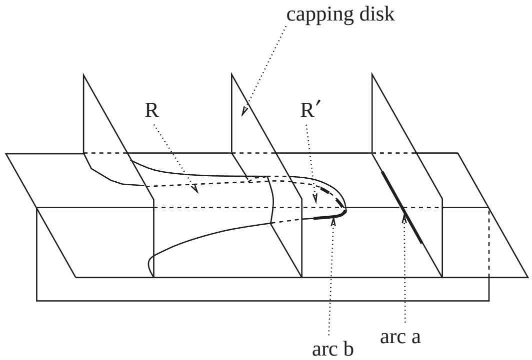

Fig. 14. Capping disks are no obstructions for moves.

step and finally covers R completely (see the bottom right of Fig. 13). Both R and D^2 are in the boundary of the 3-cell introduced by the bubble move. Thus we eventually finish with a spine dual to a distinguished triangulation and having one less bad region than P.

If R is immersed on its boundary (e.g. if it looks like an annulus with one edge that joins the boundary circles), note that it is contained inside a disk embedded in S, and as above we may still find a sequence of $2 \leftrightarrow 3$ moves ending with a spine dual to a distinguished triangulation and having one less bad region than P. Iterating this procedure, we get the conclusion. □

By using Lemma 2.10 and, for instance, a total ordering branching, we can complete any distinguished and quasi-regular triangulation (T,H) of (W,L) to a $\mathscr{D}$-triangulation for (W,L,ρ). So we have achieved the proof of Theorem 4.13.

4.4. Invariance of the QHI state sums

As bundle preserving oriented homeomorphisms of triples (W,L,ρ) transfer charged $\mathscr{D}$-triangulations, we can fix a model of W and a flat $PSL(2,\mathbb{C})$-bundle ρ on W, with the link $L \subset W$ considered up to ambient isotopy.

We need to show that the set of distinguished and quasi-regular triangulations of (W,L) is "connected". In a sense this is the main technical point. As for the existence of $\mathscr{D}$-triangulations, let us prove at first a weaker result for distinguished triangulations. Let (T,H) and (T',H') be distinguished triangulations of (W,L) such that the associated quasi-standard spines Q, Q' of M adapted to L are relative to S and S' and are of the same type t. Up to isotopy, we can assume that $S=S'$ and that the "germs" of Q and Q' at S coincide. By using Theorem 6.4.B of [32] we have the following relative version of Lemma 2.1 for adapted spines (this follows also from the argument depicted in Fig. 14 and used in Proposition 4.23):

Lemma 4.21. *Let P and P' be quasi-standard spines of M adapted to L and relative to S. There exists a spine P'' of M adapted to L and relative to S, such that P'' can be obtained from both P and P' via finite sequences of positive $0 \to 2$ and $2 \to 3$ moves, where at each step the spines are adapted to L and relative to S.*

By possibly using distinguished bubble moves, we deduce from Lemma 4.21 and the correspondence between adapted spines and distinguished triangulations that:

Lemma 4.22. *Given any two distinguished triangulations (T,H) and (T',H') of (W,L) there exists a distinguished triangulation (T'',H'') which may be obtained from both (T,H) and (T',H') via finite sequences of positive bubble, $0 \to 2$ and $2 \to 3$ distinguished moves, where at each step the triangulations of (W,L) are distinguished.*

Finally we have:

Proposition 4.23. *Any two distinguished and quasi-regular triangulations (T,H) and (T',H') of (W,L) can be connected by means of a finite sequence of distinguished and quasi-regular $2 \to 3$ moves, bubble moves and their inverses, where at each step the triangulations of (W,L) are distinguished and quasi-regular.*

Proof. We use the same terminology as in Proposition 4.20. Let $s:(T,H) \to \cdots \to (T',H')$ be a sequence of moves as in Lemma 4.22. We may assume, up to further sudivisions of s, that there are no $0 \to 2$ moves. We divide the proof in two steps. We first prove that there exists a sequence $s': P = P(T) \to \cdots \to P''$ with only quasi-regular moves and such that the spine P'' is obtained from $P' = P(T')$ by gluing some 2-disks $\{D_i^2\}$ along their boundaries. Then we show that we may construct P'' from P' just by using distinguished bubble moves and quasi-regular moves. By combining both sequences we will get the conclusion.

Bubble moves are always quasi-regular. Consider the first non quasi-regular move m in s. It produces a bad region R; see the top of Fig. 13, where we indicate R by dashed lines and we underline the sliding arc a. Alternatively, a step before m we may do a distinguished bubble move and slide a portion of its capping disk D^2 as in Proposition 4.20, until it covers a. Next, make the arc a sliding as in s; see the bottom of Fig. 13. These two moves are quasi-regular and their dual triangulations are distinguished. Starting with the moves of s and turning m into this sequence, we define the first part of s'. We wish to complete it with the following moves of s, applying the same procedure each time we meet a non quasi-regular move. But suppose that one of these moves would have affected a, and let b be the sliding arc responsible for it. Then in s' we just have first to slide b "under" D^2, pushing it up. We can do so because all the moves are purely local. This puts b in the same position w.r.t. a than it has in s; see Fig. 14. With this rule there are no obstruction to complete the desired sequence s'. The images in $T'' = T(P'')$ of all the capping disks form the set $\{D_i^2\}$. Remark that there are as many D_i^2's as there were distinguished bubble moves used to construct the sequence s'; in other words, the capping disks stay connected all along s'. This is due to the fact that in situations such as depicted in Fig. 14, once the region R has bumped into the capping disk the rest of the move is done as in s, by sliding the region R'.

Let us now turn to the second claim. In the dual cellulation $D(T')$ of W consider the boundary spheres S_j obtained by removing the disks D_i^2 one after the other. Fix one of them, S, and reversing this procedure let $D^2 \in \{D_i^2\}$ (considered with its gluings) be the first disk glued on it. By the above remark, we can do a distinguished bubble move on S and let a portion of its capping disk slide isotopically via $2 \leftrightarrow 3$ moves along the 1-skeleton of S, so that it finally reaches the position of D^2 in P''. We may repeat this argument inductively on the D_i^2's. Since all these moves are quasi-regular, this proves our claim. □

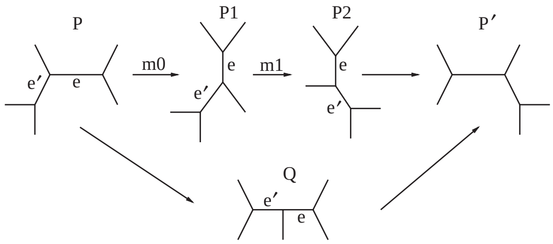

Fig. 15. The 2-dimensional analogue of Proposition 4.23.

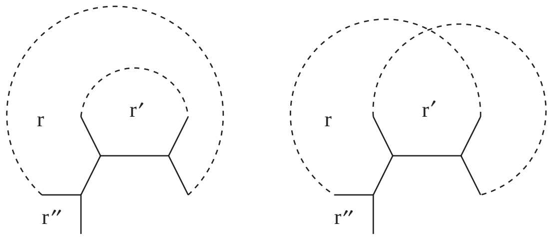

Fig. 16. The proof that $r_P(m_1)$ is quasi-regular.

We will also use a 2-dimensional analogue of the previous proposition. The main general facts about triangulations and spines of surfaces have been recalled at the end of Section 2.1.

Lemma 4.24. *Any two quasi-regular triangulations T and T' of a compact closed surface S can be connected by a finite sequence of quasi-regular $2 \to 2$ or $1 \to 3$ moves and their inverses.*

Proof. The proof is similar to the one of Proposition 4.23. In fact, it is simpler as it uses an argument of commutation of moves which is peculiar to the 2-dimensional situation. Consider any sequence s of moves m_i connecting T and T'. View it a sequence

$$s: \cdots \to P \overset{m_0}{\to} P_1 \overset{m_1}{\to} P_2 \overset{m_2}{\to} \cdots$$

between the (1-dimensional) dual spines. On a 1-dimensional standard spine dual to a quasi-regular triangulation of S, a move which is not quasi-regular is the flip of an edge that makes it the frontier of a same region. Let m_0 be the first non quasi-regular flip in s, and denote by e the corresponding edge. A step before m_0 let us first apply the "relative" $r_P(m_1)$ of m_1 on P, where by "relative" we mean the flip of the same edge e'; we get Q. Then apply $r_Q(m_0)$; see the bottom sequence of Fig. 15. (Beware that in this figure, the notations for e and e' are interchanged when following the upper or the lower sequence of flips; this is why we introduce the notion of "relative".) Note that $r_Q(m_0)$ is necessarily quasi-regular, for otherwise m_0 would not be the first non quasi-regular flip in s, since the horizontal edge below e' in the top left picture of Fig. 15 would have the same region on both sides. We claim that $r_P(m_1)$ is also quasi-regular. Indeed, in P we necessarily have one of the two situations of Fig. 16, where the dotted arcs represent boundary edges. In the first situation, $r' = r''$ is impossible. In the second one, if $r' = r''$ then $r' = r$ and m_0 is not the first non quasi-regular flip in s, thus giving a contradiction. Hence the sequence $r_Q(m_0) \circ r_P(m_1)$ is quasi-regular. Moreover we have:

$$P' = r_{P_2}(m_0) \circ m_1 \circ m_0(P) = r_Q(m_0) \circ r_P(m_1)(P).$$

This implies that we can modify s locally so as to obtain

$$s' : \cdots \to P \overset{r_P(m_1)}{\to} Q \overset{r_Q(m_0)}{\to} P' \overset{r_{P'}(m_0)}{\to} P_2 \overset{m_2}{\to} \cdots,$$

where the first possible non quasi-regular move is $r_{P'}(m_0)$. The length of s' after $r_{P'}(m_0)$ is less than the length of s after m_0. Then, working by induction on the length, replacing each non quasi-regular flip as above and noting that $1 \to 3$ moves are always quasi-regular, we get a quasi-regular sequence s'. □

4.4.1. Full invariance of the QHI state sums

Let (T,H) and (T',H') be two arbitrary distinguished and quasi-regular triangulations of (W,L). Let $(T,H) \to \cdots \to (T',H')$ be a finite sequence of distinguished and quasi-regular moves which connects (T,H) to (T',H'), as in Proposition 4.23. Any total ordering branching b on T (see Section 2.3) transits through total ordering branchings to a branching b' on T'. By Lemma 4.10, any integral charge c on (T,H) transits to an integral charge c' on (T',H'). Applying Lemma 2.14, we know that for generic 1-cocycles z on (T,b) these transits can be completed to a sequence of charged $\mathcal{D}$-transits which connects the charged $\mathcal{D}$-triangulation $(\mathcal{T},c) = (T,H,b,z,c)$ to another $(\mathcal{T}',c') = (T',H',b',z',c')$. So, by using the transit invariance of Proposition 4.17, we have proved:

Lemma 4.25. *For any triple (W,L,ρ) and every odd integer $N > 1$, given two arbitrary distinguished and quasi-regular triangulations (T,H) and (T',H') of (W,L), there exist charged $\mathcal{D}$-triangulations $(\mathcal{T},c)$ and $(\mathcal{T}',c')$ for (W,L,ρ), supported by (T,H) and (T',H'), respectively, such that*

$$H_N(\mathcal{T}_{\mathcal{I}},c) \equiv_N \pm H_N(\mathcal{T}'_{\mathcal{I}},c').$$

This statement can be complemented as follows.

Lemma 4.26. *Assume that $(\mathcal{T},c)$ and $(\mathcal{T}',c')$ are charged $\mathcal{D}$-triangulations for (W,L,ρ) which are connected by a finite sequence of $\mathcal{D}$-transits, with the possible exception of some bad cocycle transits for which the idealizability condition is lost. Nevertheless we have*

$$H_N(\mathcal{T}_{\mathcal{I}},c) \equiv_N \pm H_N(\mathcal{T}'_{\mathcal{I}},c').$$

Proof. Thanks again to Lemma 2.14, we can replace z and z' with *arbitrarily close* 1-cocycles z_1 and z_2, respectively, such that the corresponding new charged $\mathcal{D}$-triangulations $(\mathcal{T}'',c)$ and $(\mathcal{T}''',c')$ for (W,L,ρ) are actually connected by charged $\mathcal{D}$-transits. Then $H_N(\mathcal{T}''_{\mathcal{I}},c) \equiv_N \pm H_N(\mathcal{T}'''_{\mathcal{I}},c')$. Since z_1 and z_2 are arbitrarily close to z and z', and H_N is continuous as a function of idealizable 1-cocycles, we get the required conclusion. □

In the rest of this section we will tacitely use this genericity/continuity argument, so that we can always assume that the idealizability condition is never lost. So, in order to complete the proof of Theorem 4.14, it is enough to show the following proposition.

Proposition 4.27. *For any triple* (W, L, ρ) *and every odd integer* $N > 1$, *given two charged* $\mathscr{D}$*-triangulations* $(\mathscr{T}, c)$ *and* $(\mathscr{T}', c')$ *of* (W, L, ρ) *which only differ by the respective decorations of a same distinguished and quasi-regular triangulation* (T, H) *of* (W, L), *we have*

$$H_N(\mathscr{T}_{\mathscr{I}}, c) \equiv_N \pm H_N(\mathscr{T}'_{\mathscr{I}}, c').$$

Proof. The invariance with respect to the choice of branching has been already obtained in Lemma 4.15. So, from now on, we will use only total ordering branchings as they do not pose any problems of transit.

Consider the charge invariance. Let us localize the problem. Fix a triple (W, L, ρ), a $\mathscr{D}$-triangulation $(\mathscr{T}, c) = (T, H, b, z, c)$ of (W, L, ρ), and an arbitrary edge e of T. Consider the set of integral charges which differ from c only on $Star(e, T)$. It is of the form (we use the notations of Proposition 4.8)

$$C(e, c, T) = \{c' = c + \lambda w(e),\ \lambda \in \mathbb{Z}\}.$$

Thanks to Proposition 4.8, it is enough to prove that $H_N(\mathscr{T}_{\mathscr{I}}, c) \equiv_N H_N(\mathscr{T}'_{\mathscr{I}}, c')$ when c' varies in $C(e, c, T)$. Assume that $e \in T \setminus H$; the charge invariance is a consequence of the following facts:

(1) Let $(\mathscr{T}, c) \to (\mathscr{T}'', c'')$ be any $2 \to 3$ charged $\mathscr{D}$-transit such that e is a common edge of T and T''. Then the result holds for $C(e, c, T)$ if and only if it holds for $C(e, c'', T'')$.
(2) There exists a sequence of distinguished quasi-regular $2 \to 3$ moves which connects (T, H) to (T'', H''), such that e persists at each step and $Star(e, T'')$ is like the final configuration of a $2 \to 3$ move, with e playing the role of the central common edge of the 3 tetrahedra.
(3) If $Star(e, T)$ is like $Star(e, T'')$ in (2), then the result holds for $C(e, c, T)$.

By Lemmas 4.10 and 4.12 we know that $C(e, c, T)$ transits to $C(e, c'', T'')$. As the value of the QHI state sums is not altered by charged $\mathscr{D}$-transits (Lemma 4.17), the fact (1) follows.

To prove (2) it is perhaps easier to think, for a while, in dual terms. Consider the dual region $R = R(e)$ in $P = P(T)$. The final configuration of e in T'' corresponds dually to the case when R is an embedded *triangle*. More generally, there is a natural notion of *geometric multiplicity* $m(R, a)$ of R at each edge a of P, and $m(R, a) \in \{0, 1, 2, 3\}$. We say that R is *embedded* in P if for each a, $m(R, a) \in \{0, 1\}$. Call *proper* an edge with two distinct vertices. If R has a loop in its boundary, a suitable $2 \to 3$ move at a proper edge of $P(T)$ having a common vertex with the loop puts proper edges in place of the loop. Each time R has a proper edge a with $m(R, a) \in \{2, 3\}$, the (non-branched) $2 \to 3$ move along a puts new edges a' with $m(R, a') \leqslant 2$ in place of a. In the situation where this is an equality, remark that if we first blow up an edge b adjacent to a and such that $m(R, b) = 2$, and then we apply the $2 \to 3$ move along a, we get $m(R, a') = 1$ (look at Fig. 17). By induction, up to $2 \to 3$ moves, we can assume that R is an embedded polygon. To obtain the final configuration of e in T'' let us come back to the dual situation. We possibly have more than 3 tetrahedra around e. It is not hard to reduce the number to 3, via some further $2 \to 3$ moves. In the above construction we could accidently do some non quasi-regular moves, which we would like to avoid. For this, do appropriate distinguished bubble moves and slide portions of their capping disks as in Proposition 4.23. This is always possible because these moves may not increase the geometric multiplicity of the edges of the region R under consideration. In this way we eventually find sequences of distinguished and quasi-regular moves which transform R into an embedded triangle.

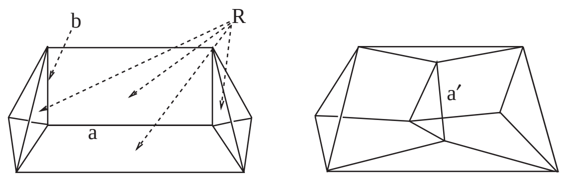

Fig. 17. Evolution of the geometric multiplicity of R when blowing-up a.

Concerning fact (3), do first a $3 \to 2$ $\mathscr{D}$-transit on e and then a $2 \to 3$ $\mathscr{D}$-transit, varying the charge transit $(T,c) \to (T',c'')$. By Lemma 4.12 we know that the charges c'' exactly describe $C(e,c',T')$. Since the value of the state sums is not altered by $\mathscr{D}$-transits, this concludes (note that, as we are using total ordering branchings, there is no problem of transit with the negative moves).

Suppose now that $e \in H$. The analogue of (1) for distinguished bubble moves is true for the same reasons. Then, applying a distinguished bubble move on a face of T containing e we are brought back to the previous situation. The charge invariance is thus proved.

Consider the cocycle invariance. Let $(\mathscr{T},c)$ and $(\mathscr{T}',c)$ be two charged $\mathscr{D}$-triangulations of (W,L,ρ) which only differ by the 1-cocycles z and z' representing ρ. We have to prove that $H_N(\mathscr{T}_{\mathscr{I}},c) \equiv_N \pm H_N(\mathscr{T}'_{\mathscr{I}},c)$. The two cocycles z and z' differ by a coboundary $\delta\lambda$, and it is enough to consider the elementary case when the 0-cochain λ is supported by one vertex v_0 of T. Again we have localized the problem. The invariance of the value of the state sums for bubble $\mathscr{D}$-transits gives us the result in the special situation when v_0 is the new vertex after the move. Let us reduce the general case to this special one by means of $\mathscr{D}$-transits. We use the notations and the facts stated at the end of Section 2.1. It is enough to show that we can modify $Star(v_0,T)$ to reach the star-configuration of the special situation. Recall that $Star(v_0,T)$ is the cone over $S = Link(v_0,T)$, which is homeomorphic to S^2. So $Star(v_0,T)$ is determined by the triangulation T_{v_0} of S. By Lemma 4.24 we know that T_{v_0} is connected to the triangulation of S corresponding to the special situation by a sequence of quasi-regular $2 \to 2$ or $3 \to 1$ moves. These can be obtained as the trace of quasi-regular $2 \leftrightarrow 3$ moves, by applying inductively the last remark in Section 2.1. Hence also the cocycle invariance is proved. □

4.5. Complements on the QHI

4.5.1. State sums over non quasi-regular triangulations

Let $\mathscr{T} = (T,H,b,z)$ be any branched distinguished (not necessarily quasi-regular) triangulation of a pair (W,L), with an idealizable cocycle z representing a bundle ρ. As T is not necessarily quasi-regular, the existence of such a z depends on ρ. For instance, if $\rho = \rho_0$ is the trivial flat bundle, it implies that T is quasi-regular. We know that $\mathscr{T}$ can be charged, by c say, so the state sum $H_N(\mathscr{T}_{\mathscr{I}},c)$ is still defined. We claim that in fact

$$H_N(W,L,\rho) \equiv_N \pm H_N(\mathscr{T}_{\mathscr{I}},c).$$

Indeed, the proof of Proposition 4.20 shows that any distinguished triangulation of (W,L) can be made quasi-regular just by using suitable distinguished bubble moves together with $2 \leftrightarrow 3$ moves done by sliding portions of their capping disks. We can complete such a sequence of moves with arbitrary charge and branching transits; the branchings may not be induced by total orderings on the vertices, so we use the general Definition 2.4. Since $H_N(\mathcal{T}_{\mathcal{I}}, c)$ is invariant for $\mathcal{I}$-transits (Lemma 4.17), we are left to prove that we can complete the above transits with idealizable cocycle transits starting from z. For that, remark that the cocycle transits are generically idealizable for bubble moves. Moreover, there is only a finite set of cocycle values on the capping disks that lead to non-idealizable cocycle transits for the moves to follow.

Combining this argument with those used in the proof of Propositions 4.23 (essentially Fig. 14) and 4.27, with some work we get the following proposition. Although we do not need it for proving Theorem 4.17, it shows that we can bypass the genericity argument of Lemma 4.26. It is necessary for proving the existence of scissors congruence classes for (W, L, ρ) in [3]. Note that it holds in greater generality, replacing $PSL(2,\mathbb{C})$ with any algebraic group G, and the idealizability condition by demanding that the cocycles take their values outside of some proper algebraic subvarieties of G.

Proposition 4.28. *Any two $\mathcal{D}$-triangulations of a same triple (W, L, ρ) may be connected by a sequence of $\mathcal{D}$-transits.*

The fact that $H_N(W, L, \rho) \equiv_N \pm H_N(\mathcal{T}_{\mathcal{I}}, c)$ for decorated triangulations which are not quasi-regular but support idealizable 1-cocycles representing ρ is of practical interest. Indeed, explicit computations are easier with non quasi-regular triangulations, since they contain in general much lesser tetrahedra than quasi-regular ones.

4.5.2. Duality

There are two natural involutions on the arguments of a triple (W, L, ρ): the first consists in changing the orientation of W, and the second is defined by passing from ρ to the complex conjugate bundle. The QHI *duality* property relates these involutions. Let $(\mathcal{T}, c)$ be a charged $\mathcal{D}$-triangulation for (W, L, ρ). Denote by z^* the complex conjugate of the 1-cocycle z of $\mathcal{T}$, and by $(\mathcal{T}^*, c)$ the corresponding charged $\mathcal{D}$-triangulation for (W, L, ρ^*), where $\rho^* = [z^*]$. We write $-W$ for the manifold W with the opposite orientation. Recall the notation $\equiv_N$ from Lemma 3.3.

Proposition 4.29. *We have $(H_N(W, L, \rho))^* \equiv_N \pm H_N(-W, L, \rho^*)$.*

Proof. If we change the orientation of W, the b-orientation of each tetrahedron Δ_i turns into the opposite, so that the pairs of faces associated to the carrying spaces $I_1 \otimes I_2$ and $O_1 \otimes O_2$ of $\mathfrak{R}_N(\Delta_i, b_i, w_i, c_i)$ are exchanged. Hence $\mathfrak{R}_N(\Delta_i, b_i, w_i, c_i)$ becomes ${}^{\mathrm{T}}\mathfrak{R}_N(\Delta_i, \bar{b}_i, w_i, c_i)$, where T is the transposition of matrices and $\bar{b}_i$ is the branching b_i for the opposite ambient orientation. But Proposition A.5 in Appendix A implies that

$${}^{\mathrm{T}}\mathfrak{R}_N(\Delta_i, \bar{b}_i, w_i, c_i)_\alpha = (\mathfrak{R}_N(\Delta_i, b_i, w_i^*, c_i)_{-\alpha})^*.$$

Here α is a state, as defined in Section 3. Since $H_N(\mathcal{T}_{\mathcal{I}}, c)$ does not depend on the states, this yields the conclusion. □

4.5.3. Some natural triples (W, L, ρ)

(1) Let M be a cusped complete hyperbolic 3-manifold of finite volume. We know from Thurston's hyperbolic Dehn fillings theorem (see e.g. [5, Chapter E]) that there exist sequences (W_n, L_n, ρ_n) of compact hyperbolic Dehn fillings of M converging geometrically to M. Here, L_n denotes the link made of the simple short geodesics in W_n forming the *cores* of the fillings, and ρ_n is the holonomy of the hyperbolic manifold W_n. These are our favorite examples of triples (W, L, ρ).

(2) Consider a compact oriented 3-manifold M with non empty boundary made of tori. Fix $\alpha \in H^1(M; \mathbb{C})$ and consider the associated flat bundle ρ_α on M, as in Section 2.2. The class α, hence the holonomy of ρ_α, may be non-trivial on the boundary of M. Here is an elementary procedure reminescent of the hyperbolic Dehn surgery, which allows us to define triples (W, L, ρ) from these pairs (M, α). To simplify the notations, let us assume that $Z = \partial M$ consists of one torus. It is well-known that the kernel of the map

$$i_* : H_1(Z; \mathbb{Q}) \to H_1(M; \mathbb{Q})$$

is a Lagrangian subspace $\mathscr{L}$ of $H_1(Z; \mathbb{Q})$ w.r.t. the intersection form. Then there exists a basis (m, l) of $\pi_1(Z) \cong H_1(Z; \mathbb{Z})$ such that $\mathscr{L}$ is generated by the homology class of $pm + ql$, where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$. Let us denote by W the closed manifold obtained from M by the Dehn filling of Z with coefficient (p, q) w.r.t. the basis (m, l). The bundle ρ_α extends to the whole of W. If L denotes the core of the filling, then (W, L, ρ_α) is a triple canonically associated to (M, ρ_α).

For example, if L is a knot in S^3 there are two families of QHI that give natural topological invariants of the knot. The first one is $K_N(S^3, L, \rho_0) = H_N^{2N}(S^3, L, \rho_0)$, where ρ_0 is the trivial flat bundle on S^3. The second one is obtained by applying the above procedure to $M = S^3 \setminus U(L)$ and a generator α of $H^1(M; \mathbb{Z}) \cong \mathbb{Z}$, where $U(L)$ is an open tubular neighbourhood of L. Similar considerations apply to links in S^3, or more generally in $\mathbb{Z}$-homology spheres.

(3) Finally, note that we can specialize the choice of the link. For example, we may take L as the trivial knot embedded in an open ball of W. In this way we formally obtain QHI for pairs (W, ρ).

Here are some further remarks.

Remark 4.30. *About the QHI phase factor.* We have prudently defined $H_N(W, L, \rho)$ only up to sign and multiplication by Nth roots of unity, which depend on the branching and the charge of the $\mathscr{I}$-triangulations used to compute it. This is due to Lemma 3.3 and Proposition A.4 in Appendix A. It is natural to ask whether this phase ambiguity is in fact not present, due to some systematic global compensations between the roots of unity coming from each tetrahedron, for a given change of branching on an $\mathscr{I}$-triangulation.

Alternatively, it is known that branchings and suitably restricted sets of branching transits can be used to encode several extra-structures on 3-manifolds, such as combings, framings, spin and Euler structures [7,8]. So we wonder about the existence of a suitable extra-structure on the pair (W, L) which, in our setup, would reflect itself in the branchings, and could serve to dominate the phase ambiguity. The models we have in mind are the Euler structures on W for which L is a pseudo-Legendrian link. As Turaev discovered, the Euler structures dominate the ambiguity, due to the action of the fundamental group on the universal covering, in the definition of Reidemeister torsions (see [31] and also [9]).

Remark 4.31. *On the B-QHI.* We already considered in [2] the QHI restricted to B-characters. In that paper we used state sum formulas differing from those in Theorem 4.14 by a scalar factor depending on the cocycle z of the $\mathscr{D}$-triangulation $\mathscr{T}$ (not only on the associated $\mathscr{I}$-triangulation $\mathscr{T}_{\mathscr{I}}$). This was a consequence of a slightly different symmetrization procedure of the quantum dilogarithms, which consisted in replacing in (10) the scalar factor in front of the matrices R' and $\bar{R}'$ by $(-q_2')^p$. (The q_j's have been defined in Remark 2.9(3), and $'$ denotes the determinations of the Nth roots of the q_j's induced by a common determination of the Nth roots of the cocycle values.) Let us write $\mathfrak{R}_N^B(\mathscr{T},c)$ for the associated state sums.

Then, the statement of Lemma 3.3 is unchanged, except that the ambiguity is only up to Nth roots of unity. However, in Proposition 3.5 we have to multiply both sides by the respective $Q_2 := \prod_i (-q_2')_i^p$. It is a remarkable but somewhat fortuitous fact that, for B-characters and for any positive $2 \to 3$ $\mathscr{D}$-transit $\mathscr{T} \to \mathscr{T}'$, we have $Q_2(\mathscr{T}')/Q_2(\mathscr{T}) = x(e)^{2p}$, where $x(e)$ is the upper-diagonal value of the cocycle z on the new edge in $T' \setminus H'$. Normalizing $\mathfrak{R}_N^B(\mathscr{T},c)$ by dividing it with $\prod_{e\in T\setminus H} x(e)^{2p}$, we eventually get a well-defined invariant $H_N^B(W,L,\rho)$ up to Nth roots of unity. The same procedure for general $PSL(2,\mathbb{C})$-characters (using the p_2''s instead of the q_2''s) does not seem to work, because the explicit formula for $P_2(\mathscr{T}')/P_2(\mathscr{T})$ heavily depends on the branching. Moreover, we believe that it is conceptually relevant that the QHI for arbitrary $PSL(2,\mathbb{C})$-characters can be computed only in terms of the idealized $\mathscr{I}$-triangulations $\mathscr{T}_{\mathscr{I}}$.

5. On the volume conjecture

Denote by $J_N(L)$ the coloured Jones polynomial of the link L in S^3, with colour N on each component of L, normalized by dividing it with the value on the unknot, and evaluated at $\zeta = \exp(2i\pi/N)$. By combining the results of [19,25] we know that

Theorem 5.1. *For every link L in S^3 we have $J_N(L) \equiv_N H_N^B(S^3,L,\rho_0)$, where ρ_0 is the necessarily trivial character of S^3, and H_N^B is the QHI for B-characters discussed in Remark* 4.31.

By using Theorem 5.1 we can state the *Volume Conjecture* of Kashaev [21] as:

Conjecture 5.2. *For every hyperbolic link L in S^3 we have*

$$\lim_{N\to\infty} (2\pi/N)\log(|J_N(L)|) = Vol(M),$$

where M is the cusped complete hyperbolic manifold (*unique up to isometry*) *homeomorphic to the complement of L in S^3.*

Recall that this conjecture has been rigorously confirmed at least for the celebrated figure-8 knot (see Ref. [33]). In this section we try to set Conjecture 5.2 against the background of the general QHI theory we have developed, also in order to find a geometric motivation for it. Our leading idea is

The hyperbolic geometry is a constitutive element of the QHI, *because they are defined as state sums over the hyperbolic ideal tetrahedra of any $\mathscr{I}$-triangulation. So their asymptotic behaviour* *should be expressable in terms of suitable 'classical' invariants of hyperbolic nature, computable over the same $\mathscr{I}$-triangulations and sharing with the QHI some basic structural features.*

This idea cannot be implemented straightforwardly. Indeed, in the case of (S^3, L), the hyperbolic geometry associated to the trivial character ρ_0 of S^3 by the idealization is trivial. On the other hand, Theorem 5.1 shows that $H_N(S^3, L, \rho_0)$ actually reflects the non-trivial geometry of $S^3 \setminus L$. In the general case (for instance, when W is hyperbolic and ρ is its holonomy) we expect that $H_N(W, L, \rho)$ combines, in a not yet understood way, the non-trivial contributions coming from both $W \setminus L$ and (W, ρ). For $S^3 \setminus L$, we can still implement our leading idea, as follows.

5.1. QHI for cusped 3-manifolds

The technology we have developed in this paper can be applied to the hyperbolic manifold $M = S^3 \setminus L$, and more generally to any non-compact complete hyperbolic 3-manifold M of finite volume. Let us call it a *cusped* manifold.

Consider a geometric triangulation of M by geodesically embedded ideal tetrahedra of non-negative volume. It is well-known that such triangulations do exist [16]. The manifold M is homeomorphic to the interior of a compact manifold Y with non-empty boundary made of tori, and the above triangulation, forgetting the hyperbolic structure, is a topological ideal triangulation of Y in the sense of Section 2.1. Assume that this triangulation admits a branching b. This is a rather mild assumption. The hyperbolic ideal tetrahedra can be encoded as usual by the cross-ratio moduli. This gives an $\mathscr{I}$-triangulation $\mathscr{T}_{\mathscr{I}}$ of M with possibly some (but not all) degenerate tetrahedra, such that for each non-degenerate (Δ_j, b_j, w_j) of $\mathscr{T}_{\mathscr{I}}$ we have $*_j = *_{w_j}$. We can endow $\mathscr{T}_{\mathscr{I}}$ with an integral charge c as in [26] (see the discussion after Proposition 4.7). So formula (12) defines a state sum $\mathfrak{R}_N(\mathscr{T}_{\mathscr{I}}, c)$. Lemma 4.15 and the statement in Lemma 4.17 concerning the $2 \to 3$ and $0 \to 2$ transits do apply to these state sums.

In spite of these facts, there are some technical problems to prove that $\mathfrak{R}_N(\mathscr{T}_{\mathscr{I}}, c)$ defines an invariant $H_N(M)$. For instance, it was important in the proof of Theorem 4.14 that the $\mathscr{I}$-transits were dominated by $\mathscr{D}$-transits. On the other hand, it may happen (as for an hyperbolic knot in S^3) that the ideal triangulation of Y only admits the trivial constant 1-cocycle, which is not idealizable. Anyway, let us postulate here that $H_N(M)$ is well defined; the details about its construction and invariance are worked out in [2]. Alternatively, the reader can replace $H_N(M)$ with $\mathfrak{R}_N(\mathscr{T}_{\mathscr{I}}, c)$ without effecting seriously the rest of the discussion.

For every cusped manifold M, set

$$\mathrm{R}(M) := CS(M) + i\, Vol(M) \bmod(\pi^2 \mathbb{Z}), \tag{15}$$

where $CS(M)$ and $Vol(M)$ are respectively the metric Chern–Simons invariant and the hyperbolic volume of the cusped manifold M. We propose the following generalization of Conjecture 5.2, that gives it a strong geometric motivation.

Conjecture 5.3. (1) *For every cusped manifold M, there exist $C \in \mathbb{C}^*$ and $D \in \mathbb{C}$ such that*

$$H_N(M)^{2N} = \left[C N^D \exp\left(\frac{N \mathrm{R}(M)}{\mathrm{i}\pi} \right) (1 + \mathcal{O}(1/N)) \right]^{2N}.$$

(2) *If L is a hyperbolic link in S^3 and $M = S^3 \setminus L$, then*

$$H_N(S^3, L, \rho_0) \equiv_N \pm H_N(M).$$

Clearly, both assertions are interesting on their own. We can relax the second, still in a meaningful way, by stating the equality up to a different normalization of $H_N(S^3, L, \rho_0)$, or even that it holds only asymptotically. Note that point (1) implies

$$\lim_{N\to\infty} (2\pi/N)\log(|H_N(M)|) = Vol(M). \tag{16}$$

Together with point (2) this generalizes Conjecture 5.2, because the QHI for B-characters have the same asymptotic behaviour than those for $PSL(2,\mathbb{C})$-characters. Conjecture 5.3 says at first that $H_N(M)^{2N}$ has an asymptotic power series expansion with, in general, an exponential growth rate. Assuming it, the invariance of $H_N(M)^{2N}$ and the uniqueness of coefficients of asymptotic expansions imply that $\exp(\mathrm{R}(M)/\mathrm{i}\pi)$, C and D are well-determined invariants of M. Then, it predicts that $\mathrm{R}(M)$ is of the form (15). We have expressed the conjecture in terms of the $2N$th power of $H_N(M)$ so as to kill an eventual multiplicative ambiguity up to $2N$th roots of unity (which is present in $H_N(W, L, \rho)$). Point (2) would make manifest the hyperbolic geometry of M hidden in $H_N(S^3, L)$.

Let M be a cusped manifold and (W_n, L_n, ρ_n) be a sequence of compact hyperbolic Dehn fillings of M converging geometrically to M, thanks to Thurston's hyperbolic Dehn filling theorem. Here, L_n denotes the link made of the simple short geodesics in W_n forming the cores of the fillings, and ρ_n is the holonomy of the hyperbolic manifold W_n. Recall that $Vol(W_n) \to Vol(M)$ when $n \to \infty$. We also propose:

Conjecture 5.4. *For every fixed N, when $n \to \infty$ we have*

$$H_N(W_n, L_n, \rho_n)^{2N} \to H_N(M)^{2N}.$$

By taking a double limit, this and (16) imply that, when $n, N \to \infty$, we have

$$(2\pi/N)\log(|H_N(W_n, L_n, \rho_n)|) \to Vol(M).$$

5.2. Motivations and comments

(1) Set $\mathrm{R}(W,\rho) := CS(\rho) + \mathrm{i}\,Vol(\rho) \bmod (\pi^2\mathbb{Z})$, where $CS(\rho)$ and $Vol(\rho)$ are, respectively, the Chern–Simons invariant and the volume of the character ρ (see [14] and the references therein for these notions). For every pair (W,ρ), we have proved in [3] that $\exp((1/\mathrm{i}\pi)\mathrm{R}(W,\rho))$ has strong structural relations with the QHI. For instance, as $\mathrm{R}(-W,\rho) = -\mathrm{R}(W,\rho)$, $CS(\rho^*) = CS(\rho)$ and $Vol(\rho^*) = -Vol(\rho)$, we see that $\exp((1/\mathrm{i}\pi)\mathrm{R}(W,\rho))$ formally verifies the duality property stated in Proposition 4.29. More substantially, $\mathrm{R}(W,\rho)$ can be computed over any $\mathscr{I}$-triangulation $\mathscr{T}_{\mathscr{I}}$ for (W,ρ) endowed with a so-called 'flattening' f as

$$\mathrm{R}(W,\rho) = \mathrm{R}(\mathscr{T}_{\mathscr{I}}, f) = \sum_j *_j\, \mathrm{R}(\Delta_j, b_j, w_j, f_j), \tag{17}$$

where the sum runs over the branched hyperbolic ideal tetrahedra of $\mathscr{T}_{\mathscr{I}}$ with induced flattenings f_i, $*_j$ is the index of the branching b_j, and $\mathrm{R}(\Delta, b, w, f)$ is a suitably 'uniformized' and symmetrized version of the Rogers dilogarithmic function $L(\varDelta, b, w)$, defined in Section 3. So $\exp((1/\mathrm{i}\pi)\mathrm{R}(W,\rho))=\exp((1/\mathrm{i}\pi)\mathrm{R}(\mathscr{T}_{\mathscr{I}}, f))$ looks very like a QHI state sum $H_N(\mathscr{T}_{\mathscr{I}}, c)$ (here it should be with $N=1$). This formula refines a description $\mathrm{mod}(\pi^2\mathbb{Q})$ of the universal second Cheeger–Chern–Simons class on $BPSL(2,\mathbb{C})$ due to Dupont–Sah [15,13], and is in agreement with the results of [26] and [29], stated for cusped and closed hyperbolic 3-manifolds (in the particular case when ρ is their holonomy).

The symmetrized quantum dilogarithms $\mathfrak{R}_N(\varDelta, b, w, c)$ and the symmetrized Rogers dilogarithm $\mathrm{R}(\varDelta, b, w, f)$ verify the same fundamental identities, that is they are invariant for all instances of charged (resp. flattened) $\mathscr{I}$-transits. Moreover, as mentioned in Section 3, the Rogers dilogarithm (also the symmetrized one) is the unique solution of these functional identities, up to a multiplicative scalar factor. Finally, the classical dilogarithms play the main role in the asymptotic expansion of the quantum dilogarithms, whence of the QHI.

On another hand, the construction of the QHI includes a link-fixing while the one of $\mathrm{R}(W,\rho)$ is link-free. This corresponds to the fact that the integral charges do not depend on the cross-ratio moduli, in contrast with the flattenings. This is a crucial difference because we know that the QHI are sensitive to the link, even asymptotically. However, this discrepancy vanishes when we work with cusped 3-manifolds, so that Conjecture 5.3(1) looks as an appropriate implementation of the leading idea stated at the beginning.

(2) The presence of the link L in $H_N(W, L, \rho)$ as well as its ambiguity up to sign and multiplication by Nth roots of unity are entirely a consequence of the specific symmetrization procedure of the basic state sums $\mathfrak{L}_N$ for (W,ρ), that we have adopted in Section 3. Suitable variations of this procedure based on *moduli-dependent charges*, similar to the flattenings, should allow us to define the QHI directly for (W,ρ). The asymptotic behaviour of such "absolute" QHI should be dominated by $R(W,\rho)$, similarly to Conjecture 5.3(1).

(3) Here we outline a possible way to approach Conjecture 5.3(2). We can use the triangulations (T,H) of (S^3,L) and T' of $Y=S^3\setminus U(L)$ constructed in Example 4.3 to compute $H_N(S^3,L,\rho_0)$ and $H_N(M)$, respectively. In both cases we have a complete decoration including an appropriate integral charge, and cross-ratio moduli of the involved $\mathscr{I}$-tetrahedra. In the first case we use as usual the idealization of an idealizable cocycle representing the trivial character ρ_0. In the second case we assume that the moduli are obtained via a sequence of $\mathscr{I}$-moves connecting T' with an hyperbolic geodesic triangulation of M. Recall that both constructions of (T,H) and T' include the selection of a same link-diagram arc, hence the selection of a $(1,1)$-tangle presentation of L. Then, developing the contributions of the diagram crossings to the state sums, we obtain for $H_N(S^3,L,\rho_0)$ and $H_N(M)$ very close expressions in terms of suitable R-matrices *depending on parameters*, and supported by that $(1,1)$-tangle presentation of L. But the values of the parameters of each R-matrix are specified by the respective global decorations (the charges give "discrete" parameters, and the cross-ratio moduli "continuous" ones).

On another hand, to compute $J_N(L)$ we can use *bare* tangle presentations of L, and, as shown in [25], a single *constant* Kashaev's R-matrix which corresponds to one *fixed* particular choice in the parameters. The proof of Theorem 5.1 includes a reduction of the above expression for $H_N^B(S^3,L,\rho_0)$ to an expression which involves only that constant R-matrix. This is due to Kashaev and is not a trivial fact. The main ingredients are indicated in [19]. [2]

[2] R.B. thanks Kashaev for having explained him the details.

So Conjecture 5.3(2) would be achieved if the same reduction to the constant R-matrix holds also for the formally similar non-constant R-matrix expressions of $H_N(S^3, L, \rho_0)$ and $H_N(M)$. This cannot be a simple adaptation of the $H_N^B(S^3, L, \rho_0)$ case, because the *global homological* properties of the integral charges as well as the fact that the moduli satisfy both edge compatibility and boundary completeness necessarily enter the proof. We believe that even eventually disproving this reduction should be very instructive.

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Appendix A. Quantum dilogarithms

In this section we present the definition of the $N^2 \times N^2$-matrix valued quantum dilogarithms as matrices of $6j$-symbols, which is originally due to Kashaev [18]. We also state their fundamental functional/symmetry relations needed for the present paper. We refer to [1, Chapter 3] for details and for the proofs.

Recall that $\zeta = \exp(2i\pi/N)$ and that $N > 1$ is an odd positive integer. Set $N = 2p+1$, $p \in \mathbb{N}$. We shall henceforth denote $1/2 := p + 1 \bmod(N)$. Fix the determination $\zeta^{1/2} = \zeta^{p+1} = -\exp(i\pi/N)$ of the square root of ζ.

A.1. Cyclic representations of $\mathscr{B}_\zeta$

Consider the $\mathbb{C}$-algebra $\mathscr{B}_\zeta$ with unity generated by elements E, E^{-1} and D such that $ED = \zeta DE$. It is well-known that $\mathscr{B}_\zeta$ can be endowed with a structure of Hopf algebra isomorphic to the simply-connected (non-restricted) integral form of a Borel subalgebra of $U_q(sl(2,\mathbb{C}))$ specialized in $q = \zeta$ [11, Section 9]. Thus it has the following co-multiplication, co-unit and antipode maps:

$$\Delta(E) = E \otimes E, \quad \Delta(D) = E \otimes D + D \otimes 1,$$

$$\varepsilon(E) = 1, \quad \varepsilon(D) = 0, \quad S(E) = E^{-1}, \quad S(D) = -E^{-1}D.$$

Given a representation ρ of $\mathscr{B}_\zeta$, denote by V_ρ the associated $\mathscr{B}_\zeta$-module. It is easily seen that if ρ is irreducible, then $dim_{\mathbb{C}}(V_\rho) \leqslant N$. We say that ρ is *cyclic* if $\rho(D) \in GL(V_\rho)$, i.e. if $dim_{\mathbb{C}}(V_\rho) = N$. Recall that the tensor product of two representations ρ and μ is defined by

$$(\rho \otimes \mu)(a) = \sum_i \rho(a_i') \otimes \mu(a_i''), \tag{A.1}$$

where $a \in \mathscr{B}_\zeta$, $\Delta(a) = \sum_i a'_i \otimes a''_i$, and the tensor product on $V_\rho \otimes V_\mu$ is over $\mathbb{C}$. We say that a sequence $\rho_1, \ldots, \rho_n$ of irreducible cyclic representations of $\mathscr{B}_\zeta$ is *regular* if $\rho_i \otimes \cdots \otimes \rho_{i+j}$ is cyclic, for any $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-i$. Two representations ρ and μ are *equivalent* if there exists an isomorphism $V_\rho \to V_\mu$ commuting with the action of $\mathscr{B}_\zeta$.

The algebra $\mathscr{B}_\zeta$ is a free module of rank N over its centre $\mathscr{Z}$, which is generated by $E^{\pm N}$ and D^N. The elements of $\mathscr{Z}$ act as scalar operators on any $\mathscr{B}_\zeta$-module V_ρ, so they define homomorphisms $\chi_\rho : \mathscr{Z} \to \mathbb{C}$ called the *central characters*. Put $e_\rho = \chi_\rho(E^N)$ and $d_\rho = \chi_\rho(D^N)$. The following lemma is an easy exercise:

Lemma A.1. *Two irreducible cyclic representations ρ and μ of $\mathscr{B}_\zeta$ are equivalent iff $(e_\rho, d_\rho) = (e_\mu, d_\mu) \in \mathbb{C}^* \times \mathbb{C}^*$.*

We find a nice parametrization of these equivalence classes $[\rho]$ by rewriting e_ρ and d_ρ as follows. Given non-zero complex numbers t_ρ and x_ρ we define a *standard* (cyclic) representation ρ of $\mathscr{B}_\zeta$ by

$$\rho(E) = t_\rho^2 Z, \quad \rho(D) = t_\rho x_\rho X, \tag{A.2}$$

where X and Z are the $N \times N$ matrices with components $X_{ij} = \delta_{i,j+1}$ and $Z_{ij} = \zeta^i \delta_{i,j}$ in the standard basis of $\mathbb{C}^N$, and $\delta_{i,j}$ is the Kronecker symbol. By Lemma A.1 any cyclic irreducible representation of $\mathscr{B}_\zeta$ is equivalent to a standard one, and two standard representations ρ and μ are equivalent iff $t_\rho^{2N} = t_\mu^{2N}$ and $t_\rho^N x_\rho^N = t_\mu^N x_\mu^N$.

For a regular pair (ρ, μ), the space $V_\rho \otimes V_\mu$ necessarily splits as the direct sum of N cyclic simple $\mathscr{B}_\zeta$-modules. Their central characters are given by $e_{\rho\otimes\mu}$ and $d_{\rho\otimes\mu}$. Then, Lemma A.1 implies that these submodules are all isomorphic. We call them the *product* submodules, and, abusing of notations, we denote them by $V_{\rho\mu}$. A direct sum decomposition of $V_\rho \otimes V_\mu$ into product submodules is obtained by choosing a linear basis of a characteristic subspace

$$E_i = Ker((\rho \otimes \mu)(E) - \zeta^i e'_{\rho\otimes\mu} id_{V_\rho \otimes V_\mu}),$$

where $e'_{\rho\otimes\mu}$ is some Nth root of $e_{\rho\otimes\mu}$. The $\mathscr{B}_\zeta$-orbit of any element of that basis is a product submodule. If ρ and μ are standard we can do these choices in a natural way, by using the standard tensor product basis of V_ρ and V_μ. Now, (A.1) gives $e_{\rho\otimes\mu} = e_\rho e_\mu$ and $d_{\rho\otimes\mu} = e_\rho d_\mu + d_\rho$. For the standard product submodules this reads

$$t_{\rho\mu}^{2N} = t_\rho^{2N} t_\mu^{2N},$$

$$x_{\rho\mu}^N = t_\rho^N x_\mu^N + x_\rho^N / t_\mu^N.$$

So, we conclude that the matrices

$$\Psi([\rho]) = \begin{pmatrix} t_\rho^N & x_\rho^N \\ 0 & t_\rho^{-N} \end{pmatrix} \tag{A.3}$$

define a one-to-one correspondence Ψ between the equivalence classes of irreducible cyclic representations of $\mathscr{B}_\zeta$, and the set of non diagonal upper triangular matrices of $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm I\}$. (The sign ambiguity is due to the choice of square root of t_ρ^{2N}.) Note that this set is open and dense in the quotient matrix topology of the upper Borel subgroup B of $PSL(2,\mathbb{C})$. Moreover, a remarkable feature of the parametrization Ψ is that for any regular pair (ρ, μ) we have $\Psi([\rho])\Psi([\mu]) = \Psi([\rho\mu])$.

A.2. 6j-Symbols

We are mainly concerned with the monoidal structure of the spaces of embeddings of cyclic simple $\mathscr{B}_\zeta$-modules. We define the *multiplicity module* of two irreducible cyclic representations ρ and μ as the complex vector space of equivariant maps from V_ρ to V_μ:

$$M_{\rho,\mu} = End_{\mathscr{B}_\zeta}(V_\rho, V_\mu) = \{U : V_\rho \to V_\mu \mid U\rho(a) = \mu(a)U,\ \forall a \in \mathscr{B}_\zeta\}.$$

We have seen above that for any regular pair (ρ,μ), we have $dim_{\mathbb{C}}(M_{v,\rho\otimes\mu}) = N$ if $[v] = [\rho\mu]$, and zero otherwise. Given a regular triple (ρ,μ,v), consider product representations $\rho\mu$, μv and $\rho\mu v$. Set

$$M_{\rho,(\mu,v)} = End_{\mathscr{B}_\zeta}(V_{\rho\mu v}, V_\rho \otimes (V_\mu \otimes V_v)),$$

$$M_{(\rho,\mu),v} = End_{\mathscr{B}_\zeta}(V_{\rho\mu v}, (V_\rho \otimes V_\mu) \otimes V_v).$$

We have vector space isomorphisms

$$M_{\rho,(\mu,v)} \cong M_{\rho\mu v,\rho\otimes\mu v} \otimes M_{\mu v,\mu\otimes v},$$

$$M_{(\rho,\mu),v} \cong M_{\rho\mu,\rho\otimes\mu} \otimes M_{\rho\mu v,\rho\mu\otimes v}.$$

Moreover, the isomorphism of $\mathscr{B}_\zeta$-modules

$$\alpha_{\rho,\mu,v} : V_\rho \otimes (V_\mu \otimes V_v) \to (V_\rho \otimes V_\mu) \otimes V_v$$

induces a vector space isomorphism between $M_{\rho,(\mu,v)}$ and $M_{(\rho,\mu),v}$. So we eventually get a linear isomorphism

$$R(\rho,\mu,v) : M_{\rho\mu v,\rho\otimes\mu v} \otimes M_{\mu v,\mu\otimes v} \to M_{\rho\mu,\rho\otimes\mu} \otimes M_{\rho\mu v,\rho\mu\otimes v}.$$

The coherence of the isomorphisms $\alpha_{.,.,.}$ for the tensor product of *four* cyclic representations making a regular sequence (ρ,μ,v,υ) implies that

$$R_{12}(\rho,\mu,v)R_{13}(\rho,\mu v,\upsilon)R_{23}(\mu,v,\upsilon) = R_{23}(\rho\mu,v,\upsilon)R_{12}(\rho,\mu,v\upsilon), \tag{A.4}$$

where $R_{12} = R \otimes id$, etc. This 3-cocycloid relation is called the *basic pentagon identity*. We can define $R(\rho,\mu,v)$ in another equivalent way. Let $\{K_\alpha(\rho,\mu)\}_{\alpha=1,\dots,N}$ denote a linear basis of $M_{\rho\mu,\rho\otimes\mu}$, and similarly for the other multiplicity modules. The families of maps $\{(id \otimes K_\delta(\mu,v)) \circ K_\gamma(\rho,\mu v)\}_{\delta,\gamma}$ and $\{(K_\alpha(\rho,\mu) \otimes id) \circ K_\beta(\rho\mu,v)\}_{\alpha,\beta}$ form two distinct linear basis of the space of embeddings of $V_{\rho\mu v}$ into $V_\rho \otimes V_\mu \otimes V_v$. Then, the isomorphism $R(\rho,\mu,v)$ may be realized as the corresponding change-of-basis matrix:

$$K_\alpha(\rho,\mu)K_\beta(\rho\mu,v) = \sum_{\delta,\gamma=0}^{N-1} R(\rho,\mu,v)_{\alpha,\beta}^{\gamma,\delta} K_\delta(\mu,v)K_\gamma(\rho,\mu v). \tag{A.5}$$

The matrix entries $R(\rho,\mu,v)_{\alpha,\beta}^{\gamma,\delta}$ are called 6j-*symbols*, and the basis vectors $K_\alpha(\rho,\mu)$ are *Clebsch–Gordan operators*. The relation (A.5) translates the coherence of the isomorphisms $\alpha_{.,.,.}$ cited above. In particular, one may prove (A.4) by applying both sides to a suitable composition of Clebsch–Gordan operators, and then using (A.5) several times.

Let us give a standardized form of the Clebsch–Gordan operators for all multiplicity modules. For that, we restrict to standard representations. By definition, each $K_\alpha(\rho,\mu)$ satisfies $(\rho\otimes\mu)(a)K_\alpha(\rho,\mu) =$

$K_\alpha(\rho,\mu)\rho\mu(a)$, for any $a\in\mathcal{B}_\zeta$. These equations are polynomials in the parameters of ρ, μ and $\rho\mu$. So, using the parametrization Ψ defined in (A.3), we see that $K_\alpha(\rho,\mu)$ is a matrix valued rational function on a branch of an N-fold ramified covering of $B\times B\times B$. Here B is the upper Borel subgroup of $PSL(2,\mathbb{C})$. More precisely, a direct computation gives the following result. Recall from (6) the definition of the function $\omega(x,y,z\,|\,n)$.

Lemma A.2. *Let (ρ,μ) be a regular pair of standard representations of $\mathcal{B}_\zeta$. The set of matrices $\{K_\alpha(\rho,\mu)\}_{\alpha=0,\dots,N-1}$ with components*

$$K_\alpha(\rho,\mu)^k_{i,j}=\zeta^{\alpha j+\alpha^2/2}\omega(t_\rho x_\mu, x_\rho/t_\mu, x_{\rho\mu}\,|\,i-\alpha)\delta(i+j-k),$$

form a linear basis of $M_{\rho\mu,\rho\otimes\mu}$.

Put $[x]=N^{-1}(1-x^N)/(1-x)$. Recall from Section 3 the definition of the complex valued functions g and h. We have:

Proposition A.3. *In the normalized basis of Clebsch–Gordan operators formed by the matrices $h(x_{\rho\mu}/t_\rho x_\mu)K_\alpha(\rho,\mu)$, the 6j-symbols read*

$$R(\rho,\mu,\nu)^{\gamma,\delta}_{\alpha,\beta}=h_{\rho,\mu,\nu}\zeta^{\alpha\delta+\alpha^2/2}\omega(x_{\rho\mu\nu}x_\mu, x_\rho x_\nu, x_{\rho\mu}x_{\mu\nu}\,|\,\gamma-\alpha)\delta(\gamma+\delta-\beta),$$

where $h_{\rho,\mu,\nu}=h(x_{\rho\mu}x_{\mu\nu}/x_{\rho\mu\nu}x_\mu)$. The matrix entries of the inverse of $R(\rho,\mu,\nu)$ are given by

$$\bar{R}(\rho,\mu,\nu)^{\alpha,\beta}_{\gamma,\delta}=\frac{[(x_{\rho\mu\nu}x_\mu)/(x_{\rho\mu}x_{\mu\nu})]}{h_{\rho,\mu,\nu}}\,\zeta^{-\alpha\delta-(\alpha^2/2)}\,\frac{\delta(\gamma+\delta-\beta)}{\omega((x_{\rho\mu\nu}x_\mu)/\zeta), x_\rho x_\nu, x_{\rho\mu}x_{\mu\nu}\,|\,\gamma-\alpha}.$$

Note that the matrices of 6j-symbols and the normalized Clebsch–Gordan operators have the same form, so that we can write $K_\alpha(\rho,\mu)^k_{i,j}=R(\rho,\mu)^{i,j}_{\alpha,k}$. This explains our choice of the normalization factor $h_{\rho,\mu}$. In fact, one can prove that both are representations of the canonical element of the Heisenberg double of $\mathcal{B}_\zeta$, acting on $M_{\rho\mu\nu,\rho\otimes\mu\nu}\otimes M_{\mu\nu,\mu\otimes\nu}$ [1, Sections 3.2–3.3, 2]. This canonical element is called a *twisted quantum dilogarithm.*

A.3. Basic pentagon identity and $\mathcal{I}$-transits

We observe that $R(\rho,\mu,\nu)$ is a matrix valued function of $x_\rho x_\nu/x_{\rho\mu}x_{\mu\nu}$ and $x_{\rho\mu\nu}x_\mu/x_{\rho\mu}x_{\mu\nu}$. Then, let us require that the standard representations ρ used for computing the Clebsch–Gordan operators are defined by taking a *same* determination of the Nth roots of t_ρ^{2N} and x_ρ^N simultaneously for all ρ. The corresponding 6j-symbols do not depend on the choice of such a determination, because they are homogeneous in the x-parameters. Hence, with this convention, we see that $R(\rho,\mu,\nu)$ is a function of, say, $(x_{\rho\mu\nu}x_\mu/x_{\rho\mu}x_{\mu\nu})^N$. Sufficient conditions for the basic pentagon identity (A.4) to be true are thus given by the relations between these ratios. We claim that they are just instances of relations between the moduli for the $\mathcal{I}$-transit shown in Fig. 8.

Indeed, associate to the edges (01), (12), (23) and (34) of this figure (for the ordering of the vertices induced, as usual, by the branching) the matrices in (A.3) for the representations ρ, μ, ν and υ, respectively. Since the sequence (ρ,μ,ν,υ) is regular, we can complete this procedure in a unique way on the other edges so that it defines an idealizable Borel valued 1-cocycle. Now, as explained in Remark 2.9 (3), the ratios of the form $(x_{\rho\mu\nu}x_\mu/x_{\rho\mu}x_{\mu\nu})^N$ are just the moduli indicated in Fig. 8. So our claim is proved.

This discussion shows that the basic pentagon identity holds true when we consider the matrices $R(\rho,\mu,v)$ more generally as functions of moduli of idealized hyperbolic tetrahedra, by using the above rule to fix the Nth roots of unity. To simplify the notations and also to keep close with those used in [18,1] (where the proofs of the results of this section are given), below we still denote by $R(\rho,\mu,v)$ the matrices of $6j$-symbols obtained in Proposition A.3, which, as we just said, essentially correspond to idealizable Borel valued 1-cocycles. However, we have to keep in mind the above generalization in terms of moduli.

A.4. Symmetries

Given a representation ρ of $\mathcal{B}_\zeta$, the *dual* representation $\bar{\rho}$ is defined by

$$\langle \bar{\rho}(a)\xi, v\rangle = \langle \xi, \rho(S(a))v\rangle$$

where $v \in V_\rho$, $\xi \in \bar{V}_\rho$ (the dual linear space), $a \in \mathcal{B}_\zeta$, S is the antipode of $\mathcal{B}_\zeta$, and $\langle .,.\rangle$ is the canonical pairing. In the case where ρ is standard, let us define the *inverse* standard representation $\bar{\rho}$ by setting $t_{\bar{p}} = 1/t_p$ and $x_{\bar{p}} = -x_p$. Clearly, $\bar{\rho}$ is equivalent to the representation dual to ρ (this explains the abuse of notation).

We can rewrite (11) as follows. For any a, $c \in \mathbb{Z}/N\mathbb{Z}$ put

$$R(\rho,\mu,v\,|\,a,c)^{\gamma,\delta}_{\alpha,\beta} = \zeta^{c(\gamma-\alpha)-ac/2} R(\rho,\mu,v)^{\gamma-a,\delta}_{\alpha,\beta-a},$$

$$\bar{R}(\rho,\mu,v\,|\,a,c)^{\alpha,\beta}_{\gamma,\delta} = \zeta^{c(\gamma-\alpha)+ac/2}\ \bar{R}(\rho,\mu,v)^{\alpha,\beta+a}_{\gamma+a,\delta}.$$

Note that in (11) we have omitted the index-independent factors $\zeta^{-ac/2}$ and $\zeta^{+ac/2}$ because of the unavoidable ambiguity of the QHI up to Nth roots of unity (see Remark 4.30). It is easy to verify that

$$R(\rho,\mu,v\,|\,a,c) = \zeta^{ac/2}(Y_1^{-a}Z_1^{-c}R(\rho,\mu,v)Z_1^cZ_2^{-a}),$$

$$\bar{R}(\rho,\mu,v\,|\,a,c) = \zeta^{-ac/2}(Z_1^cZ_2^{-a}\bar{R}(\rho,\mu,v)Z_1^{-c}Y_1^{-a}), \tag{A.6}$$

where $Y_1 = Y \otimes id$, etc., and $Y = \zeta^{1/2}XZ$ has components $Y_{m,n} = \omega^{1/2+n}\delta(m-n-1)$ (the matrices X and Z are defined in (19)). Recall from Section 3 the definition of the matrices S and T. Write $\{S^{-1}\}_{m,n} = S^{m,n}$ and so on. Normalizing the scalar factor v in T by a certain constant Nth root of unity we get:

Proposition A.4. *Put $b = 1/2 - a - c \in \mathbb{Z}/N\mathbb{Z}$. We have the following symmetry relations*:

$$\bar{R}(\bar{\rho},\rho\mu,v\,|\,a,b)^{\alpha,\delta}_{\gamma,\beta} = \left(\frac{x_{\rho\mu}x_{\mu v}}{x_\mu x_{\rho\mu v}}\right)^p \zeta^{-a/4}\sum_{\alpha',\gamma'=0}^{N-1} R(\rho,\mu,v\,|\,a,c)^{\gamma',\delta}_{\alpha',\beta}T_{\gamma,\gamma'}T^{\alpha,\alpha'},$$

$$\bar{R}(\rho\mu,\bar{\mu},\mu v\,|\,b,c)^{\alpha,\gamma}_{\beta,\delta} = \left(\frac{x_{\rho\mu}x_{\mu v}}{x_\rho x_v}\right)^p \zeta^{+c/4}\sum_{\alpha',\delta'=0}^{N-1} R(\rho,\mu,v\,|\,a,c)^{\gamma,\delta'}_{\alpha',\beta}T_{\delta,\delta'}S^{\alpha,\alpha'},$$

$$\bar{R}(\rho,\mu v,\bar{v}\,|\,a,b)^{\gamma,\beta}_{\alpha,\delta} = \left(\frac{x_{\rho\mu}x_{\mu v}}{x_\mu x_{\rho\mu v}}\right)^p \zeta^{-a/4}\sum_{\beta',\delta'=0}^{N-1} R(\rho,\mu,v\,|\,a,c)^{\gamma,\delta'}_{\alpha,\beta'}S_{\delta,\delta'}S^{\beta,\beta'}.$$

Note that, for instance, the factor $(x_{\rho\mu}x_{\mu\nu}/x_{\mu}x_{\rho\mu\nu})^p$ in the first identity is written as $(w_0')^{-p}$ with the notations of Lemma 3.3.

Given a standard representation ρ define the complex conjugate representation ρ^* by $t_{\rho^*} = (t_\rho)^*$ and $x_{\rho^*} = (x_\rho)^*$.

Proposition A.5. *We have the following unitarity property*:

$$\bar{R}(\rho^*, \mu^*, v^* \mid a, c)^{\alpha,\beta}_{\gamma,\delta} = (R(\rho, \mu, v \mid a, c)^{-\gamma,-\delta}_{-\alpha,-\beta})^*.$$

A.5. Partially symmetrized basic pentagon identity

Let us use the notations of the proof of Lemma 4.10. Consider the following set of independent charges: $i = c^4_{01}$, $j = c^2_{01}$, $k = c^0_{12}$, $l = c^1_{23}$ and $m = c^3_{12}$. They determine completely the charge transit shown in Fig. 11. We can easily show that $l + m = c^2_{13}$, $l - i = c^0_{23}$, $j + k = c^1_{02}$, $i + j = c^3_{01}$ and $m - k = c^4_{12}$. Note that the branching in Fig. 11 is the same as the one of Fig. 8. Moreover, we have seen above that the $\mathscr{I}$-transit of Fig. 8 dominates the basic pentagon identity. The following proposition describes a 'charged' generalization of this identity:

Proposition A.6. *We have*

$$R_{12}(\rho, \mu, v \mid i, m - k)R_{13}(\rho, \mu v, \upsilon \mid j, l + m)R_{23}(\mu, v, \upsilon \mid k, l - i)$$
$$= R_{23}(\rho\mu, v, \upsilon \mid j + k, l)R_{12}(\rho, \mu, v\upsilon \mid i + j, m).$$

The proof consists in using the formulas (A.6) and the commutation relations between the matrices Y, Z and $R(\rho, \mu, v)$ to reduce the statement to the basic pentagon identity.

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