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The reflexive and anti-reflexive solutions of the matrix equation $AX = B$ [☆]

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Abstract

An $n \times n$ complex matrix P is said to be a generalized reflection matrix if $P^H = P$ and $P^2 = I$. An $n \times n$ complex matrix A is said to be a reflexive (or anti-reflexive) matrix with respect to the generalized reflection matrix P if $A = PAP$ (or $A = -PAP$). This paper establishes the necessary and sufficient conditions for the existence of and the expressions for the reflexive and anti-reflexive with respect to a generalized reflection matrix P solutions of the matrix equation $AX = B$. In addition, in corresponding solution set of the equation, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm have been provided.

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1. Introduction

Throughout, C^n denotes the complex n-vector space, $C^{m\times n}$ denotes the set of $m \times n$ complex matrices. A^H, A^+ and $\|A\|_F$ stand for the conjugate transpose, the

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Moore–Penrose generalized inverse and the Frobenius norm of a complex matrix A, respectively. I_n represents the identity matrix of size n. On $C^{m\times n}$ we define inner product, $\langle A, B\rangle = \mathrm{trace}(B^{\mathrm{T}}A)$ for all $A, B \in C^{m\times n}$, then $C^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is Frobenius norm.

Chen and Chen [2], Chen [3,4], and Chen and Sameh [5] introduce the following two special classes of subspaces in $C^{m\times n}$

$$C_r^{n\times n}(P) = \{A \in C^{n\times n} | A = PAP\} \quad \text{and}$$
$$C_a^{n\times n}(P) = \{A \in C^{n\times n} | A = -PAP\},$$

where P of size n is generalized reflection matrix. By a generalized reflection matrix, say P, one mean that P satisfies the following two conditions: $P^H = P$ and $P^2 = I$. In other words, a generalized reflection matrix is an involutory Hermitian matrix. The matrices A in $C_r^{n\times n}(P)$ and B in $C_a^{n\times n}(P)$ are, respectively, said to be a reflexive and anti-reflexive matrices with respect to the generalized reflection matrix P.

The reflexive and anti-reflexive matrices with respect to a generalized reflection matrix P have many special properties and widely used in engineering and scientific computations (see, for instance, Refs. [2–5]).

In this paper, we consider the reflexive and anti-reflexive with respect to a generalized reflection matrix P solutions of the matrix equation

$$AX = B, \tag{1}$$

where A and B are given matrices in $C^{m\times n}$. We also consider the matrix nearness problem

$$\min_{X\in S_X} \|X - X^*\|_{\mathrm{F}}, \tag{2}$$

where X^* is a given matrix in $C^{n\times n}$ and S_X is the solution set of Eq. (1).

The well-known equation (1) with the unknown matrix X being symmetric, anti-symmetric, symmetric positive semidefinite and re-positive definite were studied (see, for instance, Vetter [14], Magnus and Neudecker [12,13], Don [10], Chu [7,8], Dai [9], and Wu [15]). All in these paper, the necessary and sufficient conditions for the existence of and the expressions for the solution of the equation were given by using the structure properties of matrices in required subset in $C^{m\times n}$ and the singular value decomposition of the matrix. The reflexive and anti-reflexive matrices with respect to a generalized reflection matrix P are two classes of important matrices and have engineering and scientific applications. The reflexive and anti-reflexive with respect to a generalized reflection matrix P solution of the matrix equation (1), however, has not been considered yet. In this paper, we will discuss this problem.

The matrix nearness problem (2), that is, finding the nearest matrix in the solution set of Eq. (1) to a given matrix in Frobenius norm, is initially proposed in the processes of test or recovery of linear systems due to incomplete dates or revising given dates. A preliminary estimate X^* of the unknown matrix X can be obtained by the experimental observation values and the information of statical distribution. The other form of the matrix nearness problem (2),

$$\min_{A} \|A - A^*\|_F,$$

subject to

$$AX = B,$$

where X and B are given $n \times m$ matrices, A^* is a given $n \times n$ matrix and A is an $n \times n$ symmetric, bisymmetric or symmetric positive semidefinite matrix, were discussed (see, for instance, Xie et al. [16], Zhou and Dai [17] and references therein).

The paper is organized as follows: in Section 2, the structure properties of the generalized reflection matrix P and the matrices in $C_r^{n\times n}(P)$ or $C_a^{n\times n}(P)$ will be introduced. Using these properties, together with Moore–Penrose generalized inverse, the necessary and sufficient conditions for the existence of and the expressions for the reflexive and anti-reflexive with respect to a generalized reflection matrix P solutions of Eq. (1) will be derived. In Section 3, the expression of the solution of the matrix nearness problem (2) will be provided.

2. The solution of the matrix equation (1)

In this section we first introduce some structure properties of the generalized reflection matrix P and the subsets $C_r^{n\times n}(P)$ and $C_a^{n\times n}(P)$ of $C^{m\times n}$. Then we give the necessary and sufficient conditions for the existence of and the expressions for the reflexive and anti-reflexive with respect to a generalized reflection matrix P solutions of Eq. (1).

Lemma 1. *Assume P is a generalized reflection matrix of size n. Let*

$$P_1 = \tfrac{1}{2}(I_n + P), \quad P_2 = \tfrac{1}{2}(I_n - P), \tag{3}$$

then P_1 and P_2 are orthogonal projection matrices, and satisfied $P_1 + P_2 = I_n$, $P_1 P_2 = 0$.

Proof. Notices the definition of P, P_1 and P_2, only by a direct computation, we have that the above results are held. □

Lemma 2. *Assume P_1 and P_2 are defined as* (3) *and* $\operatorname{rank}(P_1) = r$, *then* $\operatorname{rank}(P_2) = n-r$, *and there exist unit column orthogonal matrices $U_1 \in C^{n\times r}$ and $U_2 \in C^{n\times(n-r)}$ such that*

$$P_1 = U_1 U_1^H, \quad P_2 = U_2 U_2^H, \quad P = U_1 U_1^H - U_2 U_2^H, \quad U_1^H U_2 = 0.$$

Proof. Since P_1, P_2 are orthogonal projection matrices, and satisfied $P_1 + P_2 = I$, $P_1 P_2 = 0$, then the column space $R(P_2)$ of the matrix P_2 is the orthogonal complement of the column space $R(P_1)$ of the matrix P_1, i.e., $C^n = R(P_1) \oplus R(P_2)$. Hence, if

$\text{rank}(P_1) = r$, then $\text{rank}(P_2) = n - r$. On the other hand, if $\text{rank}(P_1) = r$, $\text{rank}(P_2) = n - r$, and also notice that P_1, P_2 are orthogonal projection matrices, then there exist unit column orthogonal matrices $U_1 \in C^{n\times r}$ and $U_2 \in C^{n\times(n-r)}$ such that $P_1 = U_1U_1^H$, $P_2 = U_2U_2^H$. Using $C^n = R(P_1) \oplus R(P_2)$, we have $U_1^H U_2 = 0$. Substituting $P_1 = U_1U_1^H$, $P_2 = U_2U_2^H$ into (3), we have $P = U_1U_1^H - U_2U_2^H$. $\square$

Let $U = (U_1, U_2)$. From Lemma 2, it is easy to verify that U is an unitary matrix and the generalized reflection matrix P can be expressed as

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^H. \tag{4}$$

Lemma 3. *The matrix $A \in C_r^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} U^H,$$

where $M \in C^{r\times r}$, $N \in C^{(n-r)\times(n-r)}$ and U is the same as (4).

Proof. Assume $A \in C_r^{n\times n}(P)$. By Lemma 2 and the definition of $C_r^{n\times n}(P)$, we have

$$\begin{pmatrix} U_1^H \\ U_2^H \end{pmatrix} A(U_1, U_2) = \begin{pmatrix} U_1^H \\ -U_2^H \end{pmatrix} A(U_1, -U_2),$$

which is equivalent to

$$\begin{pmatrix} U_1^H AU_1 & U_1^H AU_2 \\ U_2^H AU_1 & U_2^H AU_2 \end{pmatrix} = \begin{pmatrix} U_1^H AU_1 & -U_1^H AU_2 \\ -U_2^H AU_1 & U_2^H AU_2 \end{pmatrix},$$

which implies that $U_1^H AU_2 = 0, U_2^H AU_1 = 0$. Let $M = U_1^H AU_1$, $N = U_2^H AU_2$, then we have

$$\begin{pmatrix} U_1^H \\ U_2^H \end{pmatrix} A(U_1, U_2) = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

Hence,

$$A = U \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} U^H.$$

Conversely, for any $M \in C^{r\times r}$ and $N \in C^{(n-r)\times(n-r)}$, using the result (4), we have

$$\begin{aligned} PU \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} U^H P &= U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^H U \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} U^H U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^H \\ &= U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}^2 \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} U^H \\ &= U \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} U^H. \end{aligned}$$

This implies that $A = U \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} U^H \in C_r^{n\times n}(P)$. □

Similar to the proof of Lemma 3, we can prove the following lemma.

Lemma 4. *The matrix $A \in C_a^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} U^H,$$

where $M \in C^{r\times(n-r)}$, $N \in C^{(n-r)\times r}$ and U is the same as (4).

The following lemma is a well known result.

Lemma 5. *The matrix equation $AX = B$ with $A \in C^{p\times m}$ and $B \in C^{p\times n}$ has a solution $X \in C^{m\times n}$ if and only if $AA^+B = B$. In that case it has the general solution $X = A^+B + (I_m - A^+A)G$, where $G \in C^{m\times n}$ is arbitrary matrix.*

Theorem 1. *Given $A, B \in C^{m\times n}$ and a generalized reflection matrix P of size n. Assume P can be expressed as* (4), *and AU and BU have the following partition form*

$$AU = (A_1, A_2), \quad A_1 \in C^{m\times r}, \ \ A_2 \in C^{m\times(n-r)}, \tag{5}$$

$$BU = (B_1, B_2), \quad B_1 \in C^{m\times r}, \ \ B_2 \in C^{m\times(n-r)}, \tag{6}$$

then Eq. (1) *has a solution $X \in C_r^{n\times n}(P)$ if and only if*

$$A_1A_1^+B_1 = B_1, \quad A_2A_2^+B_2 = B_2. \tag{7}$$

In that case it has the general solution

$$X = U \begin{pmatrix} A_1^+B_1 + (I_r - A_1^+A_1)G_1 & 0 \\ 0 & A_2^+B_2 + (I_{n-r} - A_2^+A_2)G_2 \end{pmatrix} U^H, \tag{8}$$

where $G_1 \in C^{r\times r}$ and $G_2 \in C^{(n-r)\times(n-r)}$ are arbitrary matrices.

Proof. The necessity. Assume Eq. (1) has a solution $X \in C_r^{n\times n}(P)$, we have from Lemma 3 that X can be expressed as

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} U^H, \tag{9}$$

where $X_1 \in C^{r\times r}$, $X_2 \in C^{(n-r)\times(n-r)}$.

Note that U is a unitary matrix, and the definition of $A_i, B_i (i = 1, 2)$, Eq. (1) is equivalent to

$$A_1X_1 = B_1, \quad A_2X_2 = B_2.$$

It follows from Lemma 5 that

$$A_1A_1^+B_1 = B_1, \quad A_2A_2^+B_2 = B_2,$$

and

$$X_1 = A_1^+B_1 + (I_r - A_1^+A_1)G_1, \quad X_2 = A_2^+B_2 + (I_{n-r} - A_2^+A_2)G_2, \tag{10}$$

where $G_1 \in C^{r\times r}$ and $G_2 \in C^{(n-r)\times(n-r)}$ are arbitrary matrices.

Substituting (10) into (9), we know that the solution $X \in C_r^{n\times n}(P)$ of Eq. (1) can be expressed as (8).

The sufficiency. Assume $A_1A_1^+B_1 = B_1$ and $A_2A_2^+B_2 = B_2$, then we have from Lemma 5 that there exist $X_1 \in C^{r\times r}$ and $X_2 \in C^{(n-r)\times(n-r)}$ such that

$$A_1X_1 = B_1, \quad A_2X_2 = B_2,$$

which is equivalent to

$$(A_1, A_2)\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} = (B_1, B_2).$$

It in turn is equivalent to

$$AU\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}U^H = B.$$

This implies that $X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}U^H \in C_r^{n\times n}(P)$ is the solution of Eq. (1). Hence, Eq. (1) has a solution X in $C_r^{n\times n}(P)$. □

Similar to the proof of Theorem 1, we can prove the following theorem.

Theorem 2. *Given $A, B \in C^{m\times n}$ and a generalized reflection matrix P of size n. Assume P can be expressed as* (4), *and AU and BU have the partition form of* (5) *and* (6), *then Eq.* (1) *has a solution $X \in C_a^{n\times n}(P)$ if and only if*

$$A_1A_1^+B_2 = B_2, \quad A_2A_2^+B_1 = B_1. \tag{11}$$

In that case it has the general solution

$$X = U\begin{pmatrix} 0 & A_2^+B_1 + (I_{n-r} - A_2^+A_2)G_1 \\ A_1^+B_2 + (I_r - A_1^+A_1)G_2 & 0 \end{pmatrix}U^H, \tag{12}$$

where $G_1 \in C^{r\times(n-r)}$ and $G_2 \in C^{(n-r)\times r}$ are arbitrary matrices.

3. The solution of the matrix nearness problem (2)

To give the explicit expression of the solution of the matrix nearness problem (2), we first verify the following lemma.

Lemma 6. *Given matrices $A \in C^{p\times m}$ and $B \in C^{p\times n}$, then the procrustes problem* (*see Andersson and Elfving* [1] *and Higham* [11])

$$\min_{X \in C^{m\times n}} \|(I_m - A^+A)X - B\|_F \tag{13}$$

has a solution which can be expressed as $X = B + A^+G$, *where* $G \in C^{p\times n}$ *is arbitrary matrix.*

Proof. Applying the properties of Moore–Penrose generalized inverse and the inner product in space $C^{m\times n}$, we have

$$\begin{aligned} &\|(I_m - A^+A)X - B\|_F^2 \\ &\quad = \langle (I_m - A^+A)X - B, (I_m - A^+A)X - B\rangle \\ &\quad = \langle (I_m - A^+A)(X - B), (I_m - A^+A)(X - B)\rangle + \langle A^+AB, A^+AB\rangle \\ &\quad = \|(I_m - A^+A)(X - B)\|_F^2 + \|A^+AB\|_F^2. \end{aligned}$$

Hence,

$$\min_{X \in C^{m\times n}} \|(I_m - A^+A)X - B\|_F$$

if and only if

$$\min_{X \in C^{m\times n}} \|(I_m - A^+A)(X - B)\|_F.$$

It is clear that $X = B + A^+G$, with $G \in C^{p\times n}$ be arbitrary, is the solution of the above procrustes problem. So, the solution of the procrustes problem (13) can be expressed as $X = B + A^+G$. □

Theorem 3. *Given a* $n \times n$ *matrix* X^*. *Assume the solution set* $S_X \subseteq C_r^{n\times n}(P)$ *of Eq.* (1) *is nonempty, then the matrix nearness problem* (2) *has an unique solution* $\hat{X}$ *in* S_X. *Furthermore, let*

$$U^H X^* U = \begin{pmatrix} X_{11}^* & X_{12}^* \\ X_{21}^* & X_{22}^* \end{pmatrix}, \quad X_{11}^* \in C^{r\times r}, \quad X_{22}^* \in C^{(n-r)\times(n-r)}, \tag{14}$$

then $\hat{X}$ *can be expressed as*

$$\hat{X} = U\begin{pmatrix} A_1^+B_1 + (I - A_1^+A_1)(X_{11}^* - A_1^+B_1) & 0 \\ 0 & A_2^+B_2 + (I - A_2^+A_2)(X_{22}^* - A_2^+B_2) \end{pmatrix} U^H. \tag{15}$$

Proof. If the solution set $S_X \subseteq C_r^{n\times n}(P)$ of Eq. (1) is nonempty, then S_X is a closed convex cone in Hilbert space $C^{n\times n}$ with vertex at

$$X_0 = U\begin{pmatrix} A_1^+B_1 & 0 \\ 0 & A_2^+B_2 \end{pmatrix} U^H.$$

Hence, it is certain that there exists an unique $\hat{X} \in S_X$ such that the matrix nearness problem (2) holds (see Cheney [6]). Using the invariance of the Frobenius norm under unitary transformations, we have from (8) and (14) that

$$\begin{aligned}
&\|X - X^*\|_F^2 \\
&= \left\| \begin{pmatrix} A_1^+B_1 + (I - A_1^+A_1)G_1 & 0 \\ 0 & A_2^+B_2 + (I - A_2^+A_2)G_2 \end{pmatrix} - U^H X^* U \right\|_F^2 \\
&= \|(I - A_1^+A_1)G_1 - (X_{11}^* - A_1^+B_1)\|_F^2 + \|X_{12}^*\|_F^2 \\
&\quad + \|(I - A_2^+A_2)G_2 - (X_{22}^* - A_2^+B_2)\|_F^2 + \|X_{21}^*\|_F^2.
\end{aligned}$$

Hence, there exists $\hat{X} \in S_X$ such that the matrix nearness problem (2) holds is equivalent to exist $G_1 \in C^{r\times r}$ and $G_2 \in C^{(n-r)\times(n-r)}$ such that

$$\min_{G_1} \|(I - A_1^+A_1)G_1 - (X_{11}^* - A_1^+B_1)\|_F,$$
$$\min_{G_2} \|(I - A_2^+A_2)G_2 - (X_{22}^* - A_2^+B_2)\|_F.$$

It follows from Lemma 6 that

$$G_1 = X_{11}^* - A_1^+B_1 + A_1^+Y_1, \quad G_2 = X_{22}^* - A_2^+B_2 + A_2^+Y_2,$$

where $Y_1 \in C^{r\times r}$, $Y_2 \in C^{(n-r)\times(n-r)}$ are arbitrary. Taking G_1 and G_2 into (8), we obtain that the solution of the matrix nearness problem (2) can be expressed as (15). □

Similar to the proof of Theorem 3, we can prove the following theorem.

Theorem 4. *Given a $n \times n$ matrix X^*. Assume the solution set $S_X \subseteq C_a^{n\times n}(P)$ of Eq.* (1) *is nonempty, then the matrix nearness problem* (2) *has an unique solution $\hat{X}$ in S_X. Furthermore, assume $U^H X^* U$ have the partition form of* (14), *then $\hat{X}$ can be expressed as*

$$\hat{X} = U\begin{pmatrix} 0 & A_1^+B_2 + (I - A_1^+A_1)(X_{12}^* - A_1^+B_2) \\ A_2^+B_1 + (I - A_1^+A_1)(X_{21}^* - A_2^+B_1) & 0 \end{pmatrix}U^H. \quad (16)$$

4. Conclusions

In this paper, we considered the reflexive and anti-reflexive with respect to a generalized reflection P solution of the matrix equation $AX = B$. We also considered, in corresponding solution set of the equation, finding the nearest matrix to a given matrix in Frobenius norm. After providing some interesting structure characterizations, i.e., Lemmas 1–4 of the generalized reflection matrix and the reflexive (or anti-reflexive) matrix with respect to a generalized reflection matrix, the solvability conditions and the explicit formula for the solutions of the equation are given. Moore–Penrose generalized inverse is an important tool in this paper.

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