On the factoriality of some rings of complex Nash functions

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Abstract

In this paper we give a topological characterization of the factoriality of the ring \( \mathcal{N}_X(K) \) of the complex Nash functions on a semianalytic Nash affine compact of a factorial Nash subvariety of \( \mathbb{C}^n \).

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1. Introduction

The algebraic property of the rings of real Nash functions on semialgebraic submanifolds of \( \mathbb{R}^n \) are well-known (see [2]), while much less is known in the complex case.

In this paper we consider a class of compact subsets of a Nash normal subvariety \( X \) of \( \mathbb{C}^n \), which we call affine (Nash) compact sets. If \( K \) is such a compact set we give conditions for the ring \( \mathcal{N}_X(K) \) of all Nash functions on \( K \), to be factorial and noetherian. While the analogous results known in the real case have been proved under the assumption that the variety \( X \) is smooth, here we allow \( X \) to be singular, but, of course necessarily factorial.

Affine Nash compact sets \( K \) are defined as those compact sets which can be identified with the maximal spectrum of the ring \( \mathcal{N}_X(K) \). These compact sets are characterized by the property of being holomorphically convex, with respect to the analytic structure of \( X \), and by the fact that the inclusion map from \( \mathcal{N}_X(K) \) to the ring \( \mathcal{O}_X(K) \) of all analytic functions on \( K \), is dense with respect to the canonical LF topology on \( \mathcal{O}_X(K) \) (see Theorem 3.6).

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The Runge compact sets of affine algebraic subvarieties of $\mathbb{C}^n$, which are considered in [14–16], are Nash affine compact sets, but, from classical results of J. Wermer (see, e.g., [5]), it follows that Nash affine compact sets are more general. An Artin type approximation theorem (see [3,7,14]) is extended to Nash affine compact sets (see Theorem 3.5); this theorem, together with a generalization of some results of [16], allow us to prove the main result of the paper (see Theorem 5.1): if $K$ is a semianalytic Nash affine compact set of a factorial Nash subvariety of $\mathbb{C}^n$, then the canonical map $\text{Pic}(N_X(K)) \to \text{Pic}(\mathcal{O}_K(K))$ defined by $M \mapsto \mathcal{O}(K) \otimes N_X(K) M$ is an isomorphism. It follows that for such compact sets a purely topological characterization of the factoriality can be given: the ring $N_X(K)$ is factorial if and only if $H^2(K, \mathbb{Z}) = 0$ (see Theorem 5.6), this is well-known in the analytic case (see [4]).

2. Preliminary and notation

To fix our notation we first recall some well-known properties of complex Nash varieties that we shall freely use in the following (see [2,14–16]) without any further references.

A Nash (or analytic) subvariety $X$ of $\mathbb{C}^n$ is a locally closed subset, that is, a set which locally is the zero set of finitely many Nash (resp. analytic) functions. Nash (analytic) functions on open sets of $X$ are locally restrictions of Nash (analytic) functions of $\mathbb{C}^n$.

We denote by $N_X$ (resp. $\mathcal{O}_X$) the sheaf of Nash (resp. analytic) functions of $X$.

The sheaf $\mathcal{N}_X$ is a coherent subsheaf of the sheaf $\mathcal{O}_X$ and for every $x \in X$ the inclusion map $\mathcal{N}_{X,x} \to \mathcal{O}_{X,x}$ is a faithfully flat map of noetherian local rings which induces an isomorphism between their completions.

Obviously, algebraic subvarieties of $\mathbb{C}^n$, on which we shall always consider the strong topology, are also Nash subvarieties of $\mathbb{C}^n$. A Nash subvariety is irreducible (or locally irreducible) if it is irreducible (resp. locally irreducible) as an analytic subvariety. An irreducible Nash subvariety $X$ is factorial (or normal) if the ring $\mathcal{N}_{X,x}$ is factorial (resp. integrally closed in its full ring of quotients) for every $x \in X$. It is known that $X$ is a factorial (or normal) Nash subvariety if and only if it is a factorial (resp. normal) analytic subvariety. Of course every factorial subvariety is normal.

A Nash variety $X$ is a $\mathbb{C}$-ringed space that is locally isomorphic to Nash subvarieties of $\mathbb{C}^n$. A Nash (or analytic) map of Nash varieties is a morphism of $\mathbb{C}$-ringed spaces.

If $K$ is a compact subset of a Nash (or analytic) variety $X$, we simply denote by $K$ the $\mathbb{C}$-ringed space $(K, \mathcal{N}_X|_K)$ (resp. $(K, \mathcal{O}_X|_K)$). By Nash (or analytic) function on $K$ we mean a section of the sheaf $\mathcal{N}_X|_K$ (resp. $\mathcal{O}_X|_K$), that we usually simply denote by $\mathcal{N}_K$ (resp. $\mathcal{O}_K$). If $Y$ is a Nash (analytic) variety, a Nash (analytic) map $K \to Y$ is a morphism of $\mathbb{C}$-ringed spaces $(K, \mathcal{N}_K) \to (Y, \mathcal{N}_Y)$ (resp. $(K, \mathcal{O}_K) \to (Y, \mathcal{O}_Y)$). If $X$ is normal and $K$ is connected then $\mathcal{N}(K)$ and $\mathcal{O}(K)$ are normal rings.

From now on we always assume that the compact set $K$ is connected.

**Theorem 2.1.** Let $X$ be a normal Nash subvariety of $\mathbb{C}^n$ and $f : X \to \mathbb{C}^m$ a Nash map. Then, there exist a normal affine algebraic subvariety $N \subset \mathbb{C}^{n+m+p}$, an open Nash embedding $\sigma : X \to N$ and a regular map $h : N \to \mathbb{C}^m$ such that $h\sigma = f$ and $\pi\sigma = \text{id}_X$. 
where \( \pi : N \to \mathbb{C}^n \) is induced by the canonical projection \( \mathbb{C}^{n+m+p} \to \mathbb{C}^n \). Moreover, if \( a \in X \) is a regular point of \( X \) then \( \sigma(a) \) is an algebraic regular point of \( N \).

**Proof.** It is a version of the complex Artin–Mazur Theorem. (See, for instance, [2, 15].)

**Lemma 2.2.** Let \( X \) be a normal Nash subvariety of \( \mathbb{C}^n \) and let \( K \) be a compact subset of \( X \); let \( a \in K \) and \( n_a = \{ f \in \mathcal{N}_X(K) \mid f(a) = 0 \} \). Then

(i) for every \( \alpha \in K \) the map \( \mathcal{N}_X(K)_n_a \to \mathcal{N}_X,a \) is faithfully flat;
(ii) the map in (i) induces an isomorphism between the completions: \( \mathcal{N}_X(K)^{\hat{\alpha}} \cong (\mathcal{N}_X,a)^{\hat{\alpha}} \) for every \( \alpha \in K \);
(iii) \( \mathcal{N}_X,a \) is the henselianization of \( \mathcal{N}_X(K)_n_a \), i.e. \( \mathcal{N}_X,a = \mathcal{h} \mathcal{N}_X(K)_n_a \) for every \( \alpha \in K \);
(iv) If \( p \) is a prime ideal of \( \mathcal{N}_X(K)_n_a \) then there exist a finite number of primes \( q_1, \ldots, q_r \) in \( \mathcal{N}_X,a \) such that \( \mathfrak{p} \mathcal{N}_X,a = q_1 \cap \cdots \cap q_r \) and \( \text{ht}(q_i) = \text{ht}(p) \) for every \( i = 1, \ldots, r \).

**Proof.** By the Artin–Mazur Theorem we may assume that \( X \) is an irreducible algebraic subvariety of \( \mathbb{C}^n \). Let \( \mathcal{I}_X \) be the ideal of Nash functions which are zero on \( X \) and \( \mathcal{J}_X \) be the ideal of regular functions which are zero on \( X \). By a classical result of C. Chevalley (see, e.g., [11]) we have that \( \mathcal{I}_X,a = \mathcal{J}_X,a \mathcal{N}_C,a \). From this and from the well-known equality \( \mathcal{N}_C,a = \mathcal{h} \mathcal{R}_C,a \) it follows that \( \mathcal{N}_X,a = \mathcal{h} \mathcal{R}_X,a \) (see [9]). Now, all the results, follow as in the real case (see [2]).

We shall use the following result ([7,14]) that follows from a deep result of commutative algebra, the so-called Néron desingularization ([12]).

**Theorem 2.3.** Let \( Z \) be an affine algebraic subvariety of \( \mathbb{C}^n \), \( P \) a compact \( \mathcal{O}_Z(Z) \)-convex subset of \( Z \) and \( h \in \mathcal{O}_Z(P)^{\ast} \). Let \( F_1, \ldots, F_q \) be Nash functions on an open neighborhood of the graph of \( h : P \to \mathbb{C}^s \) in \( Z \times \mathbb{C}^s \) such that \( F_i(\text{id}, h) \) is zero in \( \mathcal{O}_Z(P) \) for every \( i = 1, \ldots, q \). Then, \( h \) can be uniformly approximated by a Nash map \( \tilde{h} : P \to \mathbb{C}^s \) such that \( F_i(\text{id}, \tilde{h}) \) is zero in \( \mathcal{N}_Z(P) \) for every \( i = 1, \ldots, q \).

### 3. Affine compact sets

Let \( X \) be a Nash subvariety of \( \mathbb{C}^n \) and let \( K \) be a compact subset of \( X \). We denote by \( \mathcal{C}(K) \) the uniform algebra of the continuous functions from \( K \) to \( \mathbb{C} \) and by \( \mathcal{N}(K) \) the uniform closure of the canonical image of \( \mathcal{N}_X(K) \) in \( \mathcal{C}(K) \).

If \( A \) is a topological \( \mathbb{C} \)-algebra, we denote by \( \mathcal{S}(A) \) its spectrum, i.e. the set of continuous nonzero homomorphisms (characters) from \( A \) to \( \mathbb{C} \). Let \( A \) be a uniform subalgebra of \( \mathcal{C}(K) \) and consider \( f_1, \ldots, f_s \in A \); we shall denote by \( \sigma_A(f_1, \ldots, f_s) \) the joint spectrum of \( f_1, \ldots, f_s \), i.e. the compact subset of \( \mathbb{C}^s \) which is the image of the spectrum \( \mathcal{S}(A) \) under the continuous map \( \chi \mapsto (\chi(f_1), \ldots, \chi(f_s)) \) (see, for instance, [5]). We denote by \( \tilde{\sigma}_A(f_1, \ldots, f_s) \) the polinomially convex hull of \( \sigma_A(f_1, \ldots, f_s) \).

We consider on \( \mathcal{O}_X(K) \) the finest locally convex topology which makes the functions...
$\mathcal{O}_X(U) \to \mathcal{O}_X(K)$ continuous for every open neighborhood $U$ of $K$ in $X$, where $\mathcal{O}_X(U)$ is endowed with its canonical Frechet topology. With respect to this topology, $\mathcal{O}_X(K)$ is a generalized LF space (see [17]) and each homomorphism from $\mathcal{O}_X(K)$ to $\mathbb{C}$ is continuous, as in the uniform case. We still denote by $\mathcal{S}(\mathcal{O}_X(K))$ the spectrum of the algebra $\mathcal{O}_X(K)$. $\mathcal{S}(\mathcal{O}_X(K))$ is a compact Hausdorff space with respect to the Gelfand topology (see [17]).

A compact subset $K$ of $X$, or, more generally, a compact subset of an analytic space, is said to be holomorphically convex if $K = \mathcal{S}(\mathcal{O}_X(K))$ (see [17]).

If $K$ is a Stein compact set, i.e. a compact set which has a fundamental system of neighborhoods formed by Stein open sets, then $K$ is holomorphically convex; the converse is not true in general as remarked in [17].

The following Theorem extends the well-known results of H. Cartan for Stein spaces to holomorphically convex compact subsets (see [17]).

**Theorem 3.1.** Let $X$ be a separated analytic space, i.e. global analytic functions separate points, and let $K$ be a compact holomorphically convex subset of $X$. If $\mathcal{F}$ is a coherent analytic sheaf of $\mathcal{O}_K$-modules, then

(A) $\mathcal{F}_x$ is generated by $\Gamma(K, \mathcal{F})$ as an $\mathcal{O}_{X,x}$-module for each $x \in K$.

(B) $H^q(K, \mathcal{F}) = 0$ for all $q \geq 1$.

**Lemma 3.2.** Let $X$ be an analytic subvariety of $\mathbb{C}^n$ and let $K$ be a holomorphically convex compact subset of $X$. Then

(i) $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}_X(K)$ if and only if $\mathfrak{m} = \mathfrak{m}_a = \{ f \in \mathcal{O}_X(K) \mid f(a) = 0 \}$ for some $a \in K$;

(ii) for every $a \in K$ the ideal $\mathfrak{m}_a$ is generated by $z_1 - a_1, \ldots, z_n - a_n$, where $z_1, \ldots, z_n$ are the canonical coordinates of $\mathbb{C}^n$;

(iii) for every $a \in K$ the canonical map $\mathcal{O}_X(K) \to \mathcal{O}_{X,a}$ is flat;

(iv) for every $a \in K$ the canonical map $\mathcal{O}_X(K)_{\mathfrak{m}_a} \to \mathcal{O}_{X,a}$ is a faithfully flat morphism of local rings which induces an isomorphism between the completions $(\mathcal{O}_X(K)_{\mathfrak{m}_a})^\wedge \cong (\mathcal{O}_{X,a})^\wedge$. In particular $\mathcal{O}(K)_{\mathfrak{m}_a}$ is a noetherian local ring.

**Proof.** The assertion (i) is proved in [17].

The assertions (ii) and (iii) follow from (B) and (A), respectively, of Theorem 3.1, by using classical methods.

The assertion (iv) follows from (ii) and (iii) by using the same technique used in [6] in the case of Stein compact sets. $\square$

Let $X$ be a Nash subvariety of $\mathbb{C}^n$ and $K$ a compact subset of $X$. Let $a \in K$ and let $\mathfrak{n}_a = \{ f \in \mathcal{N}_X(K) \mid f(a) = 0 \}$. We say that $K$ is an affine (Nash) compact set if the map $a \mapsto \mathfrak{n}_a$, is a bijection between the set of points of $K$ and the set of maximal ideals of $\mathcal{N}_X(K)$.

**Proposition 3.3.** Let $K$ be a holomorphically convex compact subset of a normal Nash subvariety of $\mathbb{C}^n$. Then
(i) The inclusion map $N_X(K) \to O_X(K)$ is faithfully flat if and only if $K$ is affine.

(ii) If $K$ is an affine semianalytic set then $N_X(K)$ is a noetherian ring.

**Proof.** In order to prove (i) we first observe that every maximal ideal in $O_X(K)$ is an ideal $m_a$ for some $a \in K$ (see (i) of Lemma 3.2). Hence, if $N_X(K) \to O_X(K)$ is faithfully flat and $n$ is a maximal ideal of $N_X(K)$ then $n = n_a = \{ f \in N_X(K) \mid f(a) = 0 \}$.

Vice versa, by localizing and applying (i) of Lemma 2.2 and (iii) of 3.2, because of the descending property of faithful flatness, we have that $N_X(K)_{m_a} \to O_X(K)_{m_a}$ is faithfully flat, and so $N_X(K) \to O_X(K)$ is flat. Since $K$ is affine, it is faithfully flat.

(iii) If $K$ is semianalytic then $O_X(K)$ is noetherian (see [17]), hence $N_X(K)$ is noetherian too, by the faithful flatness of $N_X(K) \to O_X(K)$. \( \square \)

**Lemma 3.4.** Let $K$ be an affine compact subset of a Nash subvariety of $\mathbb{C}^n$. Then $K$ is holomorphically convex and it is equal to the joint spectrum $\{ N_k(K)(z_1, \ldots, z_n) \}$, where $z_1, \ldots, z_n$ are the canonical coordinates of $\mathbb{C}^n$.

**Proof.** By a result of W.R. Zame (see [17, Proposition 2.2]), if for each character $\mu \in S(O_X(K))$ we have that $(\mu(z_1), \ldots, \mu(z_n)) \in K$, then $K$ is holomorphically convex. Now, if $\varepsilon : N_X(K) \to O_X(K)$ is the canonical inclusion map, then $Ker(\mu \varepsilon)$ is a maximal ideal, hence there exists $a \in K$ such that $Ker(\mu \varepsilon) = m_a$. This yields $\mu \varepsilon(z_j) = \mu(z_j) = a_j$ for $j = 1, \ldots, n$.

On the other hand, we have the equality $\sigma_{N_X(K)}(z_1, \ldots, z_n) = \{ \chi(z_1), \ldots, \chi(z_n) \mid \chi \in S(N_X(K)) \}$, which easily gives $K \subseteq \sigma_{N_X(K)}(z_1, \ldots, z_n)$. In order to see the opposite inclusion let us consider $a = (\chi(z_1), \ldots, \chi(z_n))$ with $\chi \in S(N_X(K))$, and the canonical map $\rho : N_X(K) \to N_X(K)$. By proceeding as above, it follows that $Ker(\chi \rho) = m_a$. \( \square \)

**Theorem 3.5.** Let $X$ be an affine algebraic subvariety of $\mathbb{C}^n$, let $K$ be an affine compact set of $X$ and $h \in O_X(K)^q$. Let $F_1, \ldots, F_q$ be Nash functions on an open neighborhood of the graph of $h$ in $X \times \mathbb{C}^q$, such that $F_i(id, h)$ is zero in $O_X(K)$, for every $i = 1, \ldots, q$. Then there exists $h \in N_X(K)^q$, arbitrarily close to $h$ in the $\ell F$ topology of $O_X(K)^q$, such that $F_i(id, h)$ is zero in $N_X(K)$, for every $i = 1, \ldots, q$.

**Proof.** We may assume that $h$ is defined on an open neighborhood, $V$, of $K$ in $X$ and also that $F_i(x, h(x)) = 0$, for every $x \in V$ and for each $i = 1, \ldots, q$. Let $W$ be an open set of $\mathbb{C}^n$, such that $V = W \cap X$. By Lemma 3.4 we have that $K = \sigma_{N_X(K)}(z_1, \ldots, z_n)$, hence the Theorem of Arens and Calderon (see, for instance, [5]) yields that there exist $f_1, \ldots, f_m \in N_X(K)$ such that $\pi(\delta_{N_X(K)}(z_1, \ldots, z_n, f_1, \ldots, f_m)) \subseteq W$, where $\pi$ is the canonical projection $\pi : (z_1, \ldots, z_n, y_1, \ldots, y_m) \mapsto (z_1, \ldots, z_n)$. Let us consider the closed affine subvariety of $\mathbb{C}^{n+m}$, $Z = X \times \mathbb{C}^m$, and let $Q = \delta_{N_X(K)}(z_1, \ldots, z_n, f_1, \ldots, f_m) \cap Z$; it results that $\pi(Q) \subseteq V$ and so (see, for instance, [5]) there exists a compact set $P$ of $Z$ which is $O_Z(Z)$-convex and such that $Q \subseteq P$ and $\pi(P) \subseteq V$. Since

$$\{ (z, f_1(z), \ldots, f_m(z)) \mid z \in K \} \subseteq \sigma_{N_X(K)}(z_1, \ldots, z_n, f_1, \ldots, f_m) \cap Z \subseteq Q \subseteq P,$$
then there exists an open neighborhood \( U \) of \( K \) in \( X \) and there exist Nash functions \( \phi_1, \ldots, \phi_m \in \mathcal{N}_X(U) \) such that \( \{(z, \phi(z)) \mid z \in U\} \subset \hat{P} \), where \( \phi = (\phi_1, \ldots, \phi_m) \). Let \( g = h\pi \in \mathcal{O}_Z(\pi^{-1}(V) \cap Z)^s \) and let \( G_i \) be the Nash functions defined by \( G_i(z, y, t) = F_i(z, t) \), for every \( (z, y, t) \in \mathbb{C}^{n+m+p} \) such that \( F_i(z, t) \) is defined, \( i = 1, \ldots, q \). It follows that the functions \( G_i \) are defined on an open neighborhood of the graph of \( g \) in \( Z \times \mathbb{C}^s \) and for every \( (z, y) \in \pi^{-1}(V) \cap Z \) we have the equalities \( G_i(z, y, g(z, y)) = F_i(z, h(z)) = 0 \), for every \( i = 1, \ldots, q \).

By Theorem 2.3 we can say that there exists \( \tilde{g} \in \mathcal{N}_Z(P) \) which approximates \( g \) and such that \( G_i(id, id, \tilde{g}) \) is zero in \( \mathcal{N}_Z(P) \). We set \( \tilde{h} \in \mathcal{N}_X(U) \) as the map defined by \( \tilde{h}(z) = \tilde{g}(z, \phi(z)) \) for every \( z \in U \). If the approximation is good enough then \( \tilde{h} \) approximates \( h \) on every compact neighborhood of \( K \) in \( U \) and furthermore for every \( z \in U \) we have that \( F_i(z, \tilde{h}(z)) = G_i(z, \phi(z), \tilde{g}(z, \phi(z))) = 0 \), for every \( i = 1, \ldots, q \).

\[ \blacksquare \]

**Theorem 3.6.** Let \( X \) be a normal Nash subvariety of \( \mathbb{C}^n \) and let \( K \) be a compact subset of \( X \). The following assertions are equivalent

(i) \( K \) is affine.

(ii) \( K \) is holomorphically convex and \( \mathcal{N}_X(K) \) is dense in \( \mathcal{O}_X(K) \).

**Proof.** By Theorem 2.1 we may assume that \( X \) is an affine algebraic subvariety of \( \mathbb{C}^n \). In order to show that (i) implies (ii) we observe that, by Lemma 3.4, \( K \) is holomorphically convex and so, by Theorem 3.5, \( \mathcal{N}_X(K) \) is dense in \( \mathcal{O}_X(K) \).

Now let us show that (ii) implies (i). Let \( n \) be a maximal ideal of \( \mathcal{N}_X(K) \); if \( n \mathcal{O}_X(K) \) is a proper ideal we can show, as in 3.3, that there exists \( a \in K \) such that \( n = n_a \). If \( n \mathcal{O}_X(K) = \mathcal{O}_X(K) \) then there exist \( f_i \in n, h_i \in \mathcal{O}_X(K) \), \( i = 1, \ldots, s \), such that \( 1 = \sum_i h_i f_i \). By Theorem 3.5 we can find \( \tilde{h}_1, \ldots, \tilde{h}_s \in \mathcal{N}_X(K) \) arbitrarily close to \( h_1, \ldots, h_s \) and so with the property that \( \sum_i \tilde{h}_i f_i \) is a unit in \( \mathcal{N}_X(K) \), therefore \( n = \mathcal{N}_X(K) \).

\[ \blacksquare \]

**Remark 3.7.** Obviously, every compact set which is Nash isomorphic to an affine compact set is affine. If \( X \) is an affine algebraic subvariety then each rationally convex compact subset of \( X \) is an affine compact set (see [14]). On the other hand, Wermer’s classical construction (see, e.g., [5]) easily gives examples of compact subsets of \( \mathbb{C}^3 \) which are Nash isomorphic images of closed polycylinders of \( \mathbb{C}^3 \) but they are not polynomially convex; such compact sets are not even rationally convex (see [13]). Here we only observe that Theorem 3.5 allows us to extend to affine compact sets the results on the extension and separation of [14], by following the ideas in [3].

4. Vector bundles

Let \( X \) be a Nash subvariety of \( \mathbb{C}^n \) and let \( \pi : F \to X \) be an analytic vector bundle. We say that \( F \) is a **Nash vector bundle** if \( F \) is a Nash variety, \( \pi \) is a Nash map and \( F \) has a family of trivializations such that their cocycles are Nash maps into some Grassmannian. Observe that, in general, \( F \) is not a Nash subvariety of some \( \mathbb{C}^N \) (see [16]).
Let $K$ be a compact subset of $X$, by a Nash (analytic) vector bundle on $K$ we mean a locally free sheaf, $E$, of $N_K$-modules (resp. $O_K$-modules) of finite rank.

Since the sheaf $N_K$ (resp. $O_K$) is coherent, vector bundles on $K$ can be extended to locally free sheaves of $N_X|_{\Omega}$-modules ($O_X|_{\Omega}$-modules) on an open neighborhood $\Omega$ of $K$ in $X$, isomorphisms can be extended to isomorphisms too. It follows that every Nash (analytic) vector bundle $E$ on $K$ can be seen as a usual Nash (analytic) bundle, $E$, on an open neighborhood $\Omega$ of $K$ in $X$. Indeed, the locally free sheaf $E$ of $N_K$-modules ($O_K$-modules) is the restriction to $K$ of the sheaf of the Nash (analytic) sections of $E$.

Therefore we shall refer to vector bundles on $K$ by using the same notations of vector bundles.

There is a bijection between the isomorphism classes of Nash (analytic) vector bundle of rank 1 on $K$ and $H^1(K, N^*_X)$ (resp. $H^1(K, O^*_X)$), where, we denote by $N^*_X$ (resp. $O^*_X$) the sheaf of non-vanishing Nash (resp. analytic) functions on $X$.

If $E$ is a Nash (analytic) vector bundle on $K$, we denote by $\Gamma(K, E)$ the space of its global sections. When necessary, we shall denote by $\Gamma^N(K, E)$ the Nash sections and by $\Gamma^O(K, E)$ the analytic sections.

**Lemma 4.1.** Let $K$ be a holomorphically convex compact set of a separated analytic space. Then

(i) $H^1(K, O^*_K) \cong H^2(K, \mathbb{Z})$.

(ii) Each exact sequence of coherent $O_K$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

with $\mathcal{F}''$ locally free, splits.

**Proof.** It follows from 3.1, by using classical techniques (see, for instance, [5]).

**Lemma 4.2.** Let $X$ be a normal Nash subvariety of $\mathbb{C}^n$ and let $K$ be an affine compact subset of $X$. If $E$ is an analytic vector bundle on $K$ of rank 1 then there exists a Nash vector bundle on $K$, $\tilde{E}$, of rank 1 and an analytic isomorphism $E \cong \tilde{E}$.

**Proof.** Because of 2.1 we may assume that $X$ is an algebraic subvariety of $\mathbb{C}^n$. It follows from (A) of Theorem 3.1 and from (ii) of Lemma 4.1 that $E$ can be considered as a subbundle of a trivial bundle $K \times \mathbb{C}^s$, and so it is the image of a bundle endomorphism $\sigma : (x, u) \mapsto (x, A(x)u)$, where $A$ is an $s \times s$ matrix with entries in $O_X(K)$ such that $A^2 = A$ and the rank of $A(x)$ is equal to 1, $\forall x \in K$. By Theorem 3.5 there exists an $s \times s$ matrix $\tilde{A}$ with entries in $N_X(K)$ such that $\tilde{A}^2 = \tilde{A}$; furthermore if the approximation is good enough we have that the rank of $\tilde{A}(x)$ is equal to 1 for every $x \in K$. Now let us consider the endomorphism of $K \times \mathbb{C}^s$ defined by $\tilde{\sigma} : (x, u) = (x, \tilde{A}(x)u)$: its image is a Nash vector bundle on $K$ of rank 1. Let $\psi, \tilde{\psi} : K \to \mathbb{P}^s(\mathbb{C})$ be the analytic and the Nash maps with the property that $E = \psi^*(O_{\mathbb{P}^s}(1))$ and $\tilde{E} = \tilde{\psi}^*(O_{\mathbb{P}^s}(1))$. If the approximation is good enough then we have that $\psi$ and $\tilde{\psi}$ are homotopic and so $E$ and $\tilde{E}$ are topologically equivalent. It follows from (i) of Lemma 4.1 that there exists an analytic isomorphism $E \cong \tilde{E}$. 

\[\square\]
Lemma 4.3. Let $X$ be a normal Nash subvariety of $\mathbb{C}^n$ and $K$ an affine compact subset of $X$. Let $E$ and $F$ be Nash vector bundles on $K$ such that $\text{Hom}(E, F)$ is generated by a finite number of global Nash sections. If $E$ and $F$ are isomorphic as analytic bundles then they are isomorphic as Nash bundles.

**Proof.** An analytic isomorphism between $E$ and $F$ is given by a section $s$ of the bundle $\text{Hom}(E, F)$, which is generated by a finite number of global Nash sections $s_1, \ldots, s_p$. By Theorem 3.1 we have that there exist $h_1, \ldots, h_p \in O_X(K)$ such that $s = \sum s_i h_i$. Theorem 3.5 yields that there exist $\tilde{h}_i \in O_X(K)$ which approximates $h_i$ for each $i = 1, \ldots, p$. If the approximations are good enough then the section $\tilde{s} = \sum s_i \tilde{h}_i$ defines an isomorphism between $E$ and $F$. $\square$

Lemma 4.4. Let $X$ be a normal Nash subvariety of $\mathbb{C}^n$ and $K$ an affine compact subset of $X$. A Nash bundle $E$ on $K$ is generated by a finite number of global Nash sections if and only if it is Nash isomorphic to a direct summand of a trivial bundle.

**Proof.** It follows from Lemma 4.1 that there exists an analytic vector bundle $F$ such that $E \oplus F$ is analytically isomorphic to a trivial bundle $K \times \mathbb{C}^s$. By Lemma 4.3 there exists a Nash isomorphism $E \oplus F \cong K \times \mathbb{C}^s$ and so $E$ is Nash isomorphic to a direct summand of a trivial bundle.

The converse is obvious. $\square$

Proposition 4.5. Let $X$ be a normal Nash subvariety of $\mathbb{C}^n$ and $K$ an affine compact subset of $X$. Let $E$ and $F$ be Nash vector bundles on $K$ which are generated by a finite number of global Nash sections. If $E$ and $F$ are isomorphic as analytic bundles then they are isomorphic as Nash bundles.

**Proof.** By Lemma 4.4 the vector bundle $\text{Hom}(E, F)$ is generated by a finite number of global Nash sections and then the conclusion follows from Lemma 4.3. $\square$

Proposition 4.6. Let $X$ be a normal Nash subvariety of $\mathbb{C}^n$, $K$ an affine compact subset of $X$ and $E$ a Nash vector bundle which is generated by a finite number of global Nash sections. Then there exists a canonical isomorphism between $\Gamma_K(K, E)$ and $\Gamma_K(K, E) \otimes_{N_K(K)} O_K(K)$.

**Proof.** By (B) of Theorem 3.1 we get that the map $(s, f) \mapsto sf$ from $\Gamma_K(K, E) \otimes O_K(K)$ to $\Gamma_K(K, E)$ induces a surjective map $\Gamma_K(K, E) \otimes_{N_K(K)} O_K(K) \to \Gamma_K(K, E)$ which is clearly injective if $E$ is trivial. The conclusion follows from the Lemma 4.4. $\square$

5. Factorization

Let $A$ be a noetherian (normal) ring. We denote by $\text{Pic}(A)$ the isomorphism classes of invertible projective $A$-modules of rank 1.
Let $X$ be a normal Nash subvariety of $\mathbb{C}^n$ and $K$ an affine compact subset of $X$ which is also semianalytic. The rings $\mathcal{O}_K(K)$ and $\mathcal{N}_K(K)$ are normal and noetherian (see (iii) of Lemma 3.3).

Using the same notation as in the previous section, we denote by $\tilde{M}$ the sheaf $\mathcal{N}_K \otimes \mathcal{N}_K(K) M$ (resp. $\mathcal{O}_K \otimes \mathcal{O}_K(K) M$) where $M$ is an $\mathcal{N}_K(K)$-module (resp. $\mathcal{O}_K(K)$-module). If $M$ is an $\mathcal{N}_K(K)$-module (resp. $\mathcal{O}_K(K)$-module) of finite type, then $\tilde{M}$ is a coherent sheaf of $\mathcal{N}_K$-modules (resp. $\mathcal{O}_K$-modules). If $M$ is also projective then $\tilde{M}$ is a locally free sheaf of finite rank which is generated by its global sections. Furthermore we have the following canonical isomorphisms $M \cong \Gamma(\mathcal{N}_K(K), \tilde{M})$ (resp. $M \cong \Gamma(\mathcal{O}_K(K), \tilde{M})$).

If $L$ is a locally free sheaf of $\mathcal{O}_K$-modules, then Theorem 3.1 and Lemma 4.1 imply that $L(K)$ is a projective $\mathcal{O}_K(K)$-module of finite type and we have a canonical isomorphism $L \cong \tilde{L}(K)$. We also have such an isomorphism for a locally free sheaf of $\mathcal{N}_K$-modules which is generated by a finite number of global sections (see Lemma 4.4).

By (ii) of Lemma 2.2 and (iv) of Lemma 3.2, the map $M \mapsto \tilde{M}$ induces an isomorphism $\text{Pic}(\mathcal{N}_K(K)) \rightarrow H^1(K, \mathcal{N}_K^*)$ and $\text{Pic}(\mathcal{O}_K(K)) \rightarrow H^1(K, \mathcal{O}_K^*)$.

**Theorem 5.1.** Let $X$ be a normal Nash subvariety of $\mathbb{C}^n$ and let $K$ be an affine semianalytic compact subset of $X$. Then the map $M \mapsto M \otimes_{\mathcal{N}_K(K)} \mathcal{O}_K(K)$ induces an isomorphism $\text{Pic}(\mathcal{N}_K(K)) \rightarrow \text{Pic}(\mathcal{O}_K(K))$.

**Proof.** Let $M$ and $N$ be projective $\mathcal{N}_K(K)$-modules of rank 1 and let $\tilde{M}$, $\tilde{N}$ be the associated Nash vector bundles. If $M \otimes_{\mathcal{N}_K(K)} \mathcal{O}_K \cong N \otimes_{\mathcal{N}_K(K)} \mathcal{O}_K$ then, by Proposition 4.6, $\Gamma(\mathcal{O}_K(K), \tilde{M}) \cong M \otimes_{\mathcal{N}_K(K)} \mathcal{O}_K \cong N \otimes_{\mathcal{N}_K(K)} \mathcal{O}_K \cong \Gamma(\mathcal{O}_K(K), \tilde{N})$; this implies that there exists an isomorphism $(\Gamma(\mathcal{O}_K(K), \tilde{M})) \rightarrow \Gamma(\mathcal{O}_K(K), \tilde{N})$, then the bundles $\tilde{M}$ and $\tilde{N}$ are isomorphic as analytic bundles and then by Proposition 4.5 they are isomorphic as Nash bundles, hence $M$ and $N$ are isomorphic as $\mathcal{N}_K(K)$-modules.

In order to prove that the map is surjective, let $M$ be a projective $\mathcal{O}_K(K)$-module of rank 1 and let $\tilde{M}$ be the associated analytic bundle. By Lemma 4.2 there exist a Nash bundle $E$ and an analytic isomorphism $E \cong \tilde{M}$ and so $\Gamma(\mathcal{O}_K(K), E)$ is isomorphic to $M$ as $\mathcal{O}_K(K)$-modules. By Proposition 4.6, we have $M \cong \Gamma(\mathcal{N}_K(K), E) \otimes_{\mathcal{N}_K(K)} \mathcal{O}_K(K)$. \qed

Let $A$ be a Krull domain, we denote by $D(A)$ the group of fractionary divisorial ideals of $A$ and by $\text{Prin}(A)$ the subgroup of principal divisorial ideals of $A$.

**Definition 5.2.** The Divisor Class group of $A$ is the group $D(A)/\text{Prin}(A)$ and it is denoted by $\text{Cl}(A)$.

The proofs of the two following theorems can be found in [1].

**Theorem 5.3.** If $A$ is noetherian and locally factorial then $\text{Pic}(A) \cong \text{Cl}(A)$.

**Theorem 5.4.** $A$ is factorial if and only if $\text{Cl}(A) = 0$.

**Lemma 5.5.** Let $K$ be an affine compact subset of a factorial Nash subvariety $X$ of $\mathbb{C}^n$. Then $\mathcal{N}_K(K)$ is locally factorial.
Proof. We need to prove that for every maximal ideal \( m = m_a \subset N_K(K) \) the ring \( N_K(K)_{m_a} \) is factorial. In order to do this it suffices to show that every prime ideal of height 1 of \( N_K(K)_{m_a} \) is principal. By (iii) of Lemma 2.2 we have that \( N_{X,a} = h N_K(K)_{m_a} \). Now let \( p \subset N_K(K)_{m_a} \) be a prime ideal with \( ht(p) = 1 \). By (iv) of Lemma 2.2 we have that \( p N_{X,a} = q_1 \cap \cdots \cap q_r \) with \( q_i \) prime ideals and \( ht(q_i) = 1 \) for every \( i = 1, \ldots, r \). Since \( N_K(K) \) is factorial we have that \( p N_{X,a} \) is principal and so \( p(N_K(K)_{m_a}) = p(N_K(K)_{m_a}) \) is principal too, which implies that \( p \) is a principal prime ideal (see [8]).

\[ \square \]

Theorem 5.6. Let \( X \) be a factorial Nash subvariety of \( \mathbb{C}^n \) and let \( K \) be an affine semianalytic compact subset of \( X \). The following conditions are equivalent

(i) \( N_K(K) \) is factorial.
(ii) \( H^2(K, \mathbb{Z}) = 0 \).

Proof. In order to prove that (i) and (ii) are equivalent we observe that by 5.1 there is an isomorphism \( \text{Pic}(N_K(K)) \cong \text{Pic}(O_K(K)) \). Since \( X \) is factorial the ring \( N_K(K) \) is locally factorial and so \( \text{Cl}(N_K(K)) \cong \text{Pic}(N_K(K)) \). From (i) of Lemma 4.1 and from Theorem 5.1, we have: \( \text{Pic}(N(K)) \cong \text{Pic}(O(K)) \cong H^1(K, O^*_K) \cong H^2(K, \mathbb{Z}) \) and so the conclusion follows easily. \[ \square \]

References