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Approximating smooth planar curves by arc splines[☆]

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Abstract

When a smooth curve is used to describe the path of a computer-controlled cutting machine, the path is usually approximated by many straight line segments. It is preferable to describe the cutting path as an arc spline, a tangent continuous piecewise curve made of circular arcs and straight line segments. This paper presents an algorithm for finding an arbitrarily close arc spline approximation of a smooth curve.

Keywords: Arc spline; Accuracy of approximation

1. Introduction

The standard practice for cutting a smooth curve with a computer-controlled cutting machine is to cut a polygon that is very close to the smooth curve. This method gives a continuous path with a discontinuous unit tangent vector, which causes some problems [5]. Modern cutting machines are capable of cutting a tangent continuous path called an arc spline. An arc spline is formed from straight line segments and circular arcs [4, 6, 8]. In [5] a quadratic NURBS (nonuniform rational B-spline) curve is approximated by an arc spline. In this paper, the problem of approximating a segment of a more general smooth curve is discussed. A smooth curve will be taken to mean a curve with continuous third derivatives with respect to arc length.

Here arc splines will be formed by joining biarcs. A biarc from a point $\boldsymbol{A}$ to a distinct point $\boldsymbol{B}$ is a curve made by joining two circular arcs that start at $\boldsymbol{A}$ and end at $\boldsymbol{B}$ so that their tangents match at the joining point. The angle from the tangent at $\boldsymbol{A}$ to $\boldsymbol{B}-\boldsymbol{A}$ and the angle from $\boldsymbol{B}-\boldsymbol{A}$ to the tangent at $\boldsymbol{B}$ are in $(-\pi, \pi)$. The angles of the sectors formed by each arc are in $(-2\pi, 2\pi)$. This definition of the biarc appeared in [4].

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Assume the smooth curve segment to be approximated has a finite number of curvature extrema and inflection points. These special points partition the curve into spiral segments. The theory for approximating spiral segments by arc splines appears in Section 2. An algorithm for approximating any smooth curve segment, whether it is a spiral or includes curvature extrema or inflection points, appears in Section 3.

2. Theoretical results

In this section, the approximation of smooth planar spiral segments is considered. Without loss of generality, assume the spirals have positive curvature that increases when the arc length is increasing.

Definition 1. Consider the family of circular arcs joining one given point $\boldsymbol{A}$ to another distinct given point $\boldsymbol{B}$. The two circular arcs that match a given unit tangent vector at the first point $\boldsymbol{t}_A$ and that match a given unit tangent vector at the second point $\boldsymbol{t}_B$ will be called the *bounding circular arcs* C_A and C_B (see Fig. 1).

Theorem 2. *Any biarc (as defined in the Introduction) that joins one point to another distinct point and matches given unit tangent vectors at the two points lies between the bounding circular arcs that are derived from the two points and the two unit tangent vectors.*

There is a one parameter family of biarcs that join two given points and match given unit tangent vectors at those points. The bounding curves of the family of biarcs are the bounding circular arcs of the above definition. The proof of Theorem 2 is given in [5].

Definition 3. A *convex curve* is a curve that has the property that if its endpoints are joined by a straight line, the enclosed region is a convex region [2].

Definition 4. Bounding circular arcs C_A and C_B can be derived from the two endpoints $\boldsymbol{A}$ and $\boldsymbol{B}$ and the two unit tangent vectors $\boldsymbol{t}_A$ and $\boldsymbol{t}_B$ of a segment of a spiral of positive increasing curvature.

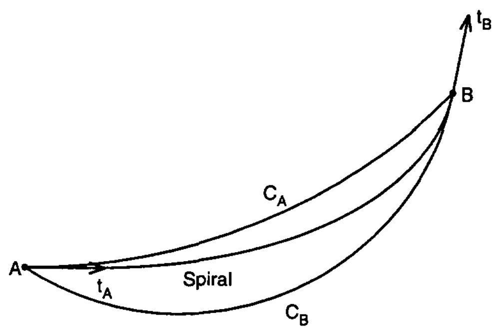

Fig. 1. Bounding circular arcs.

The spiral segment is said to satisfy the *enclosing condition* if the curvature of the spiral at $\boldsymbol{A}$ is less than or equal to the curvature of C_A and the curvature of the spiral at $\boldsymbol{B}$ is greater than or equal to the curvature of C_B (see Fig. 1).

Theorem 5. *If a convex spiral segment of positive increasing curvature satisfies the enclosing condition, then the bounding circular arcs enclose a crescent-shaped region that includes the entire spiral segment.*

Proof. Suppose a convex spiral joins $\boldsymbol{A}$ to $\boldsymbol{B}$ and the bounding circular arcs for that spiral segment are C_A and C_B as in Fig. 1. The enclosing condition means that the bounding circular arcs both have positive curvature and enclose a crescent-shaped region, and that the spiral is between the bounding circular arcs near $\boldsymbol{A}$ and $\boldsymbol{B}$. A result in [2, p.53] states that a circle and a convex spiral can have at most three points of intersection, or one point of contact and one noncontact intersection. Thus, the spiral here can intersect the bounding circular arcs only at $\boldsymbol{A}$ and $\boldsymbol{B}$, and the bounding circular arcs must enclose the entire spiral segment. □

Some notation regarding smooth curve segments will now be established. Assume the curve segment $\boldsymbol{Q}(s)$, $s_0 \leqslant s \leqslant s_1$, is parametrized in terms of arc length, and let the arc length be $h = s_1 - s_0$. Using $\boldsymbol{t} = \boldsymbol{t}(s_0)$ for the unit tangent vector at $\boldsymbol{Q}(s_0)$, $\boldsymbol{n} = \boldsymbol{n}(s_0)$ for the unit normal at $\boldsymbol{Q}(s_0)$, and employing the Frenet formulae for curves in the plane, the derivatives of $\boldsymbol{Q}(s)$ with respect to arc length are

$$\boldsymbol{Q}'(s_0) = \boldsymbol{t}, \quad \boldsymbol{Q}''(s_0) = k\boldsymbol{n}, \quad \boldsymbol{Q}'''(s_0) = -k^2\boldsymbol{t} + k'\boldsymbol{n},$$

where $k = k(s_0)$ and $k' = (\mathrm{d}/\mathrm{d}s)k(s_0)$ are the curvature and its derivative with respect to arc length at s_0. It is convenient to use a coordinate system based on $\boldsymbol{t}$ and $\boldsymbol{n}$ as the X and Y axes. Taylor series expansion gives

$$\boldsymbol{Q}(s_1) - \boldsymbol{Q}(s_0) = \boldsymbol{Q}(s_0 + h) - \boldsymbol{Q}(s_0) = \begin{pmatrix} h - \frac{1}{6}k^2h^3 + \mathrm{O}(h^4) \\ \frac{1}{2}kh^2 + \frac{1}{6}k'h^3 + \mathrm{O}(h^4) \end{pmatrix}. \tag{1}$$

Finally, the notation $\boldsymbol{A} \times \boldsymbol{B}$ shall be taken to mean the scalar

$$\boldsymbol{A} \times \boldsymbol{B} = \|\boldsymbol{A}\| \cdot \|\boldsymbol{B}\| \sin\psi = A_xB_y - A_yB_x, \tag{2}$$

where ψ is the anticlockwise angle from vector

$$\boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

to vector

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ B_y \end{pmatrix}.$$

Theorem 6. *Let $\boldsymbol{Q}(s)$, $s_0 \leqslant s \leqslant s_1$, be a smooth spiral segment of positive increasing curvature for which the derived bounding circular arcs have curvatures of the same sign, then the maximum distance between the two bounding circular arcs is $\mathrm{O}(h^3)$, where $h = s_1 - s_0$.*

Proof. Since $\boldsymbol{Q}(s)$ is a spiral of positive increasing curvature, $k > 0$ and $k' > 0$. Let $d = \|\boldsymbol{Q}(s_1) - \boldsymbol{Q}(s_0)\|$, let α be the angle from $\boldsymbol{t}$ to $\boldsymbol{Q}(s_1) - \boldsymbol{Q}(s_0)$, let β be the angle from $\boldsymbol{Q}(s_1) - \boldsymbol{Q}(s_0)$ to $\boldsymbol{t}(s_1)$, let arc E be the bounding circular arc with tangent parallel to $\boldsymbol{t}$ at $\boldsymbol{Q}(s_0)$, and let arc F be the bounding circular arc with tangent parallel to $\boldsymbol{t}(s_1)$ at $\boldsymbol{Q}(s_1)$ (see Fig. 2).

The distance between $\boldsymbol{Q}(s_0)$ and $\boldsymbol{Q}(s_1)$ is {use (1) and (A.1)},

$$d = h - \tfrac{1}{24}k^2h^3 + \mathrm{O}(h^4), \tag{3}$$

and from (2),

$$d \sin\alpha = \boldsymbol{t} \times (\boldsymbol{Q}(s_1) - \boldsymbol{Q}(s_0)).$$

Dividing the above by d {(1), (2), (3)},

$$\sin\alpha = \tfrac{1}{2}kh + \tfrac{1}{6}k'h^2 + \mathrm{O}(h^3), \tag{4}$$

and by (A.2) α is

$$\alpha = \tfrac{1}{2}kh + \tfrac{1}{6}k'h^2 + \mathrm{O}(h^3). \tag{5}$$

Similarly,

$$d \sin\beta = (\boldsymbol{Q}(s_1) - \boldsymbol{Q}(s_0)) \times \boldsymbol{t}(s_1),$$

where $\boldsymbol{t}(s_1)$ is the derivative of (1), so

$$\sin\beta = \tfrac{1}{2}kh + \tfrac{1}{3}k'h^2 + \mathrm{O}(h^3),$$

and

$$\beta = \tfrac{1}{2}kh + \tfrac{1}{3}k'h^2 + \mathrm{O}(h^3). \tag{6}$$

Consider for a moment just arc E (see Fig. 3); the radius UW of arc E is

$$\frac{1}{K_E} = \frac{d}{2\sin\alpha} \tag{7}$$

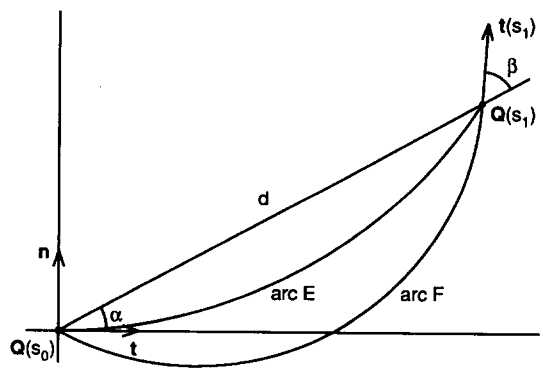

Fig. 2. Notation for bounding circular arcs of a spiral segment $\boldsymbol{Q}(s)$.

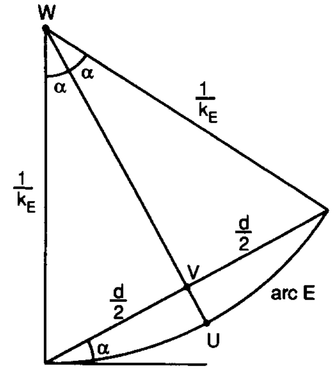

Fig. 3. Bounding circular arc E.

and VW is $d/(2\tan\alpha)$. The distance UV is

$$\frac{d}{2\sin\alpha}-\frac{d}{2\tan\alpha}=\frac{d}{2}\tan\frac{\alpha}{2}.$$

The maximum distance between the two bounding arcs E and F is the difference of two distances like UV, and is {(3), (5), (6), (A.3)}:

$$\frac{d}{2}\left(\tan\frac{\beta}{2}-\tan\frac{\alpha}{2}\right)=\frac{k'}{24}h^3+\mathrm{O}(h^4). \tag{8}$$

Formula (8) proves the theorem. □

Theorem 7. *If a given spiral of positive increasing curvature is repeatedly divided into segments such that the lengths of the segments approach zero, each segment will eventually be enclosed by the bounding circular arcs derived from that segment.*

Proof. Consider the spiral segment $\boldsymbol{Q}(s)$ from $\boldsymbol{Q}(s_0)$ to $\boldsymbol{Q}(s_1)$, and let $h=s_1-s_0$. The curvatures of $\boldsymbol{Q}(s)$ at s_0 and s_1 are k and $k(s_1)=k+k'h+\mathrm{O}(h^2)$. From the results in Theorem 6, the curvature of the bounding circular arc E is {(3), (4), (7)}:

$$k_E=\frac{2\sin\alpha}{d}=k+\frac{k'}{3}h+\mathrm{O}(h^2),$$

and similarly the curvature of bounding circular arc F is

$$k_F=\frac{2\sin\beta}{d}=k+\frac{2k'}{3}h+\mathrm{O}(h^2).$$

For small enough h, the following is true: the signs of the curvatures of the bounding circular arcs are the same; $k<k_E$ and $k_F<k(s_1)$ ($\boldsymbol{Q}(s)$ is a spiral of positive increasing curvature, $k>0$ and $k'>0$); the enclosing condition is satisfied, and the spiral is enclosed by the bounding circular arcs according to Theorem 5. □

The final two theorems are concerned with the accuracy of the approximation by arc splines. Theorem 8 gives the order of the error and Theorem 9 gives the leading term of the error.

Theorem 8. *If the bounding circular arcs enclose a given spiral segment of positive curvature* $\boldsymbol{Q}(s)$, $s_0 \leqslant s \leqslant s_1$, *and a biarc (as defined in the Introduction) that matches the same data as the bounding circular arcs is found, then the maximum distance between the biarc and the spiral is* $\mathrm{O}(h^3)$, *where* $h = s_1 - s_0$.

Proof. If a spiral segment is enclosed by bounding circular arcs, the same bounding circular arcs enclose the corresponding biarc (Theorem 2). The maximum distance between the bounding circular arcs is $\mathrm{O}(h^3)$ (Theorem 6). This proves Theorem 8. □

Theorem 9. *The maximum distance between the biarc and the spiral as described in Theorem* 8 *is*

$$\frac{k'}{324}h^3 + \mathrm{O}(h^4).$$

Proof. Consider the one parameter family of biarcs from $\boldsymbol{Q}(s_0)$ to $\boldsymbol{Q}(s_1)$ with θ being the parameter. The radii of the two circular arcs from which the biarc is formed are [4]

$$R_0 = \frac{\sin\frac{1}{2}(\beta - \alpha + \theta)}{2\sin\frac{1}{2}\theta\sin\frac{1}{2}(\alpha+\beta)}d \tag{9}$$

and

$$R_1 = \frac{\sin\frac{1}{2}(2\alpha - \theta)}{2\sin\frac{1}{2}(\alpha+\beta-\theta)\sin\frac{1}{2}(\alpha+\beta)}d, \tag{10}$$

where α, β, and d are defined in Theorem 6 (also see Fig. 2), and θ lies in the interval $(\alpha - \beta, 2\alpha)$.

It is usual to choose θ as a weighted average of α and β. For example, $\theta = \alpha$ minimizes $|R_0/R_1 - 1|$ [7], $\theta = \frac{1}{2}(\alpha+\beta)$ minimizes $|R_1 - R_0|$ [8], and $\theta = \frac{1}{2}(3\alpha - \beta)$ minimizes $|1/R_1 - 1/R_0|$ [8]. For these three choices, $\theta = \frac{1}{2}kh + \theta_2 h^2 + \mathrm{O}(h^3)$ {(5), (6)}. Only the choice $\theta = \alpha - \beta$, which is not allowed by the range for θ, would give $\theta = \mathrm{O}(h^2)$:

Without loss of generality, only the deviation between $\boldsymbol{Q}(s)$ and the first circular arc of the biarc need be calculated in detail. The formula for R_0 is {(3), (5), (6), (9), (A.4)}:

$$R_0 = \frac{1}{k} - \frac{k'}{6k^2}h + \mathrm{O}(h^2). \tag{11}$$

The above calculation is easier if you divide the two factors that depend on θ first. The radial distance or deviation between the curve $\boldsymbol{Q}(s)$ and the first circular arc is

$$D(s) = \|\boldsymbol{Q}(s) - \boldsymbol{C}_0\| - \boldsymbol{R}_0, \tag{12}$$

where the centre of the first circular arc $\boldsymbol{C}_0$ is

$$\boldsymbol{C}_0 = \boldsymbol{Q}(s_0) + R_0\boldsymbol{n}. \tag{13}$$

Let $g = s - s_0$, and let $\mathrm{O}((g, h)^k)$ stand for a power series in g and h with terms of order k and higher. The deviation $D(s)$ can be expressed {(1), (13), (A.1), (11)}:

$$
\begin{aligned}
D(s) &= \| \boldsymbol{Q}(s) - \boldsymbol{Q}(s_0) - R_0 \boldsymbol{n} \| - R_0 \\
&= \left\| \begin{pmatrix} g - \frac{1}{6}k^2 g^3 + \mathrm{O}(g^4) \\ \frac{1}{2}k\, g^2 + \frac{1}{6}k'\, g^3 + \mathrm{O}(g^4) \end{pmatrix} + \begin{pmatrix} 0 \\ -R_0 \end{pmatrix} \right\| - R_0 \\
&= R_0 \sqrt{1 + \frac{1}{R_0}\left(\frac{1}{R_0} - k\right) g^2 - \frac{k'}{3R_0} g^3 + \mathrm{O}(g^4)} - R_0 \\
&= R_0 \left(1 + \frac{1}{2R_0}\left(\frac{1}{R_0} - k\right) g^2 - \frac{k'}{6R_0} g^3 + \mathrm{O}(g^4)\right) - R_0 \\
&= \frac{1}{2}\left(\frac{1}{R_0} - k\right) g^2 - \tfrac{1}{6}k'\, g^3 + \mathrm{O}(g^4) \\
&= \tfrac{1}{12}k'\, g^2 (h - 2g) + \mathrm{O}((g, h)^4).
\end{aligned} \tag{14}
$$

Asymptotically, $D(s)$ has a double zero at $g = 0$ or $s = s_0$ because the curve and biarc match tangents there. Asymptotically, $D(s)$ also has a zero at $g = \frac{1}{2}h$, and this means the first circular arc of the biarc meets the spiral at $\boldsymbol{Q}(s_0 + \frac{1}{2}h)$. For the commonly used biarcs, $\theta = \frac{1}{2}kh + \mathrm{O}(h^2)$, the joining point of the two arcs of the biarc is {(13), (A.4), (A.5), (11), (1)}:

$$
\begin{aligned}
&\boldsymbol{C}_0 + R_0 \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
&\quad = \boldsymbol{Q}(s_0) + \begin{pmatrix} \frac{1}{2}h + \mathrm{O}(h^2) \\ \mathrm{O}(h^2) \end{pmatrix} \\
&\quad = \boldsymbol{Q}(s_0 + \tfrac{1}{2}h) + \mathrm{O}(h^2).
\end{aligned}
$$

The above show that asymptotically the first circular arc ends at $\boldsymbol{Q}(s_0 + \frac{1}{2}h)$, and thus g is in $[0, \frac{1}{2}h]$ for the first circular arc. The biarc crosses the spiral at the joining point of the two arcs that make up the biarc. Let $f(g) = g^2(h - 2g)$; the maximum value of this expression for g in $[0, \frac{1}{2}h]$ occurs when $g = \frac{1}{3}h$. The maximum deviation of the first circular arc from the spiral is from (14)

$$
\frac{k'}{324} h^3 + \mathrm{O}(h^4). \tag{15}
$$

An analogous result follows for the second circular arc, but that deviation has the opposite sign of the above. □

Corollary. *As the length of a segment of a spiral approaches zero, the distance from the spiral to the approximating biarc approaches $\frac{1}{13.5}$ of the distance between the bounding circular arcs.*

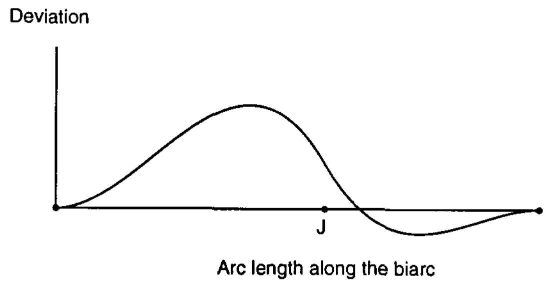

Fig. 4. Deviation between a cubic Bézier and the corresponding biarc.

Proof. The maximum distance between the bounding circular arcs (8) divided by the maximum distance from $Q(s)$ to the biarc (15) is $\frac{324}{24} = 13.5$. □

The result in the corollary could be used to reduce calculations. To find a biarc within ε of a given spiral, one could subdivide until the maximum distance between bounding circular arcs is ε. However, the corollary shows that an error of ε will be achieved (asymptotically as h approaches zero) when the maximum distance between the bounding circular arcs is 13.5ε.

An illustration of some of the results in Theorem 9 is given in Fig. 4. The radial distance (12) from a cubic Bézier spiral curve with control points $(0,0)$, $(100,0)$, $(200,2)$, $(300,8)$ to the corresponding biarc is plotted against the arc length along the biarc. The point J corresponds to the joining point of the two arcs of the biarc. Notice that the deviation and its first derivative are zero at both end points and that the deviation changes sign near J as predicted by the asymptotic analysis in Theorem 9.

3. Algorithm and examples

The goal is to approximate a smooth planar curve by an arc spline to a given tolerance. The method proposed here is to partition the curve so that the curve is between bounding circular arcs and the bounding circular arcs are within the required tolerance. Biarcs that match the same data are between the same bounding circular arcs. The biarcs join to form an arc spline that is then guaranteed to be within the required tolerance of the curve.

A short description of the algorithm follows. Let Approx(A, B) be a procedure that finds an arc spline approximation to the curve from point A to point B. The following outline shows how the curve could be partitioned so that the arc spline produced approximates the curve to a given tolerance.

procedure Approx(A, B)

case 1. If A very close to B, then return a straight line as part of the arc spline approximation.

case 2. If the enclosing condition of Definition 4 is satisfied, assume the curve is a spiral, and calculate the maximum distance between the bounding circular arcs (8).

case 2a. If the bounding circular arcs are close enough, find a biarc and return it as part of the arc spline approximation.

case 2b. If the bounding circular arcs are not close enough, invoke Approx(A, $\frac{1}{2}(A + B)$) and Approx($\frac{1}{2}(A + B), B$).

case 3. If the enclosing condition of Definition 4 is not satisfied, invoke Approx($A, \frac{1}{2}(A + B)$) and Approx($\frac{1}{2}(A + B), B$).

Several comments about the algorithm are in order. Case 1 is used if all other conditions are not met. It prevents infinite recursion when the enclosing condition has not been satisfied in a reasonable number of subdivisions. Returning a straight line may mean the loss of unit tangent continuity. For approximation to Bézier curves the condition that A be very close to B can be replaced by a flatness test [1, p. 222].

In case 2, the enclosing condition of Definition 4 is a necessary but not sufficient condition that the curve is a spiral. A nonspiral curve is not necessarily enclosed by the bounding circular arcs, so the biarc is not necessarily within the required tolerance even when the bounding circular arcs are. The curve segments with curvature extrema or inflection points could cause trouble. One solution is to ignore the problem. Assuming there are a small number of such points, only a small number of segments will cause difficulty, and, if the subdivision is fine enough, they may fall under case 1. Another solution is to search for and partition the curve at curvature extrema and inflection points before starting the arc spline approximation. A method for determining curvature extrema in cubic Bézier curves is given in [9]. Formulae for the curvature extrema of quadratic NURBS are given in [3]. This solution entails more work, but is satisfying in that the curve segments being approximated will be spirals so that the assumption in case 2 is valid.

Two numerical examples are given. The first, and simpler, curve is a cubic Bézier curve with control points (0, 0), (30, 150), (250, 120), (300, 0) (see Fig. 5). It was approximated by an arc spline (formed from Sabin's biarcs [7]) using the above method. Table 1 gives the maximum distance between bounding circular arcs, the distance from the arc spline to the cubic Bézier (estimated by taking the maximum of two hundred distances), and the number of biarcs used.

The second example is a quintic Bézier with control points (0, 0), (−20, 150), (250, 120), (300, 0), (350, 100), and (250, 300) (see Fig. 6). It was approximated by an arc spline (formed from Sabin's biarcs [7]) using the above method. Table 2 gives the maximum distance between bounding

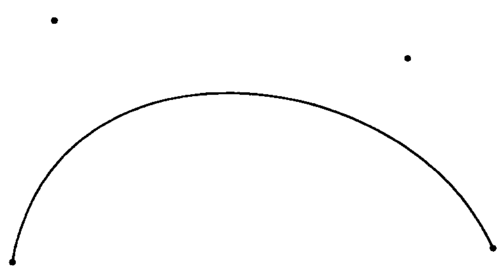

Fig. 5. A cubic Bézier.

Table 1
Accuracy of the arc spline approximation to the cubic Bézier curve

Distance between bounding arcs	Distance from arc spline to cubic Bézier	Number of biarcs
1.2180	0.32954	4
0.27735	0.035121	8
0.043782	0.0040455	16
0.0059694	0.00048843	32

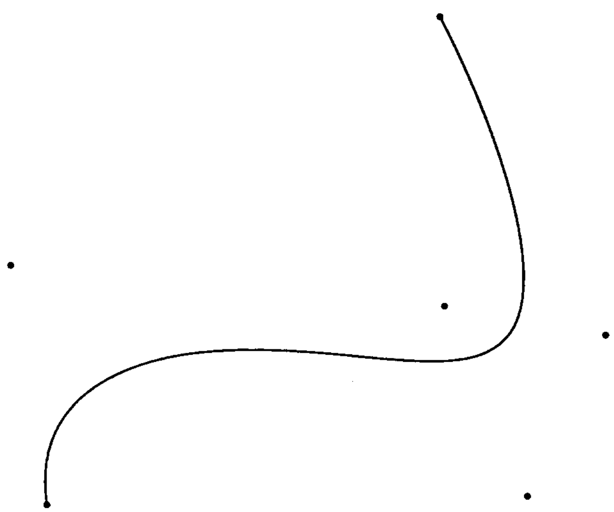

Fig. 6. A quintic Bézier.

Table 2
Accuracy of the arc spline approximation to the quintic Bézier curve

Distance between bounding arcs	Distance from arc spline to quintic Bézier	Number of biarcs
10.809	4.1602	4
1.9133	0.63756	8
0.41899	0.053943	16
0.067176	0.0061082	32

circular arcs, the distance from the arc spline to the quintic Bézier (estimated by taking the maximum of two hundred distances), and the number of biarcs used.

The numerical results for both the above examples are consistent with the theoretical results that the approximations are $O(h^3)$ and that the maximum distance between the curve and biarc is about $\frac{1}{13.5}$ of the distance between the bounding circles.

Acknowledgements

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Appendix

The following Taylor series expansions were used:

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \mathrm{O}(x^3), \tag{A.1}$$

$$\arcsin x = x + \tfrac{1}{6}x^3 + \mathrm{O}(x^5), \tag{A.2}$$

$$\tan x = x + \tfrac{1}{3}x^3 + \mathrm{O}(x^5), \tag{A.3}$$

$$\sin x = x - \tfrac{1}{6}x^3 + \mathrm{O}(x^5), \tag{A.4}$$

$$\cos x = 1 - \tfrac{1}{2}x^2 + \mathrm{O}(x^4). \tag{A.5}$$

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