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Purity in Functor Categories

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INTRODUCTION

An exact sequence $0 \to L \to M \to N \to 0$ of right A-modules (where A is a ring) is called *pure* if its exactness is preserved when tensoring it with any left A-module ([*3*]; cf. also [*10*] for discussion of different notions of purity in module categories). It is well-known that the sequence is pure if and only if every finitely presented right A-module is a relative projective for it. In this paper we will use this last characterization of pure sequences to define purity in more general categories. A new proof of the fact that the two definitions of purity in module categories are equivalent will appear here as a special case of Theorem 2, which gives alternative descriptions of purity in functor categories. The paper concludes with some characterizations of regular (in the von Neumann sense) functor categories (Theorem 4).

1. FINITELY PRESENTED OBJECTS

Throughout this paper, $\mathscr{A}$ will be a Grothendieck category, i.e. an abelian category with a generator and exact direct limits. An object M is *finitely generated* if, whenever $M = \sum M_i$ for a directed family $(M_i)_I$ of sub-objects of M, there is an $i \in I$ such that $M = M_i$. M is *finitely presented* if it is finitely generated and every epimorphism $L \to M$, where L is finitely generated, has a finitely generated kernel. $\mathscr{A}$ is said to be a *locally finitely generated* (resp. *presented*) category if it has a family of finitely generated (resp. presented) generators.

We list some useful and fairly well-known facts about finitely presented objects (cf. [*1*], [*2*], and [*5*]).

LEMMA 1. *Let* $0 \to L \to M \to N \to 0$ *be exact. Then:*

(i) *If M is finitely presented and L is finitely generated, then N is finitely presented.*

(ii) *If L and N are finitely presented, then also M is finitely presented.*

LEMMA 2. *Every finitely generated projective object is finitely presented.*

LEMMA 3. *Suppose $\mathscr{A}$ is locally finitely generated. M is finitely presented if and only if the functor* $\mathrm{Hom}(M, \cdot)$ *commutes with direct limits.*

LEMMA 4. *If $\mathscr{A}$ is locally finitely presented, then every object is a direct limit of finitely presented objects.*

LEMMA 5. *Suppose $\mathscr{A}$ is locally finitely generated and M is a finitely presented object. Let $(N_i)_I$ be the family of finitely generated subobjects of some object N. Then there is a canonical isomorphism*

$$\mathrm{Ext}^1(M, N) \cong \varinjlim \mathrm{Ext}^1(M, N_i).$$

Proof. For every extension $0 \to N \to K \to M \to 0$ there is a finitely generated subobject K' of K mapping onto M. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \cap K' & \longrightarrow & K' & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & K & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

has $N \cap K'$ finitely generated (Lemma 1). So one obtains a well-defined map $\mathrm{Ext}^1(M, N) \to \lim_{\to} \mathrm{Ext}^1(M, N_i)$, which clearly is the inverse of the homomorphism $\lim_{\to} \mathrm{Ext}^1(M, N_i) \to \mathrm{Ext}^1(M, N)$ defined by taking push-outs.

COROLLARY. *Suppose $\mathscr{A}$ is locally finitely generated. If every finitely generated object is injective, then every finitely presented object is projective.*

2. PURE SEQUENCES AND FLAT OBJECTS

In this section it is assumed that $\mathscr{A}$ is locally finitely generated.

DEFINITION. A short exact sequence in $\mathscr{A}$ is *pure* if every finitely presented object is relatively projective for it.

LEMMA 6. (i) *The pure sequences of $\mathscr{A}$ form a proper class.*

(ii) *If $\mathscr{A}$ is locally finitely presented, then there are enough pure-projective objects.*

(iii) *The class of pure sequences is closed under direct limits.*

Proof. (i) and (ii) follow from [*10*], Proposition 2.2 and 2.3; (iii) is a consequence of Lemma 3.

Two examples of purity:

(1) Let M be a finitely presented object with a finitely generated sub-object L. L is pure in M if and only if M is split by L.

(2) Let $(M_i)_1^\infty$ be a countable family of objects. $\bigoplus_1^\infty M_i$ is then a pure sub-object of $\prod_1^\infty M_i$.

Proof. If F is finitely presented, then a morphism

$$\varphi : F \to \prod_1^\infty M_i \Big/ \bigoplus_1^\infty M_i$$

factors over

$$\prod_1^\infty M_i \Big/ \bigoplus_1^n M_i$$

for some $n < \infty$ (Lemma 3). But $\bigoplus_1^n M_i$ is a direct summand of $\prod_1^\infty M_i$, so φ may be lifted to $\prod_1^\infty M_i$.

Definition. An object M is *flat* if every exact sequence $0 \to K \to L \to M \to 0$ is pure.

Lemma 7. (i) *Every projective object is flat.*

(ii) *Every finitely presented flat object is projective.*

(iii) *Every direct limit of flat objects is flat.*

Proof. Clear.

Lemma 8. (Cf. [*7*]). *Suppose $\mathscr{A}$ has enough projective objects. The following properties of an object M are equivalent*:

(a) M is flat.

(b) There exists a pure exact sequence $0 \to K \to P \to M \to 0$ with projective P.

(c) Every morphism $F \to M$, where F is finitely presented, may be factored through a projective object.

Proof. Trivial.

3. Purity in functor categories

For this section we assume that $\mathscr{A}$ has a family of finitely generated projective generators, i.e., $\mathscr{A}$ may be considered as the category of contravariant additive functors from a small preadditive category $\mathscr{D}$ to (Ab). In particular $\mathscr{A}$ is locally finitely presented, with M finitely presented if and only if there is an exact sequence $P_1 \to P_0 \to M \to 0$ with finitely generated projective P_i. For every object X in $\mathscr{D}$ we write h_X for the object $\mathrm{Hom}(\cdot, X)$ in $\mathscr{A}$.

Let $\mathscr{A}^*$ be the category of those covariant right exact functors $F : \mathscr{A} \to (Ab)$ which commute with direct sums. We call $F(M)$ the *tensor product* of $F \in \mathscr{A}^*$ and $M \in \mathscr{A}$, this terminology being justified by Theorem 1 below. Every F in $\mathscr{A}^*$ defines a functor $F' : \mathscr{D} \to (Ab)$ as $F'(X) = F(h_X)$. But for every object M in $\mathscr{A}$ there is an exact sequence

$$\oplus h_{X_j} \to \oplus h_{X_i} \to M \to 0$$

and so it is clear that F is uniquely determined by F'. Therefore we will identify $\mathscr{A}^*$ with the category of covariant additive functors $\mathscr{D} \to (Ab)$. When $\mathscr{A}$ is the category of right modules over a ring A, then of course $\mathscr{A}^*$ is the category of left modules over A.

The following result is an extension of a theorem of Watts [11] from module categories to functor categories. Note that if $M \in \mathscr{A}$ and G is an Abelian group, then $\mathrm{Hom}_Z(M(\cdot), G)$ is a covariant functor $\mathscr{D} \to (Ab)$ and thus an object of $\mathscr{A}^*$.

THEOREM 1. *There are natural isomorphisms*

$$\mathrm{Hom}_{\mathscr{A}^*}(F, \mathrm{Hom}_Z(M, G)) \cong \mathrm{Hom}_{\mathscr{A}}(M, \mathrm{Hom}_Z(F, G)) \cong \mathrm{Hom}_Z(F(M), G)$$

for any $M \in \mathscr{A}$, $F \in \mathscr{A}^*$ and Abelian group G.

Proof. For each pair M, F there is a natural transformation of functors

$$\phi : M(\cdot) \to \mathrm{Hom}_Z(F(\cdot), F(M))$$

with $\phi_X : M(X) \cong \mathrm{Hom}(h_X, M) \to \mathrm{Hom}(F(X), F(M))$ given as $\phi_X(\alpha) = F(\alpha)$. Every homomorphism $\lambda : F(M) \to G$ induces a natural transformation $\mathrm{Hom}(F(\cdot), F(M)) \to \mathrm{Hom}(F(\cdot), G)$, which composed with ϕ gives a morphism $\bar{\lambda} : M \to \mathrm{Hom}(F, G)$.

Conversely, let there be given a natural transformation

$$\psi : M(\cdot) \to \mathrm{Hom}(F(\cdot), G).$$

Thus for each $X \in \mathscr{D}$ we have $\psi_X : \mathrm{Hom}(h_X, M) \to \mathrm{Hom}(F(X), G)$. The object M may be presented by an exact sequence

$$\bigoplus h_{Y_j} \xrightarrow{\beta} \bigoplus h_{X_i} \xrightarrow{\alpha} M \longrightarrow 0$$

with α having components $\alpha_i : h_{X_i} \to M$. Applying F to this sequence, we obtain

$$\oplus F(Y_j) \to \oplus F(X_i) \to F(M) \to 0.$$

Define a homomorphism $\mu: \oplus F(X_i) \to G$ by setting $\mu_i = \psi_X(\alpha_i) : F(X_i) \to G$. But $\mu \cdot F(\beta) = 0$ because of the naturality of ψ, so μ factors over $F(\alpha)$ to give a homomorphism $\mu' : F(M) \to G$. One verifies that the mapping $\psi \to \mu'$ is the inverse of the homomorphism $\lambda \to \bar{\lambda}$.

Define contravariant "duality functors"

$$\mathscr{A} \underset{T}{\overset{S}{\rightleftarrows}} \mathscr{A}^*$$

as

$$S(M) = \mathrm{Hom}_Z(M(\cdot), Q/Z),$$

$$T(F) = \mathrm{Hom}_Z(F(\cdot), Q/Z),$$

where Q/Z denotes the Abelian group of rationals mod 1. S and T are clearly exact and faithful functors.

THEOREM 2. *The following properties are equivalent for an exact sequence* $0 \to L \to M \to N \to 0$ *in* $\mathscr{A}$:

(a) It is pure.

(b) The sequence $0 \to F(L) \to F(M) \to F(N) \to 0$ is exact for all $F \in \mathscr{A}^*$.

(c) The sequence $0 \to S(N) \to S(M) \to S(L) \to 0$ splits in $\mathscr{A}^*$.

(d) Every object $T(F)$ is a relative injective for it.

Proof. It follows from Theorem 1 that (b) is equivalent to the condition that all F in $\mathscr{A}^*$ are relative projectives with respect to the sequence $0 \to S(N) \to S(M) \to S(L) \to 0$. Hence (b) $\Leftrightarrow$ (c), and it is also clear from Theorem 1 that (b) $\Leftrightarrow$ (d).

(a) $\Leftrightarrow$ (b). In view of Lemma 4 it suffices to consider finitely presented F in $\mathscr{A}^*$. Now F is finitely presented if and only if there are finitely generated projective objects P_0, P_1 and a morphism $\alpha : P_1 \to P_0$ inducing an exact sequence [in the category of all additive functors $\mathscr{A} \to (Ab)$]

$$0 \to \mathrm{Hom}(K, \cdot) \to \mathrm{Hom}(P_0, \cdot) \to \mathrm{Hom}(P_1, \cdot) \to F \to 0,$$

where $K = \mathrm{Coker}\,\alpha$ is finitely presented. So we see that there are symmetrical presentations of finitely presented objects in $\mathscr{A}$ and in $\mathscr{A}^*$, and the assertion follows from a verification that ψ is surjective if and only if φ is injective in the following diagram:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Hom}(K,L) & \longrightarrow & \mathrm{Hom}(P_0,L) & \longrightarrow & \mathrm{Hom}(P_1,L) & \longrightarrow & F(L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & \mathrm{Hom}(K,M) & \longrightarrow & \mathrm{Hom}(P_0,M) & \longrightarrow & \mathrm{Hom}(P_1,M) & \longrightarrow & F(M) & \longrightarrow & 0 \\
 & & {\scriptstyle \psi}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(K,N) & \longrightarrow & \mathrm{Hom}(P_0,N) & \longrightarrow & \mathrm{Hom}(P_1,N) & \longrightarrow & F(N) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

COROLLARY. *There are enough pure-injective objects. An object is pure-injective if and only if it is a direct summand of $T(F)$ for some $F \in \mathscr{A}^*$. The category $\mathscr{A}$ has pure-injective envelopes.*

Proof. The first two statements are consequences of "the adjoint theorem" (cf. [*10*], Secs. 3 and 9), while the last statement follows from [*10*], Proposition 4.5.

4. FLATNESS IN FUNCTOR CATEGORIES

We assume $\mathscr{A}$ is a functor category as in the previous section. For each F in $\mathscr{A}^*$ we may define $\mathrm{Tor}_n(F, \cdot)$ as the nth left-derived functor of F, and obtain the formula ([*6*] and [*8*], chapt. 8.4)

$$\mathrm{Tor}_n(F, M) = \mathrm{Nat}(\mathrm{Ext}^n(M, \cdot), F).$$

Since $\mathscr{A}$ has enough projectives, every short exact sequence in $\mathscr{A}$ gives rise to the usual long exact sequence of Tor's.

THEOREM 3. *The following properties of an object M in $\mathscr{A}$ are quivalent:*

(a) M is flat.

(b) M is a direct limit of projective objects.

(c) For every exact sequence $0 \to F \to G \to H \to 0$ in $\mathscr{A}^*$, also the sequence $0 \to F(M) \to G(M) \to H(M) \to 0$ is exact.

(d) $S(M)$ is an injective object in $\mathscr{A}^*$.

(e) $\mathrm{Tor}_1(F, M) = 0$ for all F in $\mathscr{A}^*$.

Proof. (a) $\Rightarrow$ (b): Choose an exact sequence $0 \to K \to P \to M \to 0$ where P is a direct sum of finitely generated projectives. For each finitely generated subobject A of K one gets a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\varphi'} & P' & \longrightarrow & L & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow[\varphi]{} & P & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

where β makes P' a finitely generated direct summand of P. Since L is finitely presented, the morphism $L \to M$ may be lifted to $L \to P$. From this follows the existence of a morphism γ: $P \to K$ such that $\gamma\beta\varphi' = \alpha$. For the rest of the proof we may now proceed as in the proof given by Govorov [*4*] in the module case.

(b) $\Rightarrow$ (c): Exactness of $0 \to F \to G \to H \to 0$ in $\mathscr{A}^*$ means that the sequence $0 \to F(L) \to G(L) \to H(L) \to 0$ is exact for each finitely generated projective generator L. It is then exact for all projective L, and by hypothesis also for M.

(c) $\Rightarrow$ (a): We want to show that every exact sequence $0 \to K \to L \to M \to 0$ is pure. Consider any H in $\mathscr{A}^*$ and choose an exact sequence $0 \to F \to G \to H \to 0$ in $\mathscr{A}^*$ with projective G. We obtain a commutative diagram

$$\begin{array}{ccccccccc} & & F(K) & \longrightarrow & G(K) & \longrightarrow & H(K) & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\psi}\downarrow & & \\ & & F(L) & \longrightarrow & G(L) & \longrightarrow & H(L) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(M) & \longrightarrow & G(M) & \longrightarrow & H(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns. G is flat in $\mathscr{A}^*$, so by the implication (a) $\Rightarrow$ (c) applied to G instead of M, we conclude that φ is a monomorphism. A diagram-chase then shows that also ψ is a monomorphism.

(a) $\Leftrightarrow$ (d): Choose an exact sequence $0 \to K \to P \to M \to 0$ with projective P. The dual sequence $0 \to S(M) \to S(P) \to S(K) \to 0$ is exact and $S(P)$ is an injective object. Theorem 2 asserts that $0 \to K \to P \to M \to 0$ is pure if and only if the dual sequence splits, i.e., $S(M)$ is injective.

(a) $\Leftrightarrow$ (e) follows, of course, from the long exact sequence

$$\mathrm{Tor}_1(F, P) \to \mathrm{Tor}_1(F, M) \to F(K) \to F(P) \to F(M) \to 0$$

induced by the exact sequence $0 \to K \to P \to M \to 0$ with projective P.

5. Regular functor categories

The following theorem characterizes functor categories which are *regular*, in the sense of von Neumann.

Theorem 4. *Let $\mathscr{A}$ be a Grothendieck category with a family $\mathscr{F}$ of finitely presented generators. The following statements are equivalent*:

(a) All objects are flat.

(b) All short exact sequences are pure.

(c) All finitely presented objects are projective.

(d) Every finitely presented object is split by its finitely generated sub-objects.

(d′) Every $F \in \mathscr{F}$ is projective and split by its finitely generated sub-objects.

(e) Every finitely presented F has a regular endomorphism ring $\mathrm{Hom}(F, F)$.

(f) If F and G are finitely presented, then for every morphism $\alpha : F \to G$ there exists a $\varphi : G \to F$ such that $\alpha = \alpha\varphi\alpha$.

(f′) If F and G are in $\mathscr{F}$, then they are projective and for every morphism $\alpha : F \to G$ there exists a $\varphi : G \to F$ such that $\alpha = \alpha\varphi\alpha$.

Note: If $\mathscr{A}$ satisfies these conditions and $\mathscr{F}$ has only one member F, then $\mathscr{A}$ is equivalent to the category of modules over the regular ring $\mathrm{Hom}(F, F)$. In this case (d′) and (f′) provide Morita-invariant definitions of regular rings.

Proof. The implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) and (f) $\Rightarrow$ (e) are trivial.

(c) $\Rightarrow$ (d) and (c) $\Rightarrow$ (d′) follow directly from Lemma 1.

(c) $\Rightarrow$ (a) follows from Lemmas 4 and 7 (iii).

(d) $\Rightarrow$ (c): Let $0 \to K \to L \to M \to 0$ be any exact sequence with M finitely presented. Since $\mathscr{A}$ is locally finitely presented, there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

with finitely presented F. K' is then finitely generated, so the upper row splits which causes also the lower row to split.

(d') $\Rightarrow$ (c): For every finitely presented object F there is an exact sequence

$$0 \to L \to \bigoplus_{1}^{n} F_i \to F \to 0$$

with $F_i \in \mathscr{F}$. L is then finitely generated, and one shows by induction on n that the sequence splits (Osofsky [9], p. 897).

(d) $\Rightarrow$ (f): Consider the canonical factorization of α as

$$F \xrightarrow{\gamma} \operatorname{Im} \alpha \xrightarrow{\beta} G.$$

$\operatorname{Im} \alpha$ is finitely generated, so there exists a morphism $\lambda : G \to \operatorname{Im} \alpha$ with $\lambda\beta = 1$. But then it follows that $\operatorname{Im} \alpha$ is finitely presented, so there exists $\mu : \operatorname{Im} \alpha \to F$ with $\gamma\mu = 1$. One obtains $\alpha\mu\lambda\alpha = \beta\gamma\mu\lambda\beta\gamma = \beta\gamma = \alpha$.

(e) $\Rightarrow$ (d): Let L be a finitely generated subobject of a finitely presented object F. There exist a finitely presented object G and a sequence of morphisms

$$F \oplus G \xrightarrow{\pi} G \xrightarrow{\beta} L \xrightarrow{\alpha} F \xrightarrow{\lambda} F \oplus G,$$

where β is an epimorphism, and π and λ are canonical. By hypothesis there is an endomorphism φ of $F \oplus G$ such that $\lambda\alpha\beta\pi = \lambda\alpha\beta\pi\varphi\lambda\alpha\beta\pi$. But $\lambda\alpha$ is a monomorphism and $\beta\pi$ is an epimorphism, so $1_L = \beta\pi\varphi\lambda\alpha$ and consequently α splits.

(f) $\Rightarrow$ (f') is clear since all finitely presented objects will be projective by the already proved implication (f) $\Rightarrow$ (c).

(f') $\Rightarrow$ (d'): Let K be a finitely generated subobject of $F \in \mathscr{F}$. It clearly suffices to consider the case when K is a quotient of $F_1 \oplus F_2$ with $F_1, F_2 \in \mathscr{F}$. We have the diagram

$$\begin{array}{ccccc} F_i & \xrightarrow{\beta_i} & K_i & \xrightarrow{\alpha_i} & F \\ \downarrow & & \downarrow & & \| \\ F_1 \oplus F_2 & \xrightarrow[(\beta_1, \beta_2)]{} & K & \xrightarrow[\alpha]{} & F \end{array}, \qquad i = 1, 2.$$

with the obvious morphisms. As before it is seen that the monomorphisms α_i split F, so there exist idempotent endomorphisms e_i of F such that $\operatorname{Im} \alpha_i = \operatorname{Im} e_i$. One then shows in the usual way that $\operatorname{Im} \alpha = \operatorname{Im} e_1 + \operatorname{Im} e_2 = \operatorname{Im}(e_1 + g)$ where e_1 and g are orthogonal idempotents.

COROLLARY. *Suppose $\mathscr{A}$ is locally finitely presented and every finitely generated object is injective. $\mathscr{A}$ is then a regular functor category.*

Proof. Recall the corollary of Lemma 5.

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