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Communication

On the harmoniousness of dicyclic groups

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Abstract

A group is called harmonious if its elements can be arranged in a sequence such that the products of consecutive elements comprise all elements of the group. We give a sufficient condition for the dicyclic group of order $4n$ to be harmonious and give a specific construction by which the condition can be met when n is divisible by 4.

1. Introduction

A group G of order m is called harmonious if its elements can be listed in a sequence

$$g_1, g_2, \ldots, g_m \tag{1}$$

such that

$$g_1g_2, g_2g_3, \ldots, g_{m-1}g_m, g_mg_1$$

are all distinct. We call the sequence (1) a harmonious sequence of G.

A group G is harmonious if and only if G has a complete mapping which is also a $|G|$-cycle. Several types of group were proved to be harmonious in [2]. Harmoniousness has an interpretation in graph theory as follows. If we label vertices of the complete digraph K_m on m vertices with the elements of a group G of order m and label

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the edge joining g_i to g_j with g_ig_j, then the existence of a harmonious sequence of G is equivalent to finding a hamiltonian circuit in K_m such that each element of G occurs exactly once as an edge in the circuit. Harmoniousness also gives the appearance of being a variation of an old concept called R-sequenceability which have been discussed in great depth in [3] and [4].

The dicyclic group Q_{2n} of order $4n$ is defined by

$$Q_{2n}=\langle \alpha,\beta\colon \alpha^{2n}=1,\ \beta^2=\alpha^n,\ \alpha\beta=\beta\alpha^{-1}\rangle.$$

It was proved in [1] that Q_{2n} is sequenceable for $n>2$ and in [6] that Q_{2n} is R-sequenceable when $n\equiv 0 \bmod 4$. In this communication, we shall investigate the harmoniousness of Q_{2n}.

From a well-known theorem of Paige [5], we can deduce that Q_{2n} has no complete mappings and consequently is not harmonious if n is odd ([2, Corollary 3.2] or [6, Lemma A]). It was claimed in [2] that by computer search Q_{2n} is not harmonious if $n=2$. Hence from now on we assume that n is an even integer greater than 2. We state the following.

Theorem. *The dicyclic group Q_{2n} is harmonious if $n\equiv 0 \bmod 4$ or $n\equiv 0 \bmod 6$.*

2. Proof of the theorem

For a matrix M, the (i,j)-entry of M is denoted by $M(i,j)$. For an arbitrary n we construct two $n\times n$ matrices A and B as follows. Let

$$A(i,j)=i+j-2 \bmod 2n, \quad \text{for } i,j=1,2,\dots,n,$$

$$B(i,j)=n-i-j+1 \bmod 2n, \quad \text{for } i,j=1,2,\dots,n.$$

For a permutation π of degree n and a $n\times n$ matrix M, let

$$\pi(M)=\{M(i,\pi(i))\colon i=1,2,\dots,n\}.$$

Proposition 1. *Q_{2n} is harmonious if there exist two permutations π and θ of degree n such that $\theta\pi$ is an n-cycle and $\pi(A)\cup\theta(B)$ is a complete set of residues modulo $2n$.*

Proof. Let $f=\theta\pi$, c be an arbitrary fixed integer with $1\leqslant c\leqslant n$, and

$$b_{2i-1}=-f^{i-1}(c) \bmod 2n, \quad i=1,\dots,n,$$

$$b_{2i}=f^{i-1}(c)-1 \bmod 2n, \quad i=1,\dots,n,$$

$$a_{2i-1}=\pi f^{i-1}(c)-1 \bmod 2n, \quad i=1,\dots,n,$$

$$a_{2i}=\pi f^{i-1}(c)+n-1 \bmod 2n, \quad i=1,\dots,n.$$

Since f is a cycle, it is easy to see that

$$\{b_{2i-1}\colon i=1,\ldots,n\}=\{n,n+1,\ldots,2n-1\} \bmod 2n,$$

$$\{b_{2i}\colon i=1,\ldots,n\}=\{0,1,\ldots,n-1\} \bmod 2n,$$

$$\{a_{2i-1}\colon i=1,\ldots,n\}=\{0,1,\ldots,n-1\} \bmod 2n,$$

$$\{a_{2i}\colon i=1,\ldots,n\}=\{n,n+1,\ldots,2n-1\} \bmod 2n.$$

By direct calculation we have

$$b_{2i}-b_{2i-1}+n=2f^{i-1}(c)+n-1,$$

$$a_{2i-1}+a_{2i}=2\pi f^{i-1}(c)+n-2,$$

$$b_{2i}+a_{2i-1}=f^{i-1}(c)+\pi f^{i-1}(c)-2,$$

$$b_{2i+1}-a_{2i}=n-f^{i}(c)-\pi f^{i}(c)+1,$$

where all arithmetic is modulo $2n$. Hence,

$$\{b_{2i}-b_{2i-1}+n\colon i=1,\ldots,n\}=\{1,3,\ldots,2n-1\} \bmod 2n,$$

$$\{a_{2i}+a_{2i-1}\colon i=1,\ldots,n\}=\{0,2,4,\ldots,2n-2\} \bmod 2n.$$

By the definitions of the matrices A and B we have

$$b_{2i}+a_{2i-1}=A(f^{i-1}(c),\ \pi f^{i-1}(c)),$$

$$b_{2i+1}-a_{2i}=B(\pi f^{i-1}(c), f^{i-1}(c)).$$

Notice that f is a cycle and $f^{i}(c)=\theta\pi f^{i-1}(c)$. By the definitions of $\pi(A)$ and $\theta(B)$ we have

$$\{b_{2i}+a_{2i-1}\colon i=1,\ldots,n\}=\pi(A),$$

$$\{b_{2i+1}-a_{2i}\colon i=1,\ldots,n\}=\{B(\pi f^{i-1}(c),\ \theta\pi f^{i-1}(c))\colon i=1,\ldots,n\}=\theta(B),$$

where subscripts are read modulo $2n$.

We claim that the following sequence is a harmonious sequence of Q_{2n}:

$$\beta\alpha^{b_1}, \beta\alpha^{b_2}, \alpha^{a_1}, \alpha^{a_2}, \beta\alpha^{b_3}, \beta\alpha^{b_4}, \alpha^{a_3}, \alpha^{a_4}, \ldots, \beta\alpha^{b_{2n-1}}, \beta\alpha^{b_{2n}}, \alpha^{a_{2n-1}}, \alpha^{a_{2n}}. \tag{2}$$

This sequence comprise all elements of Q_{2n}, because both

$$\{b_j\colon j=1,2,\ldots,2n\}$$

and

$$\{a_j\colon j=1,2,\ldots,2n\}$$

are the complete set of residues modulo $2n$. The products of consecutive terms of the sequence (2) are

$$\alpha^{b_2-b_1+n},\ \beta\alpha^{b_2+a_1},\ \alpha^{a_1+a_2},\ \beta\alpha^{b_3-a_2},\ \alpha^{b_4-b_3+n},\ \beta\alpha^{b_4+a_3},\ \ldots,\ \alpha^{b_{2n}-b_{2n-1}+n},$$
$$\beta\alpha^{b_{2n}+a_{2n-1}},\ \alpha^{a_{2n-1}+a_{2n}},\ \beta\alpha^{b_1-a_{2n}}.$$

These products comprise all elements of Q_{2n} because

$$\{b_{2i}-b_{2i-1}+n\colon i=1,\ldots,n\}\cup\{a_{2i-1}+a_{2i}\colon i=1,\ \ldots,n\}$$

is a complete set of residues modulo $2n$ and by our hypothesis

$$\{b_{2i}+a_{2i-1}\colon i=1,\ \ldots,n\}\cup\{b_{2i+1}-a_{2i}\colon i=1,\ldots,n\}$$

is a complete set of residues modulo $2n$ as well. □

Proposition 2. *Q_{2n} is harmonious if $n\equiv 0 \bmod 4$.*

Proof. Let $n=4k$ for some positive integer k and let π to be the permutation of degree n defined by

$$\pi(i)=\begin{cases}2k-\frac{1}{2}(i-1) & \text{if } i \text{ is odd,}\\ 4k-\frac{1}{2}i+1 & \text{if } i \text{ is even,}\end{cases}$$

for $i=1,2,\ldots,4k$. By the definition of the matrix A, we have

$$A(i,\pi(i))=\begin{cases}2k+\frac{1}{2}(i-3) \bmod 2n & \text{if } i \text{ is odd,}\\ 4k+\frac{1}{2}i-1 \bmod 2n & \text{if } i \text{ is even,}\end{cases}$$

and

$$\pi(A)=\{2k-1,\ 2k,\ldots,4k-2,\ 4k,\ 4k+1,\ldots,6k-1\}.$$

Let θ be the permutation of degree n defined by

$$\theta(i)=\begin{cases}1 & \text{if } i=1,\\ 4k-2i+4 & \text{if } 2\leqslant i\leqslant 2k+1,\\ 8k-2i+3 & \text{if } 2k+2\leqslant i\leqslant 4k.\end{cases}$$

By the definition of matrix of B, we have

$$B(i,\theta(i))=\begin{cases}4k-1 \bmod 2n & \text{if } i=1,\\ i-3 \bmod 2n & \text{if } 2\leqslant i\leqslant 2k+1,\\ 4k+i-2 \bmod 2n & \text{if } 2k+2\leqslant i\leqslant 4k,\end{cases}$$

and

$$\theta(B)=\{8k-1,0,1,\ldots,2k-2,\ 4k-1,\ 6k,\ 6k+1,\ldots,8k-2\}.$$

Hence $\pi(A)\cup\theta(B)$ is a complete set of residues modulo $2n$. By direct calculation we deduce that

$$f=\theta\pi=(1\ \ 4\ \ 5\ \ 8\ \ 9\ldots 4k-4\ \ 4k-3\ \ 4k\ \ 2\ \ 3\ \ 6\ \ 7\ldots 4k-2\ \ 4k-1)$$

which is a cycle. Hence by Proposition 1, we conclude that Q_{2n} is harmonious. □

Lemma. *Q_{2n} is harmonious when $n=6$.*

Proof. Let π and θ be the permutations of degree 6 defined by

$$\pi=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 3 & 6 & 2 & 5 & 1 & 4\end{pmatrix},$$

$$\theta=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 1 & 5 & 6 & 2 & 3 & 4\end{pmatrix}.$$

It is easy to see

$$f=\theta\,\pi=(1\ \ 6\ \ 2\ \ 4\ \ 3\ \ 5)$$

which is a cycle. We can verify that $\pi(A)\cup\theta(B)$ is a complete set of residues modulo 12. By Proposition 1 we deduce that Q_{12} is harmonious. $\square$

It was proved in [2] that Q_{2nm} is harmonious for odd integers m provided Q_{2n} is harmonious.Therefore, by the previous lemma and Proposition 2, we have the following proposition.

Proposition 3. *Q_{2n} is harmonious if $n\equiv 0 \bmod 6$.*

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References

[1] B.A. Anderson, Sequencings of dicyclic groups II, J. Combin. Math. Combin. Comput. 3 (1988) 5–27.
[2] R. Beals, J.A. Gallian, P. Headley and D. Jungreis, Harmonious groups, J. Combin. Theory Ser. A 56 (1991) 223–238.
[3] J. Denes and A.D. Keedwell, Latin squares: New developments in the theory and applications, Ann. Discrete Math. 46 (1991).
[4] A.D. Keedwell, On the R-sequenceability and R_h-sequenceability of groups, Ann. Discrete Math. 18 (1983) 535–548.
[5] L.J. Paige, Complete mappings of finite groups, Pacific J. Math. 1 (1951) 111–116.
[6] C.-D. Wang, On the R-sequenceablity of dicyclic groups, Proc. 13th British Combin. Conf., Discrete Math., to appear.