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PARITY OF PATHS AND CIRCUITS IN TOURNAMENTS

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1. Introduction

Let T_n be a tournament on n vertices and let $x_1, x_2, ..., x_n$ be any permutation of the vertices (see end of [4]). Let $e_i = +1$ whenever $x_i \to x_{i+1}$ and $e_i = -1$ whenever $x_{i+1} \to x_i$ for $1 \leq i \leq n-1$. Traditionally the sequence $x_1, x_2, ..., x_n$ would be called a Hamiltonian path if all e_i's were +1, and the sequence would be called an antidirected Hamiltonian path (see [1]) if each $e_i = -e_{i+1}$. But we are concerned with paths which reailze other sequences $e_1, e_2, ..., e_{n-1}$ of +1's and −1's. Given such a sequence, we let $P = P(e_1, e_2, ..., e_{n-1})$ be the digraph with vertices 1, 2, ..., n and edges $i \to (i+1)$ if $e_i = 1$, and $(i+1) \to i$ if $e_i = -1$. Let $I(P, T_n)$ be the number of edge-direction-preserving maps of the vertices of P onto those of T_n in a one-to-one manner. Theorem 2.1 gives the parity of $I(P, T_n)$ purely in terms of the sequence of e_i's, thus generalizing the well-known theorem of Redei (see [3, pp. 21–24]) that the number of Hamiltonian paths in a tournament is odd. Corollary 2.2 shows that (see the conjecture at the end of [4]) when n is a power of two, each T_n realizes every sequence of e_i's. Theorem 3.1 is a circuit analogue of Theorem 2.1, and Corollary 3.2 shows that it is necessary (if one wishes the parity to be independent of the tournament) to count maps of paths rather than their images (the actual paths in the tournament).

2.

If n is a non-negative integer, let $U(n)$ denote the set of integers such that $n = \Sigma\{2^i: i \in U(n)\}$. If m, n are non-negative integers, we define a binary relation R, where $(m, n) \in R$ iff $U(m) \subseteq U(n)$.

Theorem 2.1. *If $P = P(e_1, e_2, ..., e_{n-1})$ and T_n is any tournament on n vertices, the parity of $I(P, T_n)$ equals that of the number of R-linearly ordered subsets (counting the empty set) of the set $\{i < n: (i, n) \in R, e_i = -1\}$.*

Proof. The proof is by induction on n and by induction on the largest integer k such that $e_k = -1$ (let $k = 0$ if all $e_i = 1$). The theorem is trivial when $n = 2$ since the only set to be counted is the empty set and the parity is odd. The theorem is true if $k = 0$ (for any n) by the afore-mentioned theorem of Rédei. We may therefore assume that $n > 2$ and $k > 0$ and that the theorem is true for any lesser value of n (with any k) and for the present value of n with lesser values of k. Let $P^* = P(e_1, ...$ $..., e_{k-1}, +1, +1, ..., +1)$, so P^* differs from P in the k^{th} edge. Note that $I(P, T_n) + I(P^*, T_n)$ equals the sum of all products of the form $I(P(e_i, ..., e_{k-1}), T_k) \cdot I(P(+1, +1, ..., +1), T_{n-k})$, where T_k is any k-vertex subtournament of T_n and T_{n-k} denotes the complimentary $(n-k)$ vertex subtournament in T_n. But the right-hand factor of each product is odd by the Redei theorem and the left-hand factor can be calculated (modulo 2) by our inductive hypothesis. Thus $I(P, T_n) + I(P^*, T_n)$ has the parity of $\binom{n}{k} \cdot L$, where L is the number of R-linearly ordered subsets of $\{i < k: (i, k) \in R, e_i = -1\}$. It was proved by Lucas [2] that if p is a prime and if a_i and b_i are the i^{th} coefficients of the p-ary expansions of n and k, respectively, then $\binom{n}{k}$ is congruent (modulo p) to the product of all $\binom{a_i}{b_i}$. In particular $\binom{n}{k}$ is odd iff $(k, n) \in R$. Thus the theorem is established if $(k, n) \notin R$. When $(k, n) \in R$, we have $I(P, T_n) + I(P^*, T_n)$ has the parity of L which also equals the number of R-linearly ordered subsets of $\{i < n: (i, k) \in R, e_i = -1\}$ which contain the element K (since R is transitive). The theorem follows easily by applying the second part of the inductive hypothesis (lesser values of k) to $I(P^*, T)$.

Corollary 2.2. *If n is a power of 2, then $I(P, T_n)$ is always odd, hence greater than zero. Thus every tournament on n vertices contains paths realizing every sequence of e_i's.*

Proof. The set whose R-linearly ordered subsets are to be counted is empty.

3. Circuits

Given a sequence $e_1, e_2, ..., e_n$ of -1's and $+1$'s, let $C(e_1, ..., e_n)$ be the diagraph with the same vertices and edges as $P(e_1, ..., e_{n-1})$ and one additional edge, either $n \to 1$ if $e_n = 1$ or $1 \to n$ if $e_n = -1$.

Theorem 3.1. *Let $C = C(e_1, ..., e_n)$ and let T_n be any tournament on n vertices. The parity of $I(C, T_n)$ equals that of*

$$n \cdot w(T_n) + \Sigma \{I(P(1, 1, ..., 1, e_1, e_2, ..., e_{k-1}), T_n): e_k = -1\},$$

where $w(T_n)$ denotes the number of Hamiltonian circuits in T_n. In view of Theorem 2.1, this means that the parity of $I(C, T_n)$ depends only on the e_i's and on $w(T_n)$.

Proof. The proof is a simple induction on the largest integer k such that $e_k = -1$, using the fact that if C and C^* are two circuits differing in exactly one edge, $I(C, T_n) + I(C^*, T_n) = I(P, T_n)$, where P is the path obtained by removing the disagreeable edge from either circuit. Note also that $n \cdot w(T_n) = I(C(1, 1, ..., 1), T_n)$.

Corollary 3.2. *If $P = P(e_1, e_2, ..., e_{n-1})$ and if $I(P, T_n)$ differs from the number $J(P, T_n)$ of paths in T_n which are isomorphic to P, then n is odd, P is symmetric about its midpoint (so $I(P, T_n)$ is even), and the parity of $J(P, T_n)$ is not the same for all tournaments T_n.*

Proof. The only non-trivial statement concerns the variation of $J(P, T_n)$ in different tournaments. Let P be symmetric and let C be the circuit formed by adding a directed edge from the first to the last vertex of P. Note that C contains only one copy of P for if $e_1 = e_k$, $e_2 = e_{k+1}$, etc., then $e_j = e_{n-j}$ with $j = \frac{1}{2}(n - k + 1)$ or $j = \frac{1}{2}(k - 1)$ (depending on whether k is even or odd), contradicting the symmetry of P. So we have that $J(P, T_n) = I(C, T_n)$, and (by Theorem 3.1 and the odd parity of n) we need only show that the parity of $w(T_n)$ differs for different tournaments T_n. Consider the transitive tournament TT_n and the tournament TT_n^* obtained by reversing the edge between the first and last vertices of TT_n. Clearly $w(TT_n) = 0$ and $w(TT_n^*) = 1$.

References

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