Invertibility of Toeplitz operators and corona conditions in a strip

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Abstract

A Toeplitz operator with symbol G such that det G = 1 is invertible if there is a non-trivial solution to a Riemann–Hilbert problem Gφ+ = φ− with φ+ and φ− satisfying the corona conditions in C+ and C−, respectively. However, determining such a solution and verifying that the corona conditions are satisfied are in general difficult problems. In this paper, on one hand, we establish conditions on φ± which are equivalent to the corona conditions but easier to verify, if G±1 are analytic and bounded in a strip. This happens in particular with almost-periodic symbols. On the other hand, we identify new classes of symbols G for which a non-trivial solution to Gφ+ = φ− can be explicitly determined and the corona conditions can be verified by the above mentioned approach, thus obtaining invertibility criteria for the associated Toeplitz operators.

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1. Introduction

Invertibility of Toeplitz operators with symbol G ∈ (L∞(R))2×2 is equivalent to the existence of a canonical Wiener–Hopf (or generalized) factorization of G. Let us then start by defining this type of matrix factorization.

Let H±p, 1 < p < ∞, denote the Hardy spaces Hp(C±) and let us identify each function φ± ∈ Hp with its boundary-value on R (which is a function in Lp(R)). We have then

\[ L_p(R) = H_p^+ \oplus H_p^- \]  (1.1)

and we denote by P+ the projection of Lp(R) onto H_p^+ parallel to H_p^- and by P- its complementary projection, \[ P^- = I - P^+ \].

Let moreover

\[ \lambda_\pm(\xi) = \xi \pm i, \quad r(\xi) = (\lambda_- \lambda_+^{-1})(\xi) = \frac{\xi - i}{\xi + i}, \quad \text{for} \ \xi \in \mathbb{R}. \]  (1.2)

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By a \textit{Wiener–Hopf factorization} of $G \in (L_\infty(\mathbb{R}))^{2 \times 2}$ (relative to $L_\rho(\mathbb{R})$) we mean a representation [7,10]

$$G = G_- DG_+$$

(1.3)

where $D$ is a rational diagonal matrix of the form

$$D = \text{diag}(r^{k_1}, r^{k_2}),$$

(1.4)

with $k_1, k_2 \in \mathbb{Z}$ and $k_1 \leq k_2$, and the factors $G_\pm$ are such that, for $q = \frac{p}{p-1}$,

$$\lambda_{\pm}^{-1} G_+ \in H^+_q, \quad \lambda_{\pm}^{-1} G_-^{-1} \in H^+_p,$$

(1.5)

$$\lambda_{\pm}^{-1} G_- \in H^-_p, \quad \lambda_{\pm}^{-1} G_+^{-1} \in H^-_q,$$

(1.6)

$$G_- P^+ G_-^{-1} I \quad \text{(which is defined on a dense subset of $(L_\rho(\mathbb{R}))^2$)}$$

admits a bounded extension to $(L_\rho(\mathbb{R}))^2$.

(1.7)

We will usually omit referring to $L_\rho(\mathbb{R})$ and simply say that (1.3) is a Wiener–Hopf factorization.

The integers $k_1$ and $k_2$ in (1.4) are called the \textit{partial indices} of $G$ and their sum is the \textit{total index} of $G$ (ind $G = k_1 + k_2$). If det $G \in C(\mathbb{R})$, then ind $G$ is the winding number of det $G$ relative to 0, i.e., ind $G = \text{ind}(|\text{det} G|)$.

The factorization (1.3) is said to be \textit{canonical} if $k_1 = k_2 = 0$ and \textit{bounded} if $G^\pm_+ \in$ and $G^\pm_- \in$ belong to the Hardy spaces $(H^\pm_\infty)^{2 \times 2} = (H^\pm_\infty(\mathbb{C}^+))^{2 \times 2}$ and $(H^\pm_\infty)^{2 \times 2} = (H^\pm_\infty(\mathbb{C}^-))^{2 \times 2}$, respectively.

A relevant step forward in this field was recently accomplished by using the corona theorem, which is of great value both from the point of view of Complex Analysis and Operator Algebras, as a tool to determine conditions for invertibility of Toeplitz operators (and even expressions for the inverse operator, if two associated corona problems can be solved). To state the main result in this direction, we start by defining the following classes.

\textbf{Definition 1.1.} Let $H^\pm_\infty = H_\infty(\mathbb{C}^\pm)$ denote the Hardy spaces of bounded analytic functions in $\mathbb{C}^\pm$. We define

$$CT^\pm = \left\{ \phi_\pm = (\phi_{1\pm}, \phi_{2\pm}) \in (H^\pm_\infty)^2 : \inf_{\mathbb{C}^\pm}(|\phi_{1\pm}| + |\phi_{2\pm}|) > 0 \right\}$$

(1.9)

and if $\phi_\pm \in CT^\pm$ we say that $(\phi_{1\pm}, \phi_{2\pm})$ is a corona pair in $\mathbb{C}^\pm$.

By the corona theorem (cf. [8]), if $(\phi_{1+}, \phi_{2+}) \in CT^+$, then there exists a pair $(\tilde{\phi}_{1+}, \tilde{\phi}_{2+}) \in (H^+_\infty)^2$ such that

$$\phi_{1+} \tilde{\phi}_{1+} + \phi_{2+} \tilde{\phi}_{2+} = 1 \quad \text{in } \mathbb{C}^+$$

(1.10)

(and analogously in $\mathbb{C}^-$, if $(\phi_{1-}, \phi_{2-}) \in CT^-$).

For matrix symbols $G \in (L_\infty(\mathbb{R}))^{2 \times 2}$ such that det $G$ admits a bounded canonical Wiener–Hopf factorization, in which case we can assume without loss of generality that det $G = 1$, the following result was shown in [1]:

\textbf{Theorem 1.2.} (See [1].) Let $G \in (L_\infty(\mathbb{R}))^{2 \times 2}$ and det $G = 1$. If there is a non-trivial (i.e., non-zero) solution to the Riemann–Hilbert problem

$$G \phi_+ = \phi_-, \quad \phi_+ = (\phi_{1+}, \phi_{2+}) \in (H^+_\infty)^2,$$

(1.10)

such that $\phi_\pm \in CT^\pm$, then $G$ admits a canonical Wiener–Hopf factorization (relative to $L_\rho(\mathbb{R})$).
It should be noticed that, if $G$ belongs to a class of functions such that a Wiener–Hopf factorization, when it exists, is bounded, then the conditions of Theorem 1.2 are necessary and sufficient for existence of a canonical Wiener–Hopf factorization. Furthermore, a generalization of the above result was obtained in [4] for the case where $\det G$ admits a non-canonical bounded factorization.

However, these results are useful only if a non-trivial solution to (1.10) can be found and if it is possible to check whether or not $\phi_+$ and $\phi_-$ are corona pairs. None of these difficult questions has yet an answer, except in very particular cases.

In this paper, on the one hand, we establish conditions on the solutions to (1.10), $\phi_\pm$, which are equivalent to, or imply the corona conditions, but are simpler to verify, by taking advantage of some properties of $G$. In fact, to verify that $\phi_\pm \in \mathcal{CT}_\infty$, we must show that $\phi_{1+}$ and $\phi_{2+}$ do not approach 0 simultaneously in the upper half-plane and analogously for $\phi_{1-}$ and $\phi_{2-}$ in the lower half-plane. This is generally difficult or even impossible to do directly, given the usually complicated expressions of those functions. However it turns out that, in several important cases, it is easy to see whether $\phi_{1+}, \phi_{2+}$ can approach 0 simultaneously in $\mathbb{C}^+ + i\epsilon_1$ if $\epsilon_1 > 0$ is big enough (and analogously for $\phi_{1-}, \phi_{2-}$ in $\mathbb{C}^- - i\epsilon_1$). Therefore, we are reduced, in those cases, to studying the behaviour of $\phi_{1\pm}, \phi_{2\pm}$ in a strip in the complex plane. In Section 2 we show that, if the elements of $\mathcal{G}^{\pm}$ are analytic and bounded in a strip $S = \{ z \in \mathbb{C} : \Im z \in [-\epsilon_2, \epsilon_1] \}$ with $\epsilon_1, \epsilon_2 \in [0, +\infty]$, then the conditions

(i) $\inf_S (|\phi_{1+}| + |\phi_{2+}|) > 0$,
(ii) $\inf_S (|\phi_{1-}| + |\phi_{2-}|) > 0$,
(iii) $\inf_S (|\phi_{2+}| + |\frac{1}{i}(\phi_{2-} - d\phi_{2+})|) > 0$,
(iv) $\inf_S (|\phi_{2-}| + |\frac{1}{i}(a\phi_{2-} - \gamma \phi_{2+})|) > 0$

where $\gamma = \det G$ and $a, d, c$ are elements of $G$, are equivalent. This means that, as far as approaching 0 simultaneously in $S$ is concerned, we are free to choose a particular pair of functions which is easier to study (for instance, only $\phi_{1+}$ and $\phi_{2+}$).

It should be noticed that several important classes of symbols present the above mentioned property of analyticity in a strip, namely almost-periodic symbols which have attracted great attention in mathematical publications [1,5,6, 11–13], see [3] for more references.

If the solutions to (1.10) are almost-periodic polynomials, then it is easy to know their behaviour “at infinity.” In Section 3 we show that, through an appropriate change of variables, we may be able to reduce the verification of the corona conditions in a strip to the study of the common zeros of two polynomials.

An interesting aspect here is the interplay between classical Real Analysis and Functional Analysis which is put in evidence in the way these results are proved.

On the other hand, we identify new classes of symbols $G$ for which a non-trivial solution to (1.10) can explicitly be obtained and the corona conditions can be verified, thus obtaining invertibility criteria for the associated Toeplitz operators. This is done in Section 4, where we apply the previous results in two different perspectives, allowing us to study new classes of Toeplitz operators with almost-periodic symbols, as well as some non-almost-periodic ones.

2. Corona conditions on a strip and the corona theorem

We start by establishing some notation regarding complex functions which are analytic in a strip.

**Definition 2.1.** If

$$S_{-\epsilon_2, \epsilon_1} = \{ z \in \mathbb{C} : -\epsilon_2 < \Im(z) < \epsilon_1 \}, \quad \text{with } \epsilon_1, \epsilon_2 \in [0, +\infty],$$

we say that $H_{\infty}(S_{-\epsilon_2, \epsilon_1})$ is the Hardy space of functions that are analytic and bounded in $S_{-\epsilon_2, \epsilon_1}$.

Let $G \in (L_\infty(\mathbb{R}))^{2 \times 2}$, with $\det G = \gamma$, satisfy

$$G \in \left( H_{\infty}(S_{-\epsilon_2, \epsilon_1}) \right)^{2 \times 2}, \quad \gamma^{-1} \in H_{\infty}(S_{-\epsilon_2, \epsilon_1}).$$

(2.2)
If these conditions are satisfied and \( \phi_{\pm} \in (H_{\infty}^\pm)^2 \) are non-trivial solutions to
\[
G \phi_+ = \phi_- ,
\]
then \( \phi_+ \) admits an analytic and bounded extension to \( \mathbb{C}^+ - i \varepsilon_2 \) and \( \phi_- \) admits an analytic and bounded extension to \( \mathbb{C}^- + i \varepsilon_1 \) (which we denote by \( \phi_{\pm} \) as well, respectively).

Taking this into account, we can state the following.

**Theorem 2.2.** Let
\[
G \in (H_{\infty}(S_{-\varepsilon_2, \varepsilon_1}))^{2 \times 2} , \quad \det G = \gamma \quad \text{and} \quad \gamma^{-1} \in H_{\infty}(S_{-\varepsilon_2, \varepsilon_1}) ,
\]
with \( \varepsilon_1, \varepsilon_2 \in [0, +\infty[ \) and let \( \phi_{\pm} = (\phi_{1\pm}, \phi_{2\pm}) \in (H_{\infty}^\pm)^2 \) be such that \( G \phi_+ = \phi_- \). Then (using the notation \( S = S_{-\varepsilon_2, \varepsilon_1} \) for simplicity)
\[
\begin{align*}
\inf_{S} (|\phi_1^-| + |\phi_2^-|) > 0 & \quad \Rightarrow \quad \inf_{S} (|\phi_1^+| + |\phi_2^+|) > 0 , \\
\inf_{S} (|\phi_1^-| + |\phi_2^+|) > 0 & \quad \Rightarrow \quad \inf_{S} (|\phi_1^-| + |\phi_2^-|) > 0 .
\end{align*}
\]

**Proof.** Assume that
\[
G = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .
\]
We prove only (2.5), since (2.6) can be obtained analogously. Let us assume that
\[
\inf_{S} (|\phi_1^+|) = 0 ,
\]
then there is a sequence \( (\xi_n)_{n \in \mathbb{N}} \) of points in \( S \) such that
\[
\phi_1^+(\xi_n) \to 0 , \quad \phi_2^+(\xi_n) \to 0 .
\]
Since, according to (2.3) and (2.7),
\[
\phi_- = a \phi_1^+ + b \phi_2^+ ,
\]
and both \( a \) and \( b \) are analytic and bounded in \( S \), we have also \( \phi_1^-(\xi_n) \to 0 \), which implies that
\[
\inf_{S} (|\phi_1^-|) = 0 . \quad \square
\]

The next theorem shows that several conditions that we might call of “corona type” are equivalent in a strip \( S_{-\varepsilon_2, \varepsilon_1} \), with \( \varepsilon_1, \varepsilon_2 \in [0, +\infty[ \), so that we are free to choose different pairs of functions in the set \( \{ \phi_{1\pm}, \phi_{1\pm}, \phi_{2\pm}, \phi_{2\pm} \} \) to express the same property.

**Theorem 2.3.** Let \( S = S_{-\varepsilon_2, \varepsilon_1} \) with \( \varepsilon_1, \varepsilon_2 \in [0, +\infty[ \) and let \( G \) of the form (2.7) satisfy the conditions
\[
G \in (H_{\infty}(S))^{2 \times 2} , \quad \det G = \gamma \quad \text{and} \quad \gamma^{-1} \in H_{\infty}(S) .
\]
Let moreover \( \phi_{\pm} = (\phi_{1\pm}, \phi_{2\pm}) \in (H_{\infty}^\pm)^2 \) be such that \( G \phi_+ = \phi_- \). Then the following propositions are equivalent:

(i) \( \inf_{S} (|\phi_1^+| + |\phi_2^+|) > 0 , \)
(ii) \( \inf_{S} (|\phi_1^-| + |\phi_2^-|) > 0 , \)
(iii) \( \inf_{S} (|\phi_2^+| + |\frac{1}{\varepsilon}(\phi_2^- - d \phi_2^+)|) > 0 , \)
(iv) \( \inf_{S} (|\phi_2^-| + |\frac{1}{\varepsilon}(a \phi_2^- - \gamma \phi_2^+)|) > 0 . \)
Proof. First we remark that, in (iii) and (iv), \( \frac{1}{c}(\varphi_2 - d\varphi_2) \) and \( \frac{1}{c}(a\varphi_2 - \varphi_2) \) are understood in the sense of analytic extensions, if necessary.

(i) \( \Rightarrow \) (ii). In fact, if

\[
\inf_S (|\varphi_1| + |\varphi_2|) = 0,
\]

then, for some sequence \((\xi_n)\) with \(\xi_n \in S\) for all \(n \in \mathbb{N}\), we have

\[
\varphi_1(\xi_n) \to 0, \quad \varphi_2(\xi_n) \to 0.
\]

Since

\[
\varphi_1 = \gamma^{-1}(d\varphi_1 - b\varphi_2),
\]

\[
\varphi_2 = \gamma^{-1}(-c\varphi_1 + a\varphi_2)
\]

where \(\gamma^{-1}, a, b, c, d\) are bounded in \(S\), we must have also

\[
\varphi_1(\xi_n) \to 0, \quad \varphi_2(\xi_n) \to 0,
\]

so that

\[
\inf_S (|\varphi_1| + |\varphi_2|) = 0.
\]

(ii) \( \Rightarrow \) (i) analogously.

(i) \( \Leftrightarrow \) (iii) because \(\varphi_1 = \frac{1}{c}(\varphi_2 - d\varphi_2)\), where the right-hand side of this equality is understood as the analytic extension at any point where \(c\) vanishes.

(ii) \( \Leftrightarrow \) (iv) because \(\varphi_1 = \frac{1}{c}(a\varphi_2 - \varphi_2)\).

Def. 2.4. Let \(S = S_{-\varepsilon_2,\varepsilon_1}\) with \(\varepsilon_1, \varepsilon_2 \in [0, +\infty]\). If two functions \(\psi_1, \psi_2 \in H_\infty(S)\) satisfy

\[
\inf_S (|\psi_1| + |\psi_2|) > 0,
\]

then we say that \((\psi_1, \psi_2) \in CT(S)\).

As a consequence of Theorem 2.3, we have the following.

Theorem 2.5. Let \(G\) satisfy condition (2.4) for some \(\varepsilon_1, \varepsilon_2 \in [0, +\infty]\) and let \((\varphi_+, \varphi_-)\) be a solution to (2.3). Then \(\varphi_\pm \in CT^\pm\) if

\[
\inf_{\varepsilon_1 > 0} (|\varphi_1| + |\varphi_2|) > 0, \quad \text{(2.8)}
\]

\[
\inf_{\varepsilon_2 < 0} (|\varphi_1| + |\varphi_2|) > 0, \quad \text{(2.9)}
\]

and one of the conditions (i)–(iv) in Theorem 2.3 holds for \(S = S_{-\varepsilon_2,\varepsilon_1}\).

Proof. Let \(\varphi_\pm \in CT^\pm\). Since \(\varphi_+ \in CT^+\), we have \(\varphi_1+, \varphi_2+ \in CT(S \cap \mathbb{C}^+)\). By Theorem 2.3, we conclude that

\[
\varphi_1-, \varphi_2- \in CT(S \cap \mathbb{C}^+). \quad \text{(2.10)}
\]

Analogously, as \(\varphi_- \in CT^-\), we have

\[
\varphi_1-, \varphi_2- \in CT(S \cap \mathbb{C}^-). \quad \text{(2.11)}
\]

From (2.10) and (2.11) we conclude that

\[
(\varphi_1-, \varphi_2-) \in CT(S).
\]

So condition (ii) of Theorem 2.3 is satisfied and it is clear that (2.8) and (2.9) also hold.

Conversely, if one of the conditions (i)–(iv) in Theorem 2.3 holds, then (i) and (ii) both hold which, together with (2.8) and (2.9), implies that \(\varphi_\pm \in CT^+. \quad \Box\)
We remark that for several important classes of matrix symbols $G$, the behaviour “at infinity,” which is translated by conditions (2.8) and (2.9), is easy to study for big enough $\varepsilon_1, \varepsilon_2$. Therefore, verifying that $\phi_{\pm} \in CT_{\pm}$ (which implies the invertibility of the Toeplitz operator with symbol $G$ when $\det G = 1$) is reduced to verifying one of the conditions (i)--(iv) in Theorem 2.3. This, in turn, means roughly that we should be able to compare the zeros of two particular functions in the strip $S_{-\varepsilon_2, \varepsilon_1}$. The results of the next section can be understood in this context.

3. Corona conditions and common zeros

It is clear that verifying the corona conditions for a pair of functions in $H^+_\infty$ (or $H^-\infty$) involves more than just studying their common zeros. Nonetheless in this section we show that the verification of those conditions can be closely related to the study of the common points in two algebraic curves.

Theorem 3.1. Let $S \subset \mathbb{C}$ and let $\varphi : S \rightarrow \mathbb{C}^2$ be a function such that $\varphi(S)$ is bounded. Let $D \subset \mathbb{C}^2$ be such that $D \supset \varphi(S)$ and let $F, H : D \rightarrow \mathbb{C}$ be continuous functions. Let moreover $f = F \circ \varphi, h = H \circ \varphi$. Then

$$\inf_{\xi \in S} \left( |f(\xi)| + |h(\xi)| \right) = 0 \quad (3.1)$$

iff

$$\begin{cases} F(z, w) = 0, \\ H(z, w) = 0 \end{cases} \quad (3.2)$$

admits a solution $(z_0, w_0) \in \overline{\varphi(S)}$.

Proof. If (3.1) is satisfied, then there is a sequence $(\xi_n)$, with terms $\xi_n \in S$ for all $n \in \mathbb{N}$, such that

$$f(\xi_n) \rightarrow 0, \quad h(\xi_n) \rightarrow 0.$$

Therefore, defining $(z_n, w_n) = \varphi(\xi_n)$, we have

$$F(z_n, w_n) \rightarrow 0, \quad H(z_n, w_n) \rightarrow 0. \quad (3.3)$$

Since $(z_n, w_n)$ is bounded in $\mathbb{C}^2$, it admits a convergent subsequence, so that, without loss of generality, we can assume that $(z_n, w_n)$ is convergent in $\mathbb{C}^2$.

Let

$$(z_0, w_0) = \lim_{n \rightarrow \infty} (z_n, w_n) \in \overline{\varphi(S)} \subset D.$$ 

Since $F$ and $H$ are continuous in $D$, we have, from (3.3),

$$F(z_0, w_0) = 0, \quad H(z_0, w_0) = 0.$$ 

Thus (3.2) admits a solution $(z_0, w_0) \in \overline{\varphi(S)}$.

Conversely, if (3.2) is satisfied for some $(z_0, w_0) \in \overline{\varphi(S)}$, then

$$(z_0, w_0) = \lim_{n \rightarrow \infty} \varphi(\xi_n)$$

for some sequence $(\xi_n)$ with terms in $S$.

Then

$$\lim_{n \rightarrow \infty} f(\xi_n) = \lim_{n \rightarrow \infty} F(\varphi(\xi_n)) = F(z_0, w_0) = 0$$

and, analogously,

$$\lim_{n \rightarrow \infty} h(\xi_n) = 0,$$

so that

$$\inf_{\xi \in S} \left( |f(\xi)| + |h(\xi)| \right) = 0. \quad \Box$$
Let us now consider a particular case, where $S = S_{-\varepsilon_2,\varepsilon_1}$ is defined as in (2.1), with $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, and $\varphi : S \to \mathbb{C}^2$ is given by

\begin{equation}
(z, w) = \varphi(\xi) = (e^{i\alpha\xi}, e^{-i\beta\xi}), \quad \xi \in S,
\end{equation}

for fixed $\alpha, \beta > 0$, $\alpha \neq \beta \notin \mathbb{Q}$. With these assumptions, let us establish conditions for a point $(z_0, w_0) \in \mathbb{C}^2$ to belong to $\overline{\varphi(S)}$. In order to do this, we first prove an auxiliary result.

**Lemma 3.2.** Let $\alpha, \beta > 0$, $\alpha \beta \notin \mathbb{Q}$ and $\theta, \tilde{\theta} \in \mathbb{R}$. Then there exist sequences $(m_k)$ and $(n_k)$ of integer numbers such that

\begin{equation}
\lim_{k \to \infty} u_k = 0
\end{equation}

where $u_k = \frac{1}{\alpha}(\theta + 2m_k\pi) + \frac{1}{\beta}(\tilde{\theta} + 2n_k\pi)$.

**Proof.** For every $k \in \mathbb{N}$, let $\varepsilon_k = \frac{\alpha}{2\pi k}$. From Kronecker’s theorem in one dimension [9], there are $n_k \in \mathbb{N}, m_k \in \mathbb{Z}$ such that

\begin{equation}
\left| n_k \frac{\alpha}{\beta} + m_k + \frac{\alpha\tilde{\theta} + \beta\theta}{2\pi\beta} \right| < \varepsilon_k
\end{equation}

which means that

\begin{equation}
\left| \frac{1}{\alpha}(\theta + 2m_k\pi) + \frac{1}{\beta}(\tilde{\theta} + 2n_k\pi) \right| < \frac{1}{k} \to 0. \quad \square
\end{equation}

With $S = S_{-\varepsilon_2,\varepsilon_1}$ and $\varphi$ defined by (3.4), we have the following.

**Theorem 3.3.** Let $(z_0, w_0)$ be such that

\begin{equation}
-\frac{1}{\alpha} \log |z_0| \in ]-\varepsilon_2, \varepsilon_1[.
\end{equation}

Then $(z_0, w_0) \in \overline{\varphi(S)}$ iff

\begin{equation}
-\frac{1}{\alpha} \log |z_0| = -\frac{1}{\beta} \log |w_0|.
\end{equation}

**Proof.** Let $(z_0, w_0) \in \overline{\varphi(S)}$; then there exists $(\xi_n)$, with $\xi_n \in S$, such that

\begin{equation}
(z_0, w_0) = \lim_{n \to \infty} \varphi(\xi_n) = \lim_{n \to \infty} (e^{i\alpha\xi_n}, e^{-i\beta\xi_n}).
\end{equation}

Let $z_n = e^{i\alpha\xi_n}, w_n = e^{-i\beta\xi_n}$. We have

\begin{equation}
\text{Im}(\xi_n) = -\frac{1}{\alpha} \log |z_n| = \frac{1}{\beta} \log |w_n|
\end{equation}

and since $|z_n| \to |z_0|$ and $|w_n| \to |w_0|$, it follows that $\text{Im}(\xi_n)$ converges when $n \to \infty$ and

\begin{equation}
-\frac{1}{\alpha} \log |z_0| = -\frac{1}{\beta} \log |w_0|.
\end{equation}

Conversely, let $(z_0, w_0) \in \mathbb{C}^2$ be such that the previous equality holds. Let $\theta$ and $\tilde{\theta}$ be some fixed values of $\arg z_0$ and $\arg w_0$, respectively, and let

\begin{align*}
\xi_k &= -\frac{i}{\alpha} \log |z_0| + \frac{1}{\alpha}(\theta + 2m_k\pi), \\
\tilde{\xi}_k &= \frac{i}{\beta} \log |w_0| - \frac{1}{\beta}(\tilde{\theta} + 2n_k\pi)
\end{align*}
where \((m_k)\) and \((n_k)\) are sequences of integer numbers such that 
\(\xi_k = \xi_k - u_k\) with \(u_k \to 0\), according to Lemma 3.2. We have 
\(\tilde{\xi}_k \in S\) and 
\[\varphi(\tilde{\xi}_k) = (z_0, w_0 e^{-i\beta u_k})\], 
so that 
\[\lim_{k \to \infty} \varphi(\tilde{\xi}_k) = (z_0, w_0) \in \varphi(S). \]

As a consequence of Theorems 3.1 and 3.3, we can now establish necessary and sufficient conditions for some pairs of functions \((f, h)\) to satisfy the corona conditions in a strip, meaning that \(f\) and \(h\) do not approach 0 simultaneously in the strip.

**Theorem 3.4.** Let \(F, H\) be continuous functions (in \(\mathbb{C}^2\)) and let \(S = S_{-\varepsilon_2, \varepsilon_1}\) with \(\varepsilon_1, \varepsilon_2 \in [0, +\infty[\) be such that, for all common zeros \((z_0, w_0)\) of \(F\) and \(H\), condition (3.5) is satisfied. Let moreover \(f = F \circ \varphi, h = H \circ \varphi\) with \(\varphi : S \to \mathbb{C}^2\) defined by (3.4). Then 
\[\inf_{\xi \in S} \left( |f(\xi)| + |h(\xi)| \right) = 0\]
iff there is a solution \((z_0, w_0)\) of (3.2) for which condition (3.6) is satisfied.

**Proof.** This is a direct consequence of Theorems 3.1 and 3.3.

4. Applications to some classes of Toeplitz operators

Now we use the previous results to study the existence of a canonical Wiener–Hopf factorization for matrix functions \(G\) in several classes, which is equivalent to studying the invertibility of \(TG\), applying those results in two different ways. This is the reason why we divide this section into two subsections, each one corresponding to a particular direction of application.

4.1. For all the classes of matrix functions studied in this subsection, \(G\) takes the form
\[G = \begin{bmatrix} E^{-1} & 0 \\ g & E \end{bmatrix}\]  
(4.1)
where the notation \(E^v (v \in \mathbb{R})\) stands for the function \(E^v(\xi) = e^{iv\xi} (\xi \in \mathbb{R})\) and \(g\) is an almost-periodic polynomial such that the biggest distance between two points of its Fourier spectrum is less than 1.

It is known that, in this case, a Wiener–Hopf factorization for \(G\), if it exists, is bounded, canonical, and with factors in \((\text{APW})^{2 \times 2}\) (where \(\text{APW}\) denotes the algebra of all functions \(a\) which can be written in the form \(a = \sum_j a_j E^{\lambda_j}\) with \(a_j \in \mathbb{C}, a_j \neq 0\) for at most countably many \(j, \sum_j |a_j| < \infty\) and \(\lambda_j \in \mathbb{R}\) [3]. Therefore, \(TG\) is Fredholm iff it is invertible and the existence of non-trivial \(\phi_\pm \in CT^\pm\) such that
\[G\phi_+ = \phi_-\]  
(4.2)
is a necessary and sufficient condition for Fredholmness (and invertibility) of \(TG\) [1]. Thus, we approach this problem by studying Eq. (4.2) with \(\phi_\pm \in (H^\pm)\).

It is clear that, in order to apply the preceding results, two previous questions have to be answered, namely how to determine a particular non-trivial solution to (4.2) and how to choose the best condition to verify, among (i)–(iv) in Theorem 2.3.

The answer to the second question must be given after evaluating which condition is easier to verify in each case and this obviously depends on the answer to the first question.

As to the latter, it should be noted that it has an interest of its own, independently from the fact that \(G\) admits any kind of factorization.

In the examples that we consider here, although the solutions to (4.2) are explicitly given and can be checked directly, it seems useful to understand how they were obtained. For the first and the third examples, almost-periodic
polynomial solutions were obtained in [11]. In the fourth example, the results of [4] are used. As to the second example, a table method such as that presented in [5] and developed in [6] is used to obtain an almost-periodic solution with Fourier spectrum in a group $\alpha Z + \beta Z$ with given $\alpha, \beta \in \mathbb{R}$.

In all four cases we obtain necessary and sufficient conditions for existence of a Wiener–Hopf factorization for $G$ and, thus, for invertibility of $T_G$. It should be noted that, in Example 4.1.1, such conditions have been determined before in [2], where a wider class of matrix functions is studied. However, we deduce them here in a different and simpler way and, with this approach, those conditions also appear naturally in a simpler form. In the other three examples, these conditions are obtained here (explicitly and merely in terms of the points of $\text{sp}(g)$ and the corresponding Fourier coefficients) for the first time.

Throughout we assume the notation $\sum_{j=1}^n x_j = 0$ if $i > n$.

**Example 4.1.1.** Let $g$ in (4.1) be a trinomial of the form

$$g = aE^\alpha + b + cE^{-\beta},$$

with

$$a, b, c \in \mathbb{C} \setminus \{0\}, \quad \alpha, \beta \in ]0, 1[, \quad \frac{\alpha}{\beta} \notin \mathbb{Q},$$

$$2\alpha + \beta = 1, \quad \beta < \alpha.$$  

(4.4)  

(4.5)

In this case, a non-trivial solution to the Riemann–Hilbert problem (4.2) is given by

$$\phi_{1+} = 1 + \sum_{l=0}^k \left(\frac{-a}{b}\right)^l E^{\alpha - l\beta} - \sum_{l=-1}^{k-1} \left(\frac{-a}{b}\right)^l E^{2\alpha - l\beta},$$

$$\phi_{1-} = E^{-1}\phi_{1+},$$

$$\phi_{2+} = -a^2 \frac{c}{\alpha} (k + 1) + a^3 \frac{b}{\beta} k \sum_{l=-1}^{k-1} \left(\frac{-c}{b}\right)^l E^{\alpha - (l+1)\beta},$$

$$\phi_{2-} = b + cE^{-\beta} - \frac{ac}{b} \left(\frac{-c}{b}\right)^k E^{\alpha - k\beta},$$

with $k = \lceil \frac{\alpha}{\beta} \rceil$.

Knowing a solution to the Riemann–Hilbert problem, we now want to study this solution in order to establish necessary and sufficient conditions for existence of a (canonical) Wiener–Hopf factorization for the matrix $G$.

Taking into account the expressions of $\phi_{2+}$ and $\phi_{2-}$ given by (4.8) and (4.9), respectively, it is easy to see that, for sufficiently big $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, we have

$$\inf_{\mathbb{C}^{\varepsilon_1+i\varepsilon_2}} (|\phi_{1+}| + |\phi_{2+}|) > 0,$$

$$\inf_{\mathbb{C}^{\varepsilon_2-i\varepsilon_1}} (|\phi_{1-}| + |\phi_{2-}|) > 0.$$  

(4.10)  

(4.11)

Therefore, from Theorem 2.5, to prove that $\phi_{\pm} \in \mathcal{CT}^{\pm}$ it is enough to show that one of the conditions (i)–(iv) in Theorem 2.3 holds in the strip $S = S_{\varepsilon_1, \varepsilon_2}$ for $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ such that (4.10) and (4.11) hold. So we will show that

$$\inf_S \left( |\phi_{2-}| + \left| \frac{E^{-1}\phi_{2-} - \phi_{2+}}{g} \right| \right) > 0.$$  

(4.12)

It is clear that, for any strip $S$ of the above-mentioned type, (4.12) is equivalent to

$$\inf_S \left( |\phi_{2-}| + \left| \frac{E^{-\beta}\phi_{2-} - E^{2\alpha}\phi_{2+}}{g} \right| \right) > 0.$$  

(4.13)

With the notation

$$f = \phi_{2-}, \quad h = \frac{E^{-\beta}\phi_{2-} - E^{2\alpha}\phi_{2+}}{g}$$

(4.14)
we have
\[ f = F \circ \varphi \quad \text{and} \quad h = H \circ \varphi, \]
with \( \varphi : S \rightarrow \mathbb{C}^2 \) defined by (3.4) and
\begin{align*}
F(z, w) &= b + cw - \frac{ac}{b} \left( -\frac{c}{b} \right)^k zw^{k+1}, \\
H(z, w) &= \frac{N(z, w)}{D(z, w)},
\end{align*}
with
\begin{align*}
N(z, w) &= wF(z, w) + \left( \frac{a^2}{c} (k + 1) - \frac{a^3}{b^2} \sum_{l=-1}^{k-1} (k - l) \left( -\frac{c}{b} \right)^l zw^l \right) z^2, \\
D(z, w) &= az + b + cw.
\end{align*}
Noticing that \( H \) is a polynomial given by
\begin{align*}
H(z, w) &= \left( 1 + \sum_{l=0}^{k} \left( -\frac{a}{b} \right) \left( -\frac{c}{b} \right)^l zw^l - \sum_{l=-1}^{k-1} (k - l) \left( -\frac{a}{b} \right)^2 \left( -\frac{c}{b} \right) z^2 w^l \right) w,
\end{align*}
as well as \( F \), so that \( F \) and \( H \) are continuous in \( \mathbb{C}^2 \).
Following Theorem 3.4, we now study the solutions of
\begin{align*}
\begin{cases}
F(z, w) = 0, \\
H(z, w) = 0.
\end{cases}
\end{align*}
For any solution of (4.18), we have \( w \neq 0 \) and \( z \neq 0 \) and from the first equation of (4.18) we get
\[ z = (b + cw) \frac{b}{ac} \left( -\frac{b}{c} \right)^k w^{-(k+1)}. \]
On the other hand, it is clear that all the solutions to (4.18) must also satisfy the equalities
\begin{align*}
\begin{cases}
F(z, w) = 0, \\
N(z, w) = 0, \quad \text{with} \ z \neq 0, \ w \neq 0.
\end{cases}
\end{align*}
Therefore, we start by determining all the solutions to (4.20), which, taking (4.19) into account, is equivalent to
\begin{align*}
\begin{cases}
z = (b + cw) \frac{b}{ac} \left( -\frac{b}{c} \right)^k w^{-(k+1)}, \\
(k + 1)w^{k+1} - \left( 1 + \frac{c}{b} w \right) \sum_{j=1}^{k+1} j \left( -\frac{b}{c} \right)^j w^{-j+k+1} = 0
\end{cases}
\end{align*}
(with \( z \neq 0, \ w \neq 0 \)).
This system of equations admits \( k + 1 \) solutions which are
\begin{align*}
(z_0, w_0) &= \left( -\frac{b}{a} \frac{(k + 2)^k}{(k + 1)^{k+1}}, -\frac{b}{c} \frac{k + 1}{k + 2} \right),
\end{align*}
and
\begin{align*}
(z_j, w_j) &= \left( -\frac{b}{a} (1 - \alpha_j^{-1}), -\frac{b}{c} \alpha_j^{-1} \right), \quad j = 1, 2, \ldots, k.
\end{align*}
where \( \alpha_j = e^{i \frac{2\pi j}{k+1}}, j = 1, 2, \ldots, k. \)
It is clear that $H(z_0, w_0) = 0$ (since $N(z_0, w_0) = 0, D(z_0, w_0) \neq 0$). As to the points $(z_j, w_j), j = 1, \ldots, k$, it is easy to see that $H(z_j, w_j) \neq 0$.

So we conclude that the unique solution to the system (4.18) is $(z_0, w_0)$ given by (4.22).

Let $S = S_{-\epsilon_2, \epsilon_1}$ be such that (3.5) is satisfied, as well as (4.10) and (4.11). From Theorem 3.4 we conclude that

$$
\inf_S \left( |\phi_{2-} - \phi_{2+} - \frac{E^{-1} \phi_{2-} - \phi_{2+}}{g} | \right) > 0
$$

iff

$$
\left| \frac{a}{b} \right| \frac{c}{b} \neq \frac{k + 1}{k + 2} \left( \frac{k + 2}{k + 1} \right)^{\frac{\alpha}{\beta}}.
$$

(4.24)

Therefore, $T_G$ is invertible (and $G$ admits a canonical bounded Wiener–Hopf factorization) iff (4.24) holds, which, although in a different form, is equivalent to the condition obtained in [2].

**Example 4.1.2.** For a trinomial $g$ in (4.1) of the form $g = aE^\alpha + b + cE^{-\beta}$ a complete answer to the problem of existence of a Wiener–Hopf factorization for the matrix $G$, when $2\alpha + \beta = 1$ and $\frac{\alpha}{\beta} \notin \mathbb{Q}$, was given in [2] (whether $\beta > \alpha$ or $\beta < \alpha$) and, in particular, it coincides with (4.24), when $\beta < \alpha$.

However, when we look for explicit solutions to the Riemann–Hilbert problem (4.2), we see that the case where $\beta > \alpha$ admits simpler almost-periodic polynomial solutions (cf. [6,11]). Therefore, taking the approach of the present paper, it is simpler to verify if a non-trivial solution to (4.2) satisfies the corona conditions when $\beta > \alpha$.

Now, it seems natural to proceed to a next step in this study by adding more points to the spectrum of $g$, thus obtaining conditions for existence of a Wiener–Hopf factorization of matrix functions in a wider class.

Therefore, we take now $g$, not as a trinomial, but as an almost-periodic polynomial admitting more than three points in its spectrum. In fact, if

$$
g = aE^\alpha + b + \sum_{j=1}^{k} c_j E^{-j\beta}, \quad k = \left[ \frac{1}{\beta} \right],
$$

(4.25)

with

$$
a, b \in \mathbb{C} \setminus \{0\}, \quad \sum_{j=1}^{k} |c_j| \neq 0, \quad \alpha, \beta \in ]0, 1[, \quad \frac{\alpha}{\beta} \notin \mathbb{Q},
$$

(4.26)

$$
n\alpha + \beta = 1, \quad \beta > (n - 1)\alpha, \quad n \in \mathbb{N},
$$

(4.27)

a non-trivial almost-periodic polynomial solution always exists and can be determined using a table method of the type presented in [5] and [6].

We consider here only the case where $n = 2$ and some additional conditions are satisfied. A similar study can be carried out in the general case (4.25) but this study goes beyond the scope of the present paper.

So let $g$ in (4.1) be of the form (4.25) with $n = 2$ (which implies that $k = 1$ or $k = 2$),

$$
g = aE^\alpha + b + c_1 E^{-\beta} + c_2 (k - 1) E^{-2\beta}
$$

(4.28)

where

$$
2\alpha + \beta = 1, \quad \beta > \alpha, \quad 3\alpha - 2\beta < 0.
$$

(4.29)

We remark that, with these conditions, we have $\alpha + \beta > \frac{1}{2}$, but, in our case, we cannot use the results of [11] to conclude immediately from this inequality that (4.2) admits an almost-periodic polynomial solution, because $g$ is not, in general, a trinomial. Here we will show that there is indeed such a solution and study it to obtain conditions for existence of a generalized factorization of $G$.

In fact, a first solution to the Riemann–Hilbert problem (4.2) can be obtained as we said before. In Table 1, the Fourier spectrum of $\phi_{1+}$ (with points of the form $j\alpha - l\beta$) is represented for the case where $k = 2$ and $c_1, c_2 \neq 0$. In all other cases, $\text{sp} \phi_{1+}$ is contained in the subset of $\alpha\mathbb{Z} + \beta\mathbb{Z}$ represented there.
The Fourier coefficients of $\varphi_{1+}$ can easily be obtained from Table 1 (using the same reasoning as in [5] and [6]). A non-trivial solution to the Riemann–Hilbert problem (4.2) is thus determined and we get, with $\tilde{c}_2 = c_2(k-1)$,

$$\varphi_{1+} = (2bc_1^2 - b^2\tilde{c}_2) + (a\tilde{c}_2 - 2ac_1^2)E^\alpha - a^2\tilde{c}_2E^{2\alpha} + \frac{a^3\tilde{c}_2}{b}E^{3\alpha} + 2a^2c_1E - \frac{a^2c_1\tilde{c}_2}{b}E^{2\alpha - \beta},$$  \hspace{1cm} (4.30)

$$\varphi_{1-} = 2a^2c_1 + (ab\tilde{c}_2 - 2ac_1^2)E^{-\alpha - \beta} - a^2\tilde{c}_2E^{-\beta} + \frac{a^3\tilde{c}_2}{b}E^{-\alpha - \beta} + (2bc_1^2 - b^2\tilde{c}_2)E^{-1} - \frac{a^2c_1\tilde{c}_2}{b}E^{-2\beta},$$  \hspace{1cm} (4.31)

$$\varphi_{2+} = -2a^2c_1b - 2a^3c_1E^\alpha - \frac{a^4\tilde{c}_2}{b}E^{2\alpha - \beta},$$  \hspace{1cm} (4.32)

$$\varphi_{2-} = (2b^2c_1^2 - b^3\tilde{c}_2) + (2bc_1^3 - b^2c_1\tilde{c}_2)E^{-\beta} + (abc_1\tilde{c}_2 - 2ac_1^3)E^{-\alpha - \beta} + \left(-\frac{a^2c_1^2\tilde{c}_2}{b}\right)E^{2\alpha - 2\beta}$$  
$$+ (2bc_1^2\tilde{c}_2 - b^2\tilde{c}_2^2)E^{-2\beta} + (ab\tilde{c}_2 - 2ac_1^2\tilde{c}_2)E^{\alpha - 2\beta} - \frac{a^2c_1\tilde{c}_2}{b}E^{-2\beta} + \frac{a^3c_2}{b}E^{3\alpha - 2\beta}. $$  \hspace{1cm} (4.33)

Knowing a non-trivial solution to the Riemann–Hilbert problem, we now want to establish necessary and sufficient conditions for existence of a Wiener–Hopf factorization for the matrix $G$.

It is easy to see that, for sufficiently big $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, we have

$$\inf_{\mathbb{C}^+ + i\varepsilon_1} (|\varphi_{1+}| + |\varphi_{2+}|) > 0, $$  \hspace{1cm} (4.34)

$$\inf_{\mathbb{C}^- - i\varepsilon_2} (|\varphi_{1-}| + |\varphi_{2-}|) > 0, $$  \hspace{1cm} (4.35)

because, taking into account that $|c_1| + |c_2| \neq 0$, if $c_1 = 0$, then $\varphi_{1+}$ and $\varphi_{2-}$ have non-trivial constant terms and, if $c_1 \neq 0$, then $\varphi_{1-}$ and $\varphi_{2+}$ have non-trivial constant terms.

So, from Theorem 2.5, to prove that $\varphi_{\pm} \in CT^\infty$ it is enough to show that one of the conditions (i)–(iv) in Theorem 2.3 holds in a strip $S = S_{-\varepsilon_2, \varepsilon_1}$ for $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ such that (4.34) and (4.35) hold. So we will show that

$$\inf_S (|\varphi_{1+}| + |\varphi_{2+}|) > 0, $$  \hspace{1cm} (4.36)

It is clear that, for a strip $S$ of the above-mentioned type, (4.36) is equivalent to

$$\inf_S (|E^{-\beta}\varphi_{1+}| + |\varphi_{2+}|) > 0. $$  \hspace{1cm} (4.37)

With the notation

$$f = E^{-\beta}\varphi_{1+}, \quad h = \varphi_{2+} $$  \hspace{1cm} (4.38)

we have

$$f = F \circ \varphi \quad \text{and} \quad h = H \circ \varphi, $$  \hspace{1cm} (4.39)

with $\varphi : S \to \mathbb{C}^2$ defined by (3.4) and

$$F(z, w) = (2bc_1^2 - b^2\tilde{c}_2)z + (ab\tilde{c}_2 - 2ac_1^2)zw - a^2\tilde{c}_2z^2w + \frac{a^3\tilde{c}_2}{b}z^3w + 2a^2c_1z^2 - \frac{a^2c_1\tilde{c}_2}{b}zw^2, $$  \hspace{1cm} (4.40)

$$H(z, w) = -2a^2c_1b - 2a^3c_1z - \frac{a^4\tilde{c}_2}{b}z^2w. $$  \hspace{1cm} (4.41)
Let us assume firstly that $c_1 \neq 0$.
The functions $F$ and $H$ are continuous in $C^2$, so, following Theorem 3.4, we now study the solutions of
\[
\begin{align*}
F(z, w) &= 0, \\
H(z, w) &= 0.
\end{align*}
\] (4.42)
For any solution of (4.42), we have $w \neq 0$ and $z \neq 0$ because $c_1 \neq 0$. The second equation of (4.42) is equivalent to
\[
a^2 c_2 z^2 w = -2bc_1 (az + b).
\] (4.43)
We remark that, if $b \tilde{c}_2 = 4c_1^2$, then there is no solution to the system (4.42). So, in this case, we conclude that the
matrix $G$ admits a canonical factorization.

Let us consider now $b \tilde{c}_2 \neq 4c_1^2$. Replacing (4.43) in the first equation of (4.42), we obtain
\[
z_0 = b \left( \frac{b \tilde{c}_2 - 4c_1^2}{4c_1^2} \right).
\] (4.44)
Replacing (4.44) in (4.43), we determine the solution to the system (4.42), which is given by
\[
\begin{align*}
z_0 &= b \left( \frac{b \tilde{c}_2 - 4c_1^2}{4c_1^2} \right), \\
w_0 &= -\frac{8bc_1^3}{(b \tilde{c}_2 - 4c_1^2)^2}.
\end{align*}
\] (4.45)
Let $S = S_{\tilde{c}_2, c_1}$ be such that (3.5) is satisfied, as well as (4.34) and (4.35). From Theorem 3.4 we conclude that
\[
\inf_S (|\varphi_1| + |\varphi_2|) > 0
\] iff
\[
|a|^\beta |2c_1|^{\frac{2\beta}{\alpha} - 3} \neq |b|^\beta + 1 |b \tilde{c}_2 - 4c_1^2|^\frac{\beta}{\alpha} - 2.
\] (4.46)
Therefore, $T_G$ is invertible (and $G$ admits a canonical bounded Wiener–Hopf factorization) iff (4.46) holds.

If $c_1 = 0$ and $k = 2$, then we have a trinomial $g = a E^\alpha + b + c_2 E^{-2\beta}$ with $2\beta + \alpha > 1$ and $G$ admits a canonical
Wiener–Hopf factorization (cf. [5]).

We have thus proved the following theorem.

**Theorem 4.1.** Let $G$ be given by (4.1), where $g$ takes the form (4.28), with $\alpha, \beta$ satisfying (4.26) and (4.29). Then a
solution to the Riemann–Hilbert problem (4.2) is given by (4.30)–(4.33) and $G$ admits a canonical bounded Wiener–Hopf factorization iff
\[
|a|^\beta |2c_1|^{\frac{2\beta}{\alpha} - 3} \neq |b|^\beta + 1 |b \tilde{c}_2 - 4c_1^2|^\frac{\beta}{\alpha} - 2
\] (4.47)
where we replace the right-hand side by 0 if $b \tilde{c}_2 - 4c_1^2 = 0$ and $\frac{\beta}{\alpha} - 2 < 0$.

**Remark 4.2.** If we take $c_2 = 0$ in (4.28), we obtain from (4.47) the condition
\[
\left| \frac{a}{b} \right|^{\frac{\beta}{\alpha}} \left| \frac{c_1}{b} \right| \neq \frac{1}{2},
\]
which is the same condition that was already obtained in [2] for the trinomial case.

**Example 4.1.3.** Let us now consider a matrix function $G$ of the form (4.1), where $g$ is given by
\[
g = a E^\alpha + b + c E^{-\beta}.
\] (4.48)
with

\[ a, b, c \in \mathbb{C} \setminus \{0\}, \quad \alpha, \beta \in ]0, 1[, \quad \alpha \neq \frac{\beta}{n}. \tag{4.49} \]

\[ \alpha + \beta < 1 \quad \text{and} \quad n\alpha - \beta = 1 \quad \text{for} \ n \geq 2. \tag{4.50} \]

Let us assume moreover that

\[ \alpha + \beta > \max\{1 - \alpha, 1 - \beta\}. \tag{4.51} \]

Taking into account (4.50), we see that (4.51) is equivalent to

\[ \begin{cases} 
\frac{3}{2n+1} < \alpha < \frac{2}{n+1} & \text{if} \ 2 \leq n < 4, \\
\frac{2}{n+2} < \alpha < \frac{2}{n+1} & \text{if} \ n \geq 4.
\end{cases} \]

With these conditions, it is easy to see that \( \alpha + \beta > \frac{1}{2} \) and we can conclude that (4.2) admits an almost-periodic polynomial solution and determine it according to [11]. Nevertheless, to determine this solution, it is also possible (and it may be advantageous in this case) to use a table method such as that used in the previous example, since it gives what might be called a “graphical” insight of the problem, namely as to understanding the differences between the even and odd cases. In Tables 2 and 3, the Fourier spectrum of \( \phi_{1+} \) (with points of the form \( j\alpha - l\beta \)) is represented, considering the cases where \( n \) is even or odd separately.

The Fourier coefficients of \( \phi_{1+} \) can be obtained easily from Tables 2 and 3 (similarly to [5] and [6]). We have the following solution to the Riemann–Hilbert problem (4.2):

\[
\phi_{1+} = \sum_{j=0}^{[\frac{n+1}{2}]} \left( -\frac{a}{b} \right)^j E^{ja} + \sum_{j=0}^{n-1} p \left( -\frac{a}{b} \right)^j \frac{c}{b} E^{ja-n} + p \left( -\frac{a}{b} \right)^n \frac{c}{b} E^{n} + p \left( -\frac{a}{b} \right)^{n-1} \frac{c}{b} E^{2} + \frac{(p-1) (n-1)E^{1-\alpha-\beta} + \left( -\frac{a}{b} \right)^n \frac{c}{b} E^{1-\alpha-\beta}}{\left( -\frac{a}{b} \right)^n \frac{c}{b} E^{1-\alpha-\beta}}, \tag{4.52} \]

\[ E \]

\[ \text{Table 2} \]

\begin{tabular}{|c|c|c|}
\hline
\( j \backslash l \) & 0 & 1 \\
\hline
\hline
0 & 0 & \\
\hline
1 & \( \alpha \) & \( \alpha \) \\
\hline
2 & \( 2\alpha \) & \( 2\alpha \) \\
\hline
\vdots & \vdots & \vdots \\
\hline
\frac{n-1}{2} & \frac{n-1}{2} \alpha & \frac{n-1}{2} \alpha - \beta \\
\hline
\hline
\frac{n+1}{2} & \frac{n+1}{2} \alpha & \frac{n+1}{2} \alpha - \beta \\
\hline
\vdots & \vdots & \vdots \\
\hline
n-1 & (n-1)\alpha - \beta & (n-1)\alpha - 2\beta \\
\hline
n & n\alpha - \beta & \\
\hline
\end{tabular}

\[ \text{Table 3} \]

\begin{tabular}{|c|c|c|}
\hline
\( j \backslash l \) & 0 & 1 \\
\hline
\hline
0 & 0 & \\
\hline
1 & \( \alpha \) & \( \alpha \) \\
\hline
2 & \( 2\alpha \) & \( 2\alpha \) \\
\hline
\vdots & \vdots & \vdots \\
\hline
\frac{n-1}{2} & \frac{n-1}{2} \alpha & \frac{n-1}{2} \alpha - \beta \\
\hline
\hline
\frac{n+1}{2} & \frac{n+1}{2} \alpha & \frac{n+1}{2} \alpha - \beta \\
\hline
\vdots & \vdots & \vdots \\
\hline
n-1 & (n-1)\alpha - \beta & (n-1)\alpha - 2\beta \\
\hline
n & n\alpha - \beta & \\
\hline
\end{tabular}
\( \phi_{1-} = E^{-1} \phi_{1+} \), \( \phi_{2+} = 2p \left( -\frac{a}{b} \right)^n c + \left( -\frac{a}{b} \right)^{\left\lfloor \frac{n+1}{2} \right\rfloor} aE^{[\frac{n+1}{2}]+1-1} - p \left( -\frac{a}{b} \right)^{n+1} E^{\alpha}, \) \( \phi_{2-} = b + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left( -\frac{a}{b} \right)^j cE^{j\alpha} \) - \sum_{j=\left\lceil \frac{n+1}{2} \right\rceil}^{n-2} p \left( -\frac{a}{b} \right)^j c^2 E^{j\alpha - 2\beta} - (p-1) \left( -\frac{a}{b} \right)^{\frac{n-1}{2}} c^2 E^{\frac{n+1}{2} \alpha - 2\beta} - p \left( -\frac{a}{b} \right)^{n-1} c^3 ab E^{(n-1)\alpha - 3\beta} \) (4.55)

where

\[ p = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 2 & \text{if } n \text{ is odd}. \end{cases} \]

Having obtained a first solution to the Riemann–Hilbert problem, we will now use the results of Sections 2 and 3 to establish necessary and sufficient conditions for the existence of a Wiener–Hopf factorization for the matrix \( G \).

By (4.54) and (4.55), we can see that, for sufficiently big \( \varepsilon_1, \varepsilon_2 \in [0, +\infty[ \), we have

\[ \inf_{\mathbb{C}^+ + i\varepsilon_1} (|\phi_{1+}| + |\phi_{2+}|) > 0, \]

(4.56)

\[ \inf_{\mathbb{C}^- - i\varepsilon_2} (|\phi_{1-}| + |\phi_{2-}|) > 0. \]

(4.57)

So, from Theorem 2.5, to prove that \( \phi_\pm \) are corona pairs, it is enough to show that one of the conditions (i)–(iv) in Theorem 2.3 holds in a strip \( S = S - \varepsilon_2, \varepsilon_1 \) for \( \varepsilon_1, \varepsilon_2 \in [0, +\infty[ \) such that (4.56) and (4.57) hold. Therefore, we just have to show that

\[ \inf_S \left( |\phi_{2+}| + \left| \frac{\phi_{2-} - E\phi_{2+}}{g} \right| \right) > 0. \]

(4.58)

For a strip \( S \) of the above-mentioned type, (4.58) is equivalent to

\[ \inf_S \left( |E\phi_{2+}| + \left| \frac{\phi_{2-} - E\phi_{2+}}{g} \right| \right) > 0. \]

(4.59)

With the notation

\[ f = E\phi_{2+}, \quad h = \frac{\phi_{2-} - E\phi_{2+}}{g} \]

(4.60)

we have

\[ f = F \circ \varphi \quad \text{and} \quad h = H \circ \varphi, \]

(4.61)

with \( \varphi : S \to \mathbb{C}^2 \) defined by (3.4) and

\[ F(z, w) = 2p \left( -\frac{a}{b} \right)^n cz^n w + \left( -\frac{a}{b} \right)^{\left\lfloor \frac{n+1}{2} \right\rfloor} a z^{\left\lfloor \frac{n+1}{2} \right\rfloor+1} - p \left( -\frac{a}{b} \right)^{n+1} cz^{n+1} w, \]

(4.62)

\[ H(z, w) = \frac{N(z, w)}{D(z, w)}, \]

(4.63)

with

\[ N(z, w) = b + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left( -\frac{a}{b} \right)^j cz^j w - \sum_{j=\left\lceil \frac{n+1}{2} \right\rceil}^{n-2} p \left( -\frac{a}{b} \right)^j c^2 z^j w^2 - (p-1) \left( -\frac{a}{b} \right)^{\frac{n+1}{2}} c^2 z^{\frac{n+1}{2}} w^2 - p \left( -\frac{a}{b} \right)^{n-1} c^3 ab z^{n-1} w^3 - F(z, w) \]

and

\[ D(z, w) = az + b + cw. \]
Notice that $H$ is a polynomial given by
\[
H(z, w) = \sum_{j=0}^{n+1} \left( -\frac{a}{b} \right)^j \frac{c^j}{z^j} + \sum_{j=n+1}^{n-1} p \left( -\frac{a}{b} \right)^j \left( -\frac{c}{b} \right)^j \frac{w^j}{z^j} + p \left( -\frac{a}{b} \right)^n \frac{c^n}{b^n} \frac{w^n}{z^n}
\]
as well as $F$, so that $F$ and $H$ are continuous in $\mathbb{C}^2$.

Following Theorem 3.4, we now study the solutions of
\[
\begin{align*}
F(z, w) &= 0, \\
H(z, w) &= 0.
\end{align*}
\]
For any solution of (4.64), we have
\[
F(z, w) = H(z, w) = \frac{F(z, w)}{H(z, w)} = F(z, w) = 0.
\]
On the other hand, it is clear that all the solutions to (4.64) must also satisfy
\[
\begin{align*}
F(z, w) &= 0, \\
N(z, w) &= 0, \\
\text{with } z \neq 0, w \neq 0.
\end{align*}
\]
Therefore, we start by determining all the solutions to (4.66), which, taking (4.65) into account and using the change of variable $x = \frac{c}{b} z$, is equivalent to
\[
\begin{align*}
\begin{cases}
-w^{-1} x^{-[\frac{d}{2}] - 1} = -p \frac{c}{b} \left( -1 \right)^{[\frac{d}{2}]} (x + 2), \\
-p([x]) \frac{c}{b} [\frac{d}{2}] - 7 p(-x) \left( [\frac{d}{2}] \right) - 14 p(-x) \left( [\frac{d}{2}] \right) + 8 p(-x) \left( [\frac{d}{2}] \right) + x + 4 = 0
\end{cases}
\end{align*}
\]
with $x \neq 0, w \neq 0$.
For $x = -4$, this system admits a solution given by
\[
(z_0, w_0) = \left( -\frac{b}{a}, -2^{1-n} \frac{b}{c} \right).
\]
It is clear that $H(z_0, w_0) = 0$ (since $N(z_0, w_0) = 0, D(z_0, w_0) \neq 0$).
As to the points $(z_j, w_j), j = 1, \ldots, [\frac{d}{2}]$, which satisfy
\[
p(-x) \left( [\frac{d}{2}] \right) - 3 p(-x) \left( [\frac{d}{2}] \right) + 2 p(-x) \left( [\frac{d}{2}] \right) + 1 = 0,
\]
it is easy to see that $H(z_j, w_j) \neq 0$.
So we conclude that the unique solution to the system (4.64) is $(z_0, w_0)$ given by (4.69).
Let $S = S_{-\frac{d}{2}, \frac{e_1}{2}}$ be such that (3.5) is satisfied, as well as (4.56) and (4.57). From Theorem 3.4 we conclude that
\[
\inf_S \left( |\phi_2 + | + |\phi_2 - E\phi_2 + | g \right) > 0
\]
iff
\[
\left| \frac{b}{a} \right| \neq \left| 2^{n-1} \frac{c}{b} \right|.
\]
We have proved the following theorem.
Theorem 4.3. Let $G$ be given by (4.1) where $g$ is of the form (4.48), with $\alpha, \beta$ satisfying (4.50) and (4.51) for $n \geq 2$. Then $T_G$ is invertible iff

$$\left| \frac{b}{a} \right|^\frac{\beta}{\alpha} \neq \left| 2^{n-1} \frac{c}{b} \right|.$$  

Example 4.1.4. Let us now consider $G$ given by (4.1) where $g$ is an almost-periodic polynomial of the form

$$g = aE^{\alpha} + bE^{\mu} + cE^{-\sigma} + dE^{\gamma},$$

with

$$a, b, c, d \in \mathbb{C} \setminus \{0\}, \quad \alpha, \mu, \sigma, \gamma \in ]0, 1[, \quad \mu < \gamma < \alpha.$$  

Let $n \in \mathbb{N}$ be defined by

$$\frac{1}{n} \leq \mu + \sigma < \frac{1}{n - 1}.$$  

Moreover, we assume some further conditions on $\alpha, \mu$ and $\sigma$,

$$\mu > \frac{\alpha(n-1)}{n} \quad \text{and} \quad \alpha + \sigma < \frac{1}{n - 1}.$$  

The case where $\gamma = 0$ was studied in [4] where necessary and sufficient conditions for the matrix $G$ to admit a Wiener–Hopf factorization were established. In that paper, it was suggested that its results could be extended to cases with more than three points in $\text{sp} \ g$, under some additional conditions. Here we study the invertibility of $T_G$, with symbol $G$ of that type, using the results of Section 2 in the present paper.

A first solution to the Riemann–Hilbert problem (4.2) can be obtained following the same steps as in [4] and we get

$$\phi_1^+ = E^{1-v} \sum_{j=0}^{n-1} (-1)^j \left( aE^{\alpha-v} + bE^{\mu-v} + dE^{\gamma-v} \right) ^{n-1-j} \left( cE^{-(\sigma-\beta)} \right) ^j E^{-\frac{j}{n}},$$

$$\phi_2^+ = -(aE^{\alpha-v} + bE^{\mu-v} + dE^{\gamma-v}) ^n,$$

$$\phi_1^- = E^{-1} \phi_1^+, \quad \phi_2^- = (-1)^{n+1} (cE^{-(\sigma-\beta)}) ^n,$$

where $\nu, \beta$ satisfy the conditions $\nu + \beta = \frac{1}{n}$ and

$$\max \left\{ \frac{(n-1)}{n}, \frac{1}{n}, \frac{1}{n} \right\} \leq v \leq \min \left\{ \mu, \frac{1 - \sigma(n-1)}{n} \right\}.$$  

We can state the following.

Theorem 4.4. Let $G$ be given by (4.1) where $g$ is of the form (4.70), with $\alpha, \mu, \sigma$ satisfying (4.73) for $n$ defined by (4.72).

(i) If $\mu + \sigma \neq \frac{1}{n}$, $G$ does not admit a Wiener–Hopf factorization.

(ii) If $\mu + \sigma = \frac{1}{n}$, a solution to (4.2) is given by

$$\phi_1^+ = E^{1-\mu} \sum_{j=0}^{n-1} ((-1)^j (aE^{\alpha-\mu} + b + dE^{\gamma-\mu}) ^{n-1-j} \left( cE^{-\sigma} \right) ^j E^{-\frac{j}{n}}),$$

$$\phi_2^+ = -(aE^{\alpha-\mu} + b + dE^{\gamma-\mu}) ^n,$$

$$\phi_2^- = (-1)^{n+1} c^n,$$

and $G$ admits a canonical bounded Wiener–Hopf factorization.
Proof. (i) If \( \mu + \sigma \neq \frac{1}{n} \), we can write, from (4.74) and (4.75),
\[
\phi_1^+ = E^\delta \tilde{\phi}_1^+ \quad \text{and} \quad \phi_2^+ = E^\delta \tilde{\phi}_2^+,
\]
with \( \delta > 0, \tilde{\phi}_1^+, \tilde{\phi}_2^+ \in H^\infty_\mathbb{C} \).

From Theorem 2.13 of [4] we conclude that \( G \) does not admit a Wiener–Hopf factorization.

(ii) If \( \mu + \sigma = \frac{1}{n} \), then the solution to the Riemann–Hilbert problem given by (4.74)–(4.76) takes the form given by (4.78)–(4.80).

Taking into account the particular form of \( \phi_2^- \) given by (4.80), it is easy to see that condition (iv) of Theorem 2.3 holds and, for any \( \varepsilon_2 \in [0, +\infty[ \),
\[
\inf_{\mathbb{C}^- - i\varepsilon_2} (|\phi_1^-| + |\phi_2^-|) > 0.
\]

Therefore, from Theorem 2.5, to prove that \( \phi_\pm \in CT \) it is enough to show that
\[
\inf_{\mathbb{C}^+ + i\varepsilon_1} (|\phi_1^+| + |\phi_2^+|) > 0 \quad \text{for some} \ \varepsilon_1 > 0.
\]

This is an immediate consequence of the fact that \( \phi_2^+ \), given by (4.79), has a non-zero constant term \( b_2 \).

It is clear that the present approach of the factorization problem for \( G \) could easily be applied when \( \text{sp} \ g \) has any number of points \( \gamma_j \), i.e., when \( g \) has the form
\[
g = a E^\alpha + b E^\mu + c E^{-\sigma} + \sum_{j=1}^k d_j E^{\gamma_j}, \quad \mu \leq \gamma_j \leq \alpha, \ j = 1, \ldots, k.
\]

4.2. The results of Section 2 can also be used to study the invertibility of some classes of Toeplitz operators by reducing it to the study of other classes with simpler symbols. Let us study this question in connection with the corona theorem, following Theorem 1.2 on one hand and Theorem 2.5 on the other hand. We see that if \( G, \phi_+ \) and \( \phi_- \) satisfy the assumptions of Theorem 2.5 and moreover
\[
\phi_+ \in CT(S_{-\varepsilon_2, +\infty}), \quad \phi_- \in CT(S_{-\infty, \varepsilon_1}), \quad \tilde{\gamma} = \det \tilde{G} \quad \text{be such that they admit a bounded canonical factorization and} \ \gamma^{-1}, \tilde{\gamma}^{-1} \in H^\infty(S_{-\varepsilon_2, \varepsilon_1}).
\]

Let moreover \( \tilde{G} \) admit a bounded canonical factorization.

Theorem 4.5. Let \( G, \tilde{G} \in (H^\infty(S_{-\varepsilon_2, \varepsilon_1}))^{2 \times 2} \) with \( \varepsilon_1, \varepsilon_2 \in [0, +\infty[ \) and let \( \gamma = \det G, \tilde{\gamma} = \det \tilde{G} \) be such that they admit a bounded canonical factorization and \( \gamma^{-1}, \tilde{\gamma}^{-1} \in H^\infty(S_{-\varepsilon_2, \varepsilon_1}). \)

Let moreover \( \tilde{G} \) admit a bounded canonical factorization.

(i) If there are non-zero functions \( \phi_+ \in (H^\infty_\mathbb{C})^2, \phi_- \in (H^\infty_\mathbb{C})^2 \) such that
\[
G\phi_+ = \phi_- \quad \text{and} \quad \tilde{G}\phi_+ = \tilde{\phi}_-, \quad \text{then} \ G \ \text{admits a canonical Wiener–Hopf factorization if condition (2.9) holds.}
\]

(ii) If there are non-zero functions \( \phi_+, \phi_- \in (H^\infty_\mathbb{C})^2 \) such that
\[
G\phi_+ = \phi_- \quad \text{and} \quad \tilde{G}\phi_+ = \phi_- \quad \text{then} \ G \ \text{admits a canonical Wiener–Hopf factorization if (2.8) holds.}
\]

Proof. If \( \tilde{G} \) admits a bounded canonical Wiener–Hopf factorization and \( \phi_+, \phi_- \) are non-zero solutions of the second equation in (4.83), then \( \phi_+ \in CT^+, \phi_- \in CT^- \) and it follows from Theorems 2.5 and 2.3, \( \phi_+ \) satisfies condition (4.81).
Thus $\phi_- \in CT^-$ if (2.9) holds and, from Theorem 1.2, we conclude that $G$ admits a canonical Wiener–Hopf factorization.

The second part of the theorem can be proved analogously. □

We apply this result to obtain conditions for existence of a canonical Wiener–Hopf factorization in the following example and we remark that in this example we are no longer restricted to almost-periodic symbols (contrary to what happened in the examples studied in the first part of this section).

**Example 4.2.1.** Here we use the results of [5] as a starting point of our study. Let $\alpha, \beta \in [0, 1]$, $\frac{1}{\beta} \notin \mathbb{Q}$, $\alpha + \beta > 1$.

We assume that $N$, which will be used below, is a non-negative integer depending on the values of $\alpha$ and $\beta$, defined in (4.9) of [5] as follows:

$$N = \min J,$$

with

$$J = \left\{ j \in \mathbb{N} \cup \{0\} : \frac{1 + j\beta}{\alpha} = \frac{(j + 1)\beta}{\alpha} \text{ or } n\alpha - j\beta = 1 \text{ for some } n \in \mathbb{N} \right\}$$

(where $[X]$ denotes the integer part of the real number $X$).

Now let $g$ in (4.1) take the form

$$g = cE^{-\beta} + b + \sum_{j=1}^{n} a_j E^{j\alpha} + d_- E^{-\mu}$$

(4.85)

where $n = \lfloor \frac{1}{\alpha} \rfloor$, if $\frac{1}{\alpha} \notin \mathbb{N}$ and $n = \lfloor \frac{1}{\alpha} \rfloor - 1$, if $\frac{1}{\alpha} \in \mathbb{N}$; $d_- \in H^-_{\infty}$; $c, b, a_j$ ($j = 1, 2, \ldots, n$) are complex numbers with $b \neq 0$, $\sum_{j=1}^{k} a_j \neq 0$; $\mu \in [0, 1]$ is such that

$$\mu > \max\{S_{l+1}\alpha - l\beta : l = 0, 1, \ldots, N\} = M,$$  \hspace{1cm}  (4.86)

with

$$S_l = \left\lfloor \frac{1 + (l - 1)\beta}{\alpha} \right\rfloor \text{ for } l = 0, 1, \ldots, N,$$  \hspace{1cm}  (4.87)

$$S_{N+1} = \left\lfloor \frac{(N + 1)\beta}{\alpha} \right\rfloor$$  \hspace{1cm}  (4.88)

(cf. [5, Definition 4.8]).

Let us consider the question of existence of a canonical Wiener–Hopf factorization for $G$ of the form (4.1) with $g$ given by (4.85), which we can write as

$$g = p + d_- E^{-\mu}$$

(4.89)

for

$$p = cE^{-\beta} + b + \sum_{j=1}^{n} a_j E^{j\alpha}.$$  \hspace{1cm}  (4.90)

The subclass of matrix functions $\widetilde{G}$ for which $d_- = 0$ in (4.85) (i.e., $g = p$) was completely studied in [5], where it was shown that $\widetilde{G}$ admits a canonical Wiener–Hopf factorization, which was explicitly obtained.

A solution to the Riemann–Hilbert problem

$$pq_+ = Eu_+ + u_-,$$  \hspace{1cm}  (4.91)

with $q_+, u_+ \in H^+_{\infty}$, $u_- \in H^-_{\infty}$, $sp q_+ \subset [0, 1]$, is known from [5] and, in particular, we have

$$q_+ = \sum_{l=0}^{N} \sum_{j=S_l}^{S_{l+1}} \tilde{A}_{j,l} E^{j\alpha - l\beta}$$

(4.92)
where the Fourier coefficients \( \tilde{A}_{j,l} \) are defined in Theorem 4.15 of [5] and \( u_+ \) and \( u_- \) are uniquely defined by \( p \) and \( q_+ \), according to (4.91). We remark here that \( 0 \in \text{sp}(u_-) \) (cf. [5]).

Therefore, the Riemann–Hilbert problem
\[
\tilde{G} \phi_+ = \tilde{\phi}_-
\]
(admits a non-zero solution \( \phi_+ = (q_+, -u_+) \), \( \tilde{\phi}_- = (E^{-1}q_+, u_-) \).

Taking (4.89) into account, it is clear that
\[
\tilde{G} \phi_+ = \tilde{\phi}_- + \begin{bmatrix} 0 \\ d_- E^{-\mu} q_+ \end{bmatrix}.
\]

Denoting the right-hand side of (4.94) by \( \phi_- \) we have \( \phi_- \in (H^{-1}_\infty)^2 \),
(4.95)

since condition (4.86) implies that \( \mu \) is greater than \( M = \text{max} \, \text{sp}(q_+) \), as we can see from (4.92).

Since \( \tilde{G} \in (H^\infty_\infty(S_{\varepsilon_2},0))^{2 \times 2} \) (for any \( \varepsilon_2 > 0 \)), \( \gamma = 1 \), (4.83) is satisfied and \( \tilde{G} \) admits a bounded canonical Wiener–Hopf factorization, then it follows from Theorem 4.5 that \( G \) admits a canonical Wiener–Hopf factorization if (2.9) holds.

Since, in this case,
\[
\phi_2_- = u_- + d_- E^{-\mu} q_+ = u_- + (d_- E^{-M} q_+) E^{-(\mu - M)}
\]
and \( d_- E^{-M} q_+ \in H^\infty_\infty, \mu - M > 0 \) and \( 0 \in \text{sp}(u_-) \), we see that, in fact, (2.9) holds (for sufficiently big \( \varepsilon_2 \)) and thus we get the following.

**Theorem 4.6.** Let \( G \) take the form (4.1) where \( g \) is given by (4.85) and \( \mu \) satisfies (4.86). Then \( G \) admits a canonical Wiener–Hopf factorization.

Analogously, if we define \( \tilde{N} \) and \( \tilde{S}_l \) \( (l = 0, 1, \ldots, \tilde{N} + 1) \) in the same way as \( N \) and \( S_l \), respectively, with \( a \) replaced by \( \beta \) and vice versa, we have the following result.

**Theorem 4.7.** Let \( G \) take the form (4.1) where
\[
g = \sum_{j=1}^{n} c_j E^{-j\beta} + b + a E^\alpha + d_+ E^\mu
\]
where \( n = \left\lfloor \frac{1}{\beta} \right\rfloor \), if \( \frac{1}{\beta} \notin \mathbb{N} \) and \( n = \left\lfloor \frac{1}{\beta} \right\rfloor - 1 \), if \( \frac{1}{\beta} \in \mathbb{N} \); \( d_+ \in H^\infty_\infty; a, b, c_j \ (j = 1, 2, \ldots, n) \) are complex numbers with \( b \neq 0, \sum_{j=1}^{k} |c_j| \neq 0, \alpha, \beta, \mu \in ]0, 1[ \) with \( \frac{\alpha}{\beta} \notin \mathbb{Q}, \alpha + \beta > 1 \) and
\[
\mu > \max \{ \tilde{S}_{l+1} \beta - l \alpha: l = 0, 1, \ldots, \tilde{N} \} = \tilde{M}.
\]

Then \( G \) admits a canonical Wiener–Hopf factorization.

**Remark 4.8.** A particular case of matrix functions \( G \) satisfying the conditions of the last theorem, with \( g = c E^{-\beta} + b + a E^\alpha + d E^{2\alpha - \beta} \), where \( a, b, c, d \in \mathbb{C}, b \neq 0, \frac{1}{2} \leq \beta < 2\alpha - \beta < 1 \) and \( 3\alpha - 2\beta > 1 \), was considered in [5], where an explicit (almost-periodic polynomial) canonical Wiener–Hopf factorization was obtained. Indeed we have, in this case, \( \tilde{N} = 0, \tilde{M} = \beta \) and, taking \( \mu = 2\alpha - \beta \), we see that the conditions of Theorem 4.7 are satisfied. As regards the mere question of existence of a canonical Wiener–Hopf factorization, Theorems 4.6 and 4.7 can be seen as a generalization of those obtained in [5].

It should be remarked that a direct verification of the corona conditions \( \phi_\pm \in CT^\pm \) is in general quite difficult, even if \( d_\pm \) is an almost-periodic polynomial. For instance, taking
\[
g = c E^{-\beta} + b + \sum_{j=1}^{n} a_j E^{j\alpha} + d E^{n\alpha - 2\beta},
\]
with \( d \in \mathbb{C}, \ 2\beta - n\alpha < 1 \) and \( \beta > n\alpha \), it can be easily verified that the conditions of Theorem 4.6 are satisfied (and therefore \( T_G \) is invertible), while a direct study of the solutions \( \phi_{\pm} \) of (4.2) would be much more difficult. For instance, for \( n = 4 \), a solution to (4.2) is given by

\[
\begin{align*}
\phi_{1+} &= q_+ = 1 + A_1 E^\alpha + A_2 E^{2\alpha} + A_3 E^{3\alpha} + A_4 E^{4\alpha}, \\
-E\phi_{2+} &= Eu_+ = (a_4 A_1 + a_3 A_2 + a_2 A_3 + a_1 A_4) E^{5\alpha} + (a_4 A_2 + a_3 A_3 + a_2 A_4) E^{6\alpha} \\
&\quad + (a_4 A_3 + a_3 A_4) E^{7\alpha} + a_4 A_4 E^{8\alpha}, \\
\phi_{1-} &= E^{-1} \phi_{1+}, \\
\phi_{2-} &= b + c E^{-\beta} + c A_1 E^{\alpha-\beta} + c A_2 E^{2\alpha-\beta} + c A_3 E^{3\alpha-\beta} + c A_4 E^{4\alpha-\beta} + d E^{4\alpha-2\beta} + d A_1 E^{5\alpha-2\beta} \\
&\quad + d A_2 E^{6\alpha-2\beta} + d A_3 E^{7\alpha-2\beta} + d A_4 E^{8\alpha-2\beta}
\end{align*}
\]

where

\[
\begin{align*}
A_1 &= -\frac{a_1}{b}, \quad A_2 = -\frac{a_2}{b} + \frac{a_1^2}{b^2}, \quad A_3 = -\frac{a_3}{b} + 2 \frac{a_1 a_2}{b^2} - \frac{a_1^3}{b^3}, \\
A_4 &= -\frac{a_4}{b} + 2 \frac{a_1 a_3}{b^2} + \frac{a_2^2}{b^2} - 3 \frac{a_1 a_2}{b^3} + \frac{a_4^4}{b^4}.
\end{align*}
\]

References