Segal–Bargmann Transform, Functional Calculus on Matrix Spaces and the Theory of Semi-circular and Circular Systems

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Let $M_d(\mathbb{C})$ be the space of $d \times d$ complex matrices, and $\mathcal{H}_d$ be the subspace of hermitian matrices. We study the Segal–Bargmann transform of functions on $\mathcal{H}_d$, with values in $M_d(\mathbb{C})$, which are given by functional calculus. We show that when $d \to \infty$, the transform of such a map becomes close to the space of holomorphic functional calculus on $M_d(\mathbb{C})$, and that this yields, in the limit, an isometry between the $L^2$ space of Wigner’s semi-circle distribution and the Hardy space of the disk.

We relate this isometry to Voiculescu’s theory of circular and semi-circular systems, and we study its analogue when the Segal–Bargmann transform is replaced by the Hall transform on unitary groups of large dimensions.

INTRODUCTION

The Segal–Bargmann transform is an integral operator which intertwines the Fock and the Schrödinger models for the irreducible representations of the Heisenberg group, it is a useful tool in quantum field theory as well as in some parts of analysis (see, e.g., [F]). This transform yields a unitary isomorphism between the $L^2$ space of a gaussian measure on some finite dimensional real Hilbert space and the space of holomorphic functions on the complexification of this Hilbert space, which are square integrable with respect to a gaussian measure on this space.

In this paper we will investigate the limit of the classical Segal–Bargmann transform acting on functions which are given by functional calculus on matrix spaces of large dimension. More precisely, let $\mathcal{H}_d$ be the space of hermitian $d \times d$ matrices, we shall consider maps from $\mathcal{H}_d$ to the space $M_d(\mathbb{C})$ of $d \times d$ complex matrices, with the Hilbert space structure given by $\langle A, B \rangle = (1/d) \text{tr}(AB^*)$. The Segal–Bargmann transform of such a map will be a holomorphic map from $M_d(\mathbb{C})$, the complexification of $\mathcal{H}_d$, to

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itself. A particular class of such functions is given by the functional calculus, namely, for any Borel function \( f \) on \( \mathbb{R} \), there is a well defined function \( \Theta_{f}^{d} \) from \( X_{d} \) to \( M_{d}(\mathbb{C}) \) given by the functional calculus, and similarly, on \( M_{d}(\mathbb{C}) \), holomorphic functional calculus can be defined, and for any entire function \( F \) on \( \mathbb{C} \), there is a function \( \Theta_{F}^{d} \) from \( M_{d}(\mathbb{C}) \) to \( M_{d}(\mathbb{C}) \). The spaces given by functional calculus are very small subspaces of the space of matrix valued functions, square integrable with respect to the gaussian measure on \( X_{d} \), or \( M_{d}(\mathbb{C}) \), nevertheless, we shall see that, as \( d \to \infty \) these spaces are asymptotically mapped one into another by the Segal-Bargmann transform, and that this gives rise in the limit to an integral transform which is a Hilbert space isomorphism between the \( L^{2} \) space of a semi-circular distribution and the Hardy space of a disk whose radius is equal to the square root of the variance of the semi-circular distribution. This integral transform has the explicit form

\[
\mathcal{F}^{d}f(z) = \int_{-2 \sqrt{t}}^{2 \sqrt{t}} \frac{tf(x)}{t-x^{2}+z^{2}} \mu'(dx)
\]

for \( f \in L^{2}(\mu') \) and \( z \) in the disk of radius \( \sqrt{t} \), where \( \mu' \) is the semi-circular distribution of variance \( t \), see Section 1.1.7 below.

In the recently developed theory of free random variables (see [VDN] for an introduction to that subject), circular and semi-circular systems have been shown to be suitable non-commutative analogues of the gaussian families, respectively complex and real gaussian, and some realizations of them on free Fock space have been developed. In this context, there is a natural analogue of the Segal–Bargmann transform, which can be informally described as a unitary isomorphism between the space of square integrable “functions” of a semi-circular system and the space of square integrable “holomorphic functions” of a circular system, where the term function here refers to the functional calculus of operators, and operator algebra theory. We shall see that the transform \( \mathcal{F}^{d} \) can be interpreted as the one-dimensional free analogue of the Segal–Bargmann transform. We shall also extend this result to the case of functional calculus of several independent gaussian matrices and recover, in the \( d \to \infty \) limit, the multi-dimensional version of the free Segal–Bargmann isomorphism. This result should not come as a surprise, indeed, as Voiculescu [V2] has shown, large gaussian matrices can be used to approximate semi-circular and circular systems, so the fact that the Segal–Bargmann transform goes to the limit is quite natural.

Recently, a remarkable extension of the Segal–Bargmann isomorphism was introduced by Hall [Ha], in which the gaussian measure on a Hilbert space is replaced by the heat kernel measure on a compact Lie group. In this context the Segal–Bargmann isomorphism becomes a unitary
isomorphism between the $L^2$ space of this heat kernel measure and the space of holomorphic functions on the complexification of the group, square integrable with respect to a suitable heat kernel on this complex group. The Segal–Bargmann and Hall's transforms can in fact be brought together in the context of Lie groups of compact type (see Driver [Dr]). As shown by Gross and Malliavin [GM], Hall's transform can be recovered from an infinite dimensional version of the Segal–Bargmann transform, by imbedding the $L^2$ space of the heat kernel measure on a compact type Lie group into the $L^2$ space of the Wiener measure associated with brownian motion with values in its Lie algebra. We shall show in this paper that an analogue of the Gross–Malliavin result holds in the free case and that it yields an interesting unitary isomorphism between two Hilbert spaces. One of these Hilbert spaces is the $L^2$ space of a measure on the unit circle, which is the multiplicative analogue of the semi-circular distribution, whereas the other one is a Hilbert space of analytic functions in some domain of the complex plane, which is a subspace, with a different norm however, of the Hardy space of this domain. These spaces depend on a parameter, which comes from the time parameter in the heat kernel, and the nature of the domain depends of this parameter, since for $t \leq 4$ the domain is simply connected, while for $t > 4$, it is conformally equivalent to an annulus.

This paper is organized as follows. In Section 1 we recall the definition of the Segal–Bargmann transform, we study some properties of Gaussian measures on matrix spaces, and we introduce the transform $\mathcal{F}_t$, then we prove that $\mathcal{F}_t$ is the limit of the Segal–Bargmann transform for functional calculus on matrix spaces. Results are stated in Subsection 1.1, while proofs are given in 1.2. The second section deals with Hall's extension of the Segal–Bargmann transform. We give an overview of Hall's transform then state the theorem of Gross and Malliavin, and we explain our strategy for finding the analogue of the transform $\mathcal{F}_t$ for the functional calculus on unitary groups. This strategy relies upon the theory of semi-circular and circular systems, which we review in Section 3. We extend the results of Section 1 to the polynomial functional calculus of several independent gaussian matrices, thus recovering the free analogue of the Segal–Bargmann isomorphism. Finally, in Section 4, we introduce the stochastic calculus on free Fock space, due to Kümmerer and Speicher [KS] and use it to construct free multiplicative brownian motions, by solving an exponential stochastic differential equation. Then we prove an analogue of the Gross–Malliavin theorem. Note that the way we prove this result is by explicit computations, whereas Gross and Malliavin use abstract arguments from quasi sure analysis. These computations allow us to give explicit descriptions of the Hilbert spaces and the integral transforms arising from Hall's transform on unitary groups.
1. SEGAL–BARGMANN TRANSFORM FOR FUNCTIONAL CALCULUS IN MATRIX SPACES OF LARGE DIMENSION

1.1. Preliminaries and Statement of the Results

1.1.1. Let $H$ be a finite dimensional real Hilbert space, and denote $H^\mathbb{C} = H \otimes \mathbb{C}$ its complexification. Unless the contrary is explicitly stated, the Hilbert spaces we consider, in particular $L^2$ spaces, will be assumed to be complex.

Let $p_\mathbb{R}$ and $q_\mathbb{C}$ be the Gaussian densities on $H$ and $H^\mathbb{C}$ such that

$$\int_H e^{\langle z, y \rangle} p_\mathbb{R}(y) \, dy = e^{-\frac{1}{2} \langle z, x \rangle} \quad \text{and} \quad \int_{H^\mathbb{C}} e^{\langle z, w \rangle} q_\mathbb{C}(w) \, dw = e^{-\frac{1}{2} \langle z, z \rangle}$$

for all $x \in H$, $z \in H^\mathbb{C}$, where $dy$ and $dw$ are the standard Lebesgue measures on $H$ and $H^\mathbb{C}$ respectively. Let $L^2(H, p_\mathbb{R})$ be the space of square integrable functions for the measure $p_\mathbb{R}(x) \, dx$ on $H$ and $L^2_{hol}(H^\mathbb{C}, q_\mathbb{C})$ be the space of holomorphic functions on $H^\mathbb{C}$, square integrable for the measure $q_\mathbb{C}(w) \, dw$.

**Theorem 1.** Let $f \in L^2(H, p_\mathbb{R})$, then the integral

$$\int_H f(y) \, p_\mathbb{R}(x-y) \, dy \quad (1.1.1)$$

converges for all $x \in H$ and this function has a unique analytic continuation to $H^\mathbb{C}$, denoted by $\mathcal{S}(f)$. The map $f \mapsto \mathcal{S}(f)$ is an isometry from $L^2(H, p_\mathbb{R})$ onto $L^2_{hol}(H^\mathbb{C}, q_\mathbb{C})$.

This result is one way of introducing the transformation $\mathcal{S}$, known as the Segal–Bargmann transform, see [Ba], [F] or [GM] for a proof. We brought this point of view from the paper [GM], to which we refer for a thorough discussion of the Segal–Bargmann transform and the Hall transform.

1.1.2. Let $L$ be a finite dimensional complex Hilbert space, we can identify the space $L^2(H, p_\mathbb{R}, L)$ of square integrable functions on $H$, with values in $L$, and the tensor product $L^2(H, p_\mathbb{R}) \otimes L$. Also there is an isomorphism between $L^2_{hol}(H^\mathbb{C}, q_\mathbb{C}, L)$, the space of holomorphic square integrable functions on $H^\mathbb{C}$, with values in $L$, and $L^2_{hol}(H^\mathbb{C}, q_\mathbb{C}) \otimes L$. Using these identifications we can dilate the isometry $\mathcal{S}$ to an isometry $\mathcal{S} \otimes Id_L$ from $L^2(H, p_\mathbb{R}, L)$ onto $L^2_{hol}(H^\mathbb{C}, q_\mathbb{C}, L)$.

We now take for $H$ the space $X_d$ of $d \times d$ complex hermitian matrices, with the real Hilbert space structure given by the inner product $\langle A, B \rangle = tr_d(AB)$, where $tr_d$ denotes the normalized trace (i.e., the normalized trace of the identity matrix is 1). The decomposition of $d \times d$ complex matrices
into hermitian and skew hermitian parts gives a natural identification of \( \mathcal{A}_d \) with \( M_d(C) \) the space of complex \( d \times d \) matrices, with inner product \( \langle A, B \rangle = \text{tr}_d(AB^*) \). We shall also take \( L \) to be equal to \( M_d(C) \), we shall denote by \( \mathcal{F} \) the Segal–Bargmann transform from \( L^2(\mathcal{A}_d, p_\mu) \) to \( L^2_{\text{hol}}(M_d(C), q_\mu) \) and by \( \mathcal{F}_d \) its dilation to \( L^2(\mathcal{A}_d, p_\mu, M_d(C)) \).

1.1.3. Let \( f \) be a complex Borel function on \( \mathbb{R} \), for every hermitian matrix \( A \), there is a normal matrix \( f(A)^* \) obtained by functional calculus so we get a map \( \theta_d^f : A \mapsto f(A) \) from \( \mathcal{A}_d \) to \( M_d(C) \). There is a unique probability measure \( \mu_\mu^f \) on \( \mathbb{R} \) such that, for every bounded Borel function \( f \) on \( \mathbb{R} \), one has

\[
\int f(x) \text{d}\mu_\mu^f(x) = \int \text{tr}_d(\theta_d^f(A)) p_\mu(A) \text{d}A.
\]

Of course, for \( d = 1 \), \( \mu_\mu^f \) is just a gaussian measure on \( \mathbb{R} \), of variance \( \mu \). The map \( f \mapsto \theta_d^f \) is an isometry from \( L^2(\mu_\mu^f) \) into \( L^2(\mathcal{A}_d, p_\mu, M_d(C)) \). The measures \( \mu_\mu^f \) have a strictly positive density with respect to Lebesgue measure on \( \mathbb{R} \), which can be computed explicitly in terms of Hermite functions, see subsection 1.2.1 below. Since the measures \( \mu_\mu^f \) belong to the class of Lebesgue measure, one can identify functions in these spaces with their class with respect to Lebesgue null sets. We shall prove

**LEMMA 1.** For every integer \( d \geq 1 \), and \( \mu \in \mathbb{R}_+^* \), one has a continuous inclusion with dense range \( L^2(\mu_\mu^f) \subset L^2(\mu_\mu^f) \).

1.1.4. Let \( F \) be an entire function on \( \mathbb{C} \), by holomorphic functional calculus we have a map \( \theta_d^f : A \mapsto f(A) \) from \( M_d(C) \) to \( M_d(C) \). Consider the inner product

\[
\langle F, G \rangle_d = \int_{M_d(C)} \text{tr}_d(\Theta_d^f(w) \Theta_d^f(w)^*) q_\mu(w) \text{d}w
\]

on the space of \( \mathcal{E}^f_\mu \) of entire functions such that

\[
\int_{M_d(C)} \text{tr}_d(\Theta_d^f(w) \Theta_d^f(w)^*) q_\mu(w) \text{d}w < \infty.
\]

Let \( \mathcal{L}_\mu^f \) be the gaussian measure on \( \mathbb{C} \) with density \( (1/\pi \mu)^{\frac{d}{2}} e^{-|z|^2} \) with respect to Lebesgue measure, identifying \( M_d(C) \) with \( \mathbb{C} \) one has \( L^2_{\text{hol}}(\mathcal{L}_\mu^f) = \mathcal{E}^f_\mu \).

**LEMMA 2.** For every integer \( d \geq 1 \), and \( \mu \in \mathbb{R}_+^* \), the space \( (\mathcal{E}^f_\mu, \langle \cdot, \cdot \rangle_d) \) is a Hilbert space, one has \( \|f\|_{\mathcal{E}^f_\mu} \leq \|f\|_{L^2_{\text{hol}}(\mathcal{L}_\mu^f)} \), and \( L^2_{\text{hol}}(\mathcal{L}_\mu^f) \subset \mathcal{E}^f_\mu \) is a continuous inclusion with dense range.
By construction, the map $\Theta^d$ is an isometry from $L^2_{hol}(M_d(C), p, M_d(C))$, and thus $\Theta^d(\mathcal{E}^d)$ is a closed subspace of $L^2_{hol}(M_d(C), p, M_d(C))$.

1.1.5. The subspaces of $L^2(\mathcal{X}_d, p, M_d(C))$ and $L^2_{hol}(M_d(C), q, M_d(C))$ given by functional calculus are very small subspaces (they are of infinite codimension), but nevertheless we shall see that, when $d \to \infty$, the Segal-Bargmann transform maps asymptotically these subspaces one into another.

By Lemmas 1 and 2, for every $f \in L^2(\mathcal{X}_d)$ the map $\Theta^d$ is in $L^2(\mathcal{X}_d, p, M_d(C))$, and similarly, for $F \in L^2_{hol}(\mathcal{X}_d)$, the map $\Theta^2_F$ is in $L^2_{hol}(M_d(C), q, M_d(C))$, for all $d \geq 1$.

Theorem 2. (1) Let $f \in L^2(\mu'_1)$, then the distance of $T^d(\Theta^2_f)$ to the space $\Theta^d(\mathcal{E}^d)$ goes to zero as $d \to \infty$.

(2) Let $F \in L^2_{hol}(\lambda'_1)$, then the distance of $(T^d)^{-1}(\Theta^2_F)$ to the space $\Theta^d(L^2(\mu'_d))$ goes to zero as $d \to \infty$.

1.1.6. We shall give a more precise formulation of this result for a suitable class of functions, namely we shall see that the transforms $T^d$, acting on the subspaces of functional calculus actually converge to a certain integral transform between two Hilbert spaces of functions.

Let us first recall that, as $d \to \infty$, the measures $\mu'_d$ converge weakly towards the semi-circular distribution

$$\mu'(dx) = \frac{1}{2\pi t} \sqrt{4t - x^2} \, dx$$

on the interval $[-2\sqrt{t}, 2\sqrt{t}]$ (this is a well-known result of Wigner, see e.g. [M]). The parameter $t$ is the variance of the distribution.

Analogously, we shall see that

Lemma 3. For any entire functions $F, G$ one has

$$\langle F, G \rangle_d \to \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sqrt{t} e^{i\theta}) \overline{G(\sqrt{t} e^{i\theta})} \, d\theta \quad \text{when} \quad d \to \infty.$$  

Recall that the completion of the space of entire functions for the limit inner product in Lemma 3 is the Hardy space $H^2_t$ of the disk of radius $\sqrt{t}$.

1.1.7. We see that the spaces $L^2(\mu'_d)$ and $H^2_t$ appear as limiting objects of the spaces $L^2(\mu'_d)$ and $\mathcal{E}^d$, as $d \to \infty$. Let us now define the transform that plays the role of the Segal-Bargmann transform for functional calculus, and which exchanges these two spaces.
Proposition 1. Let \( f \in L^2(\mu') \) then the integral
\[
\mathcal{F} f(z) = \int_{-\sqrt{t}}^{\sqrt{t}} f(x) \mu'(dx)
\]
converges for all \( z \in \mathbb{C} \) such that \( |z| < \sqrt{t} \), and the function \( \mathcal{F} f \) belongs to the Hardy space of the disk with radius \( \sqrt{t} \). Furthermore, \( \mathcal{F} \) is an isometry from \( L^2(\mu') \) onto this Hardy space.

Proof. Observe that, if \( |z|^2 < t \), then the function
\[
x \mapsto \frac{t}{t - xz + z^2}
\]
is bounded on the interval \([-2\sqrt{t}, 2\sqrt{t}]\). This implies that \( \mathcal{F} f(z) \) is well defined and analytic in the open disk of radius \( \sqrt{t} \). Let \( T_m \) be the \( m \)th Tchebycheff polynomial defined by
\[
T_m(2 \cos \lambda) = \frac{\sin(m+1) \lambda}{\sin \lambda}
\]
and let \( T_m(t, x) = t^m T_m(x/\sqrt{t}) \). The polynomials \( T_m(t, \cdot) \) form an orthogonal basis of the space \( L^2(\mu') \), and have norms \( \|T_m(t, \cdot)\|_{L^2(\mu')} = t^m \). Any function \( f \in L^2(\mu') \) has an \( L^2 \)-convergent expansion
\[
f = \sum_{n=0}^{\infty} a_n T_n(t, \cdot)
\]
where
\[
a_n = \left[ t^{\frac{-m}{2}} \int_{-\sqrt{t}}^{\sqrt{t}} f(x) T_m(t, x) \mu'(dx) \right]
\]
and
\[
\left[ \int_{-\sqrt{t}}^{\sqrt{t}} |f(x)|^2 \mu'(dx) \right]^{\frac{1}{2}} = \sum_{n=0}^{\infty} |a_n|^2 t^m.
\]
The generating function for Tchebycheff polynomials (see e.g., Szego [Sz]), yields the identity
\[
\frac{t}{t - xz + z^2} = \sum_{n=0}^{\infty} z^n T_n(t, x) \quad \text{for} \quad |z|^2 < t,
\]
where the convergence is uniform for $x$ on the interval $[-2 \sqrt{t}, 2 \sqrt{t}]$. It follows from Parseval identity that

$$\int_{-2 \sqrt{t}}^{2 \sqrt{t}} \frac{tf(x)}{t-xz+z^2} \mu'(dx) = \sum_{n=0}^{\infty} a_n z^n$$

for $z$ in the disk of radius $\sqrt{t}$. This proves the proposition, since the functions $z \mapsto z^n$ form an orthogonal basis of the Hardy space of the disk with radius $\sqrt{t}$, with norms $\|z^n\|_{H^2_t} = t^n$. Observe that from the proof we have

$$\mathcal{F}(T_{\mu}(t, \cdot))(z) = z^n.$$  

The map $\mathcal{F}'$ is a free counterpart to the Segal–Bargmann transform. It has been considered in [LM], as well as some $q$-deformations of it. We shall come back to this in more details in Section 3, and explain how the formula (1.1.7) is the free counterpart of the convolution formula defining de Segal–Bargmann transform on a Gaussian space.

### 1.1.8

Let $f \in L^2(\mu'_1)$ be such that $\mathcal{F}'f$ has an analytic continuation to all of $\mathbb{C}$, which belongs to $L^2_{\text{hol}}(\mathcal{L}'_1)$. This is true, for example, if $f$ is a polynomial hence this property holds for $f$ in a dense subspace of $L^2(\mu'_1)$. The transform $\mathcal{F}'_d(f)$ is then well defined for all $d \geq 1$, and we can take its orthogonal projection onto the closed subspace $\Theta(\mathcal{E}'_d)$. By Theorem 2, we know that the distance of $\mathcal{F}'_d(f)$ to this projection, measured in $\mathcal{E}'_d$ norm, goes to zero as $d \to \infty$. Taking the inverse image, by $\Theta^d_1$ of this projection we obtain an element $\mathcal{F}'_d f$ in $\mathcal{E}'_d$. We now come to the main result of this section.

**Theorem 3.** Let $f \in L^2(\mu'_1)$ be such that $\mathcal{F}'f$ has an analytic continuation to all of $\mathbb{C}$, which is in $L^2_{\text{hol}}(\mathcal{L}'_1)$, then

$$\|\mathcal{F}'_d f - \mathcal{F}'f\|_{\mathcal{E}'_d} \to 0 \quad \text{when} \quad d \to \infty.$$  

This result means that the map $\mathcal{F}'$ is the limit of the maps $\mathcal{F}'_d$ acting on the maps given by functional calculus. It can also be expressed by saying that the following diagram is asymptotically commutative

$$\begin{array}{ccc}
L^2(\mu'_1) & \xrightarrow{\mathcal{F}'} & L^2(\mathcal{E}'_d, \mathcal{F}_d(\mathbb{C})) \\
\downarrow & & \downarrow \\
L^2_{\text{hol}}(\mathcal{E}'_d, \mathcal{F}_d(\mathbb{C})) & \xrightarrow{\mathcal{F}'} & L^2_{\text{hol}}(\mathcal{E}'_d, \mathcal{F}_d(\mathbb{C}))
\end{array}$$
This means that for an \( f \in L^2(\mu_d) \), satisfying the assumptions of Theorem 3, the distance between the two images in \( L^2(\mathbb{M}, \mathbb{M}(C)) \) goes to zero as \( d \to \infty \). Note that three of the maps in the diagram are isometries.

1.2. Proof of the Results

We keep the notations of Section 1.1.

1.2.1. We shall start with some information on the measures \( \mu_d' \). According to Mehta [M], the measure \( \mu_d' \) has a density \( \psi_d' \) with respect to Lebesgue measure, which can be expressed in terms of Hermite functions. Namely, let

\[
\psi_d(x) = (2^{j!} \sqrt{\pi})^{-1/2} e^{-x^2} (-d/dx)^j e^{-x^2}
\]

be the \( j \)th Hermite function, then one has

\[
\psi_d'(x) = (2/dt)^{1/2} \sum_{j=0}^d \phi_j((2d/t)^{1/2} x)
\]

\[
= (2/dt)^{1/2} \phi_d((2d/t)^{1/2} x)
\]

\[
- (2(d+1)/t)^{1/2} \phi_{d-1}((2d/t)^{1/2} x) \phi_{d+1}((2d/t)^{1/2} x)
\]

see formula 5.2.16 in [M] (note that we use a different scaling of the measure). The second equality in this identity is the Christoffel-Darboux formula.

Lemma 4. The spaces \( L^2(\mu_d') \) form a scale of Hilbert spaces, which means that the identification of class functions gives continuous embeddings with dense range \( \mu_d' : L^2(\mu_d') \to L^2(\mu_d') \), for \( d \leq d' \), furthermore, for every \( d \), the norms of the embedding \( \mu_d' \) for \( d' \geq d \) are uniformly bounded.

Proof. Since each measure \( \mu_d' \) has a strictly positive continuous density \( \psi_d' \), it is enough to prove that \( \sup_{x \in \mathbb{R}} (\psi_d'(x)/\psi_d'(x)) \) is bounded independently of \( d' \). We shall use this result only when \( d = 1 \), in which case it amounts to proving that

\[
\sup_{d' \geq 1} \sup_{x \in \mathbb{R}} \psi_d'(x) e^{-x^2/2} < +\infty.
\]

It is easy to check, using the asymptotic formula of Plancherel-Rotach type for Hermite polynomials, that

\[
\sup_{d' \geq 1} \sup_{x \in [2 \sqrt{1+\varepsilon}, 2 \sqrt{1-\varepsilon}]} \psi_d'(x) < +\infty
\]
and
\[
\sup_{d \geq 1} \sup_{x \in [2 \sqrt{t} + \epsilon, A]} \psi_d(x) e^{x^2/2t} < +\infty
\]
for any \( \epsilon > 0 \) and \( A > 2 \sqrt{t} \), (see 8.22.12 and 8.22.13 in [Sz]). In order to extend the estimate to the whole of \( \mathbb{R} \), one needs the more refined estimates given by Erdelyi [E]. This is rather cumbersome but does not present any difficulty.

We observe that Lemma 1 is a direct consequence of Lemma 4.

1.2.2. We now pass to Lemma 2 and some further information on the spaces \( E_t^d \).

**Lemma 5.** Let \( d \geq 1 \) and \( m, n \geq 0 \) be integers, then one has
\[
\int_{M_d(C)} \text{tr}_d(w^m(w^*)^n) q_j(w) \, dw = 0 \quad \text{if} \quad n \neq m
\]
\[
t^d n! \leq \int_{M_d(C)} \text{tr}_d(w^m(w^*)^n) q_j(w) \, dw \leq t^d n!.
\]
(1.2.2a)

**Proof.** The first equation follows at once from the fact that \( q_j(e^{i\theta}w) = q_j(w) \) for every complex number with modulus one \( e^{i\theta} \). For the inequalities of the second line we expand
\[
\int_{M_d(C)} \text{tr}_d(w^m(w^*)^n) q_j(w) \, dw
\]
\[
= \frac{1}{d} \int_{M_d(C)} \sum_{1 \leq j_1, \ldots, j_k \leq d} w_{i_1}^m w_{n_1}^* \cdots w_{i_k}^m w_{n_k}^* q_j(w) \, dw. \quad (1.2.2b)
\]
Under the measure \( q_j(w) \, dw \) the coordinates \( w_{i_j} \) have independent Gaussian complex distribution, and
\[
\int_{M_d(C)} w_{i_1}^m w_{n_1}^* q_j(w) \, dw = \delta_{i_1} d^{-k} k!.
\]
In particular, all terms in the sum are nonnegative. The sum is thus larger than the sum of terms corresponding to \( i_1 = i_2 = \cdots = i_k = j_1 = \cdots = j_n \), which is equal to \( d^{-k} n! \), hence the first inequality.

Each term in the sum (1.2.2b) can be written as
\[
\frac{1}{d} \int_{M_d(C)} w_{i_1} w_{i_2} \cdots w_{i_k} w_{n_1}^* w_{n_2}^* \cdots w_{n_k}^* \, dw.
\]
where the $a_k$ and $b_k$ are pairs of indices among $i_1, i_2, ..., i_{n-1}, j_r$. The term is non zero only if each pair $ij$ appears among $\beta_1, ..., \beta_n$ with same multiplicity $k_{ij}$ as in $x_1, ..., x_n$. In this case, the term has the value $d^{-n-1} \prod_k k_{ij}!$. For each choice of $(x_1, ..., x_n)$ there are at most $n! \prod_k k_{ij}!$ values of $(\beta_1, ..., \beta_n)$ which can produce a nonzero term, corresponding to the different ways of placing the pairs $ij$ with the required multiplicities. Since the total number of choices for $x_1, ..., x_n$ is $d^{n+1}$, we see that the sum is less than $\! n!$ as claimed.

**1.2.3. Using Lemma 5 we now have**

**Proof of Lemma 2.** Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, then

$$\langle F, F \rangle_d = \sup_{K > 0} \int_{|w| \leq K} tr_d(\Theta^*_d(w) \Theta^*_d(w)) q_d(w) dw.$$  

One has again $\int_{|w| \leq K} tr_d(w^k(w^*)^* q_d(w) dw = 0$ if $k \neq l$, hence using Fubini theorem to interchange summation and integration,

$$\langle F, F \rangle_d = \sum_{n=0}^{\infty} |a_n|^2 \int_{|w| \leq K} tr_d(w^n(w^*)^* q_d(w) dw$$

$$= \sum_{n=0}^{\infty} |a_n|^2 \int_{C} tr_d(w^n(w^*)^* q_d(w) dw.$$  

It follows from Lemma 5 that

$$\langle F, F \rangle_d \leq \sum_{n=0}^{\infty} d^{-n-1} |a_n|^2 n! = \frac{d}{\pi t} \int_{C} |F(z)|^2 e^{-d|z|^2} dz.$$  

A standard argument using Cauchy’s formula shows that for each $R > 0$ there exists a constant $C_R$ such that $|F(z)|^2 \leq C_R \langle F, F \rangle_d$ for all entire function $F$, and all $z$ with $|z| \leq R$, and this implies that the space $\delta_d$ is a Hilbert space. Furthermore, by (1.2.2a) again, we have the inequality

$$\langle F, F \rangle_d \leq \sum_{n=0}^{\infty} t^n |a_n|^2 n! = \frac{1}{\pi t} \int_{C} |F(z)|^2 e^{-\frac{|z|^2}{t}} dz = \|F\|_{L^2(\delta_d)},$$

which shows that there is a contractive embedding $L^2_{hol}(\delta'_d) \subset \delta_d$.

**Remark.** We do not know whether there exists a probability measure $\lambda'_d$ on $C$, such that $\langle F, F \rangle_d = \int_{C} |F(z)|^2 d\lambda'_d(z)$ for all $F \in \delta'_d$.

**1.2.4. Lemma 3 follows from a more general result of Voiculescu on the approximation of circular systems by random gaussian complex**
matrices (see [V2]). For the convenience of the reader, we give the following simple proof.

Proof of Lemma 3. Using the estimates in the proof of Lemma 5, we see that it is enough to prove that \( \int_{M_d(\mathbb{C})} \text{tr}_d(w^n(n^*)^* q_d(w)) \, dw \to t^n \) for each \( n \), as \( d \to \infty \). Using again the expansion (1.2.2b), suppose that a non zero term in this expansion corresponds to indices \( i_1, ..., i_n, j_1 \) which take \( n + 1 \) distinct values. This happens if and only if one has \( i_1 = j_n \), \( i_2 = j_n - 1 \), ..., \( i_n = j_2 \), and the value of this term is equal to \( t^n d^{-n-1} \). The total contribution of such terms is thus \( t^n d^{-n-1} (d-1) \cdots (d-n) \) which converges to \( t^n d^{-n-1} \) as \( d \to \infty \). Let us count now the number of non zero terms which correspond to a given sequence of indices \( i_1, ..., i_n, j_1 \) which takes at most \( n \) distinct values. The sequence of pairs \((j_2, j_1) \cdots (j_1, j_n)\) must be a permutation of the sequence \((i_1, i_2, ..., in, j_1)\), hence there are at most \( n! \) such terms. Since each term in the sum (1.2.2b) is less than \( t^n d^{-n-1} \), and there are less than \( n^d \) sequences \( i_1, ..., i_n, j_1 \), which take at most \( n \) distinct values, we see that the total contribution of such sequences is \( O(1/d) \) as \( d \to \infty \).

1.2.5. Denote by \( X_{ij}, Z_{ij} \), for \( 1 \leq i, j \leq d \), the coordinate functions on \( \mathcal{X}_d \) and \( M_d(\mathbb{C}) \) respectively, and \( X, Z \) the matrix-valued corresponding functions. We shall now prove Theorem 3 in the particular case where \( f \) is a polynomial. By linearity, it is enough to consider the case where \( f(x) = T_m(t, x) \), the \( m \)th rescaled Tchebycheff polynomial. Since \( \mathcal{F}'(T_m(t, x)) = z^m \), we see that what we have to prove is that \( \| \mathcal{F}'(T_m(t, X)) - Z^m \|_{L^2(M_d(\mathbb{C})), q_d} \to 0 \) as \( d \to \infty \), or, since \( \mathcal{F}' \) is unitary, \( \| T_m(t, X) - (\mathcal{F}'_d)^{-1}(Z^m) \|_{L^2(\mathcal{X}_d, p_d, M_d(\mathbb{C})))} \to 0 \) as \( d \to \infty \). Let us start with the following two lemmas.

**Lemma 6.** Let \( i, j, k, l \) be integers between \( 1 \) and \( d \), and let \( m, n \) be integers, with \( m \geq 1, n \geq 0 \), then one has

\[
(\mathcal{F}'_d)^{-1}(Z_{ij}) = X_{ij}
\]

and

\[
(\mathcal{F}'_d)^{-1}(Z_{ij} Z_{kl}) = (\mathcal{F}'_d)^{-1}(Z_{ij}) (\mathcal{F}'_d)^{-1}(Z_{ij} Z_{kl})
\]

\[
- \frac{n! m!}{d} \delta_{ij} \delta_{kl} (\mathcal{F}'_d)^{-1}(Z_{ij} Z_{kl}^{-1}).
\]

**Proof.** Let \( H(t, x) \) be the normalized Hermite polynomials, such that

\[
\sum_{m=0}^{\infty} \frac{m^m}{m!} H_m(t, x) = e^{tx} - 1.
\]
Let \((x_1, \ldots, x_n)\) denote the coordinates in \(\mathbb{R}^n\) with its standard Hilbert space inner product, \((z_1, \ldots, z_n)\) the coordinates in the complexification \(\mathbb{C}^n\), and \(\mathcal{S}\) the Segal–Bargmann transform on \(\mathbb{R}^n\), then for any integers \(k_1, k_2, \ldots, k_n\) one has

\[
\mathcal{S}(H_{k_1}(t, x_1) \cdots H_{k_n}(t, x_n))(z_1, \ldots, z_n) = z_1^{k_1} \cdots z_n^{k_n}
\]

Furthermore, the Hermite polynomials satisfy the following recursion formula

\[
H_m(t, x) = x H_{m-1}(t, x) - (m - 1) t H_{m-2}(t, x).
\]

The random variables \(\text{Re}(X_{ij})\) and \(\text{Im}(X_{ij})\) (for \(i < j\)) are independent gaussian variables, of variance \(t^2d\), which are independent of all \((X_{kl})_{k < l}\) for \((i, j) \neq (k, l)\). The formula of Lemma 6 follows from these facts, after some straightforward computation.

1.2.6. From Lemma 6 we deduce

**Lemma 7.** Let \(r \geq 0\), and \(m, k_1, \ldots, k_r \geq 1\) be integers, then

\[
(\mathcal{F}_r)\mathbb{Z}_m r \mathbb{Z}_k = (\mathbb{Z}_m \mathbb{Z}_k)^{-1} (\mathbb{F}_r)^{-1} (\mathbb{Z}_m^{-1} \mathbb{Z}_k^{-1} - tr_d(\mathbb{Z}_k) - tr_d(\mathbb{Z}_m))
\]

\[
- t (\mathbb{F}_r)^{-1} (\mathbb{Z}_m^{-2} tr_d(\mathbb{Z}_k) \cdots tr_d(\mathbb{Z}_m))
\]

\[
- t \sum_{k_0=1}^{m-2} (\mathbb{F}_r)^{-1} (\mathbb{Z}_m^{m-2-k_0} tr_d(\mathbb{Z}_m) \cdots tr_d(\mathbb{Z}_m))
\]

\[
- \frac{t^2}{d} \sum_{j=1}^{r} k_j (\mathbb{F}_r)^{-1} (tr_d(\mathbb{Z}_k) \cdots tr_d(\mathbb{Z}_m^{m-k_0}))
\]

\[
\times Z^{m-k_0-2} tr_d(\mathbb{Z}_m^{m-k_0-1}) \cdots tr_d(\mathbb{Z}_m^{m-k_0-k_0})
\]

**Proof.** For simplicity we consider only the case \(r = 1\), \(k_1 = k \geq 1\), the other cases being similar. The \(ij\) coordinate of the matrix \(\mathbb{Z}_m \mathbb{Z}_k\) is equal to

\[
\frac{1}{d} \sum_{1 \leq i_2, \ldots, i_{m-1} \leq d} Z_{m, i_1} Z_{i_2, j} \cdots Z_{i_{m-1}, i} Z_{i_m, 1} \cdots Z_{i_m, i_m}
\]  

(1.2.6)
Let $A$ be the matrix function such that

$$(\mathcal{T})^{-1}\left(\frac{1}{d}\sum_{1 \leq i_1, \ldots, i_{u-1} \leq d} Z_{u_1} \cdots Z_{u_{i-1}} Z_{u_{i+2}} \cdots Z_{u_{m}}\right)$$

$$= \frac{1}{d}\sum_{1 \leq i_1, \ldots, i_{u-1} \leq d} (\mathcal{T})^{-1}(Z_{i_1}) (\mathcal{T})^{-1}(Z_{i_2}) \cdots (\mathcal{T})^{-1}(Z_{i_{u-1}}) Z_{u_{i+2}} \cdots Z_{u_{m}}$$

$$+ (\mathcal{T})^{-1}(A).$$

We have

$$(\mathcal{T})^{-1}(Z^{m}tr_d(Z^{k})) = (\mathcal{T})^{-1}(Z(\mathcal{T})^{-1}(Z^{m-1}tr_d(Z^{k}))) + (\mathcal{T})^{-1}(A)$$

and we must evaluate $(\mathcal{T})^{-1}(A)$ using Lemma 6. Let us write $Z_{u_1} \cdots Z_{n_{u_1}}$, a typical term occurring in the sum (1.2.6), and let $x_{u_1}, \ldots, x_{u_r}$, with $2 \leq u_1 < u_2 < \cdots < u_r$, be the indices which are equal to the pair $i_1 i$. According to Lemma 6, the contribution of such a term to the coefficient $A_{i_1 i_2}$ will be

$$- \frac{1}{d^2} \sum_{a=1}^{r} Z_{u_2} Z_{u_3} \cdots \widehat{Z_{u_a}} \cdots Z_{n_{u_1}},$$

where the hat means that we have omitted the term in the product. It follows that one has

$$A_{i_1 i_2} =$$

$$- \frac{1}{d^2} \sum_{r=1}^{m-1} \sum_{1 \leq i_1, \ldots, i_{r-1}, i_{r+2}, \ldots, i_{r} \leq d} Z_{u_2} \cdots Z_{u_{i-1}} Z_{u_{i+2}} \cdots Z_{u_{r-1}} Z_{u_{r+2}} \cdots Z_{u_{m}}$$

$$- \frac{1}{d^2} \sum_{r=1}^{k} \sum_{1 \leq i_1, \ldots, i_{r-1} \leq d} Z_{u_2} \cdots Z_{u_{i-1}} Z_{u_{i+2}} \cdots Z_{u_{r-1}} Z_{u_{r+2}} \cdots Z_{u_{m}}.$$

It is now easy to recognize the term $A_{i_1 i_2}$ as the $ij$ coefficient of the matrix

$$- \frac{m-2}{r=0} Z^{m-2-\epsilon} tr_d(Z^{r}) tr_d(Z^{k}) - k \frac{t}{d^2} Z^{m+k-2}.$$

1.2.7. The Tchebycheff polynomials verify the recursion relations

$$T_{m+1}(t, x) = xT_{m}(t, x) - tT_{m-1}(t, x).$$  (1.2.7)
By Lemma 1 one has \((\mathcal{F}^n_\omega)^{-1}(Z) = X\), and we deduce from Lemma 2 and (1.2.7) that for \(m \geq 2\) one has

\[
(\mathcal{F}^n_\omega)^{-1}(Z^m) - T_m(t, X) = X((\mathcal{F}^n_\omega)^{-1}(Z^{m-1}) - T_{m-1}(t, X))
- t((\mathcal{F}^n_\omega)^{-1}(Z^{m-2}) - T_{m-2}(t, X))
- \frac{t}{d^2} \sum_{k=1}^{m-2} k(\mathcal{F}^n_\omega)^{-1}(Z^{m-2-k}).
\]

By induction on \(m\), using Lemma 7, we see that \((\mathcal{F}^n_\omega)^{-1}(Z^m)\) is equal to a sum of terms of the form

\[
C(d, t) X^{k_0} q^{r_0} \left( \sum_{i=1}^{k_0} Z_{ij_0} \right) \cdots \sum_{i=1}^{k_0} Z_{ij_0} \left( \sum_{i=1}^{k_0} Z_{ij_0} \right)
\]
with \(k_0 \geq 0, r_0 \geq 0, k_1, \ldots, k_r \geq 1\), where the coefficient \(C(d, t)\) is \(O(1/d^2)\) for \(r = 0\), and is bounded in every case. We shall prove that the norm in \(L^2(X^d, \mu, M_d(\mathbb{C}))\) of each of these terms goes to zero as \(d \to \infty\), and this will prove the Theorem 3 for polynomials.

**Lemma 8.** (1) For all \(m \geq 1\), \(\int_{x^d} tr_d(X^m(x))^2 p_1(x) \, dx\) is bounded uniformly in \(d\).

(2) For all \(k \geq 1\), one has \(\int_{M(d; \mathbb{C})} |tr_d(Z(w)^k)|^2 q(w) \, dw = O(1/d^2)\) as \(d \to \infty\).

**Proof.** (1) One has \(tr_d(X^{m^2}) \leq tr_d(X^{2m})\), by Cauchy-Schwarz inequality, and the integral \(\int_{x^d} tr_d(X^{2m}(x)) p_1(x) \, dx = \int_{x^d} \chi^{2m} \mu(dx)\) converges to \(\int \chi^{2m} \mu(dx)\) as \(d \to \infty\), hence is bounded.

(2) One has

\[
\int_{M(d; \mathbb{C})} |tr_d(Z(w)^k)|^2 q(w) \, dw
= \frac{1}{d^2} \int_{M(d; \mathbb{C})} \sum_{1 \leq i_1 \leq d} Z_{ij_1} \cdots \sum_{1 \leq i_k \leq d} Z_{ij_k} \cdots \sum_{1 \leq i_k \leq d} Z_{ij_k} q(w) \, dw. \tag{1.2.8}
\]

Orthogonality relations for the \(Z_{ij}\) imply that a term of the form (1.2.8) is nonzero only if for each variable \(Z_{ij}\) appearing in the product, the variable \(Z_{ij}\) appears as many times, so that for each choice of \(l_1, l_2, \ldots, l_k\) there are at most \(k!\) choices of \(i_1, \ldots, i_k\) which yield a non zero term. All these terms are bounded by \(1/d^2 \int_{M(d; \mathbb{C})} |Z(w)^k|^2 q(w) \, dw = d^{-\frac{1}{2} - \frac{k}{2}'} \) by Hölder’s inequality, and there are \(d^k\) of them, so we see that the total sum is less than \((t'/d^2)k!\)!
1.2.8. We can now give an estimate of the expression
\[ \| X^m (S')^{-1} (tr_d(Z^k)) \|_{L^2(x_0, \rho_l, M_d(C))} \]
namely, one has
\[ \| X^m (S')^{-1} (tr_d(Z^k)) \|_{L^2(x_0, \rho_l, M_d(C))}^2 = \int \left| \# \right|^2 p_d(x) \, dx \]
\[ \leq \left\| tr_d(X^m) \right\|_{L^2(x_0, \rho_l)} \| X^m (S')^{-1} (tr_d(Z^k)) \|_{L^2(x_0, \rho_l)}^2 . \]
The map \( tr_d(Z^k) \) is a polynomial function on \( M_d(C) \), of degree \( k \), hence the map \( (S')^{-1}(tr_d(Z^k)) \) on \( \mathcal{A}_d \) is a polynomial of degree \( k_1 + \ldots + k_r \). It is a well known consequence of the hypercontractivity of the Ornstein–Uhlenbeck semi-group that, for each \( N \geq 1 \), the \( L^p \) norms, for \( 2 \leq p < \infty \), are equivalent on the space of polynomials of degree less than \( N \) in a gaussian space, with constants independent of the dimension of the gaussian space, so that the \( L^4 \) norm of \( (S')^{-1}(tr_d(Z^k)) \) is controlled by its \( L^4 \) norm which is equal to the \( L^4 \) norm of \( tr_d(Z^k) \). Using H"older’s inequality and hypercontractivity again we see that the \( L^4 \) norm of \( tr_d(Z^k) \) is controlled by the product of the \( L^2 \) norms of \( tr_d(Z^k), \ldots, tr_d(Z^k) \), so by Lemma 8, the quantities \( \| X^m (S')^{-1}(tr_d(Z^k)) \|_{L^2(x_0, \rho_l, M_d(C))} \) are uniformly bounded in \( d \) and converge to 0 as soon as \( r \geq 1 \). We deduce Theorem 3 for polynomial functions from this.

1.2.9. End of proof of Theorem 3. Let now \( f \in L^2(\mu'_1) \) satisfy the assumptions of Theorem 3, and let \( \varepsilon > 0 \), then there exists a polynomial \( P \) such that \( \| f - P \|_{L^2(\mu'_1)} < \varepsilon \). By Lemma 4, there exists a constant \( K > 0 \), independent of \( d \) and \( \varepsilon \), such that \( \| f - P \|_{L^2(\mu'_1)} < K \varepsilon \) for all \( d \geq 1 \), hence since \( \mathcal{F}_d' \) is a contraction from \( L^2(\mu'_1) \) to \( \mathcal{E}'_d \), one has \( \| \mathcal{F}_d'f - \mathcal{F}_d'P \|_{\mathcal{E}'_d} \leq \| f - P \|_{L^2(\mu'_1)} < K \varepsilon \). On the other hand, by Lemma 5, we know that
\[ \| \mathcal{F}_d'f - \mathcal{F}_d'P \|_{\mathcal{E}'_d} \rightarrow \| \mathcal{F}f - \mathcal{F}P \|_{\mathcal{E}} = \| f - P \|_{L^2(\mu'_1)} \leq K \| f - P \|_{L^2(\mu'_1)} \]
as \( d \rightarrow \infty \), so that \( \| \mathcal{F}f - \mathcal{F}P \|_{\mathcal{E}} \leq 2K \varepsilon \) for \( d \) large enough. Using the triangle inequality we have
\[ \| \mathcal{F}_d'f - \mathcal{F}_d'P \|_{\mathcal{E}'_d} \leq \| \mathcal{F}_d'f - \mathcal{F}_d'P \|_{\mathcal{E}'_d} + \| \mathcal{F}_d'P - \mathcal{F}_d'P \|_{\mathcal{E}'_d} + \| \mathcal{F}P - \mathcal{F}P \|_{\mathcal{E}} \]
\[ \leq 3K \varepsilon + \| \mathcal{F}_d'P - \mathcal{F}P \|_{\mathcal{E}} \]
for \( d \) large enough. Since \( P \) is a polynomial, by what we have proved before, the term \( \| \mathcal{F}_d'P - \mathcal{F}P \|_{\mathcal{E}} \) goes to zero as \( d \rightarrow \infty \), hence...
1.2.10. Proof of Theorem 2. The argument is almost the same as in Section 1.2.9. Let \( f \in L^2(\mu_t) \), and choose a polynomial \( P \) such that \( \| f - P \|_{L^2(\mu_t)} < \varepsilon \), then \( \| f - P \|_{L^2(\mu_{t+d})} < K\varepsilon \) for all \( d \geq 1 \). Since \( \mathcal{F}_t \) is an isometry, the distance of \( \mathcal{F}_t(\theta_{t+d}^*\phi) \) to the space \( \mathcal{H}(\mathcal{S}^d) \) is less than \( K\varepsilon + \) the distance of \( \mathcal{F}_t(\theta_{t+1}^*\phi) \) to the same space \( \), and this last distance goes to zero as \( d \to \infty \). This proves part (1) of Theorem 2.

The argument for part (2) is analogous, using Lemma 5 to approximate an entire function by polynomials, uniform in all spaces \( \mathcal{S}^d \).

2. HALL’S TRANSFORM AND A THEOREM OF GROSS AND MALLIAVIN

In this section we introduce Hall’s transform, which is an extension of the Segal-Bargmann transform to compact type Lie groups. We shall be interested in a theorem of Gross and Malliavin which reduces the proof of the unitarity of Hall’s transform to that of the infinite dimensional version of the Segal-Bargmann transform. The material in this section, except that in 2.3, is discussed in details in [GM], so we shall be rather sketchy and refer to this paper for a more thorough discussion.

2.1. Hall’s Transform

Let \( K \) be a compact type Lie group, so \( K \) is a Lie group, whose Lie algebra has an \( Ad \ K \)-invariant inner product. Let \( \Lambda \) be the Laplace operator associated to this inner product, and let \( p_t \), for \( t > 0 \), be the density of the integral operator \( e^{-t\Lambda} \), with respect to the Haar measure.

Let \( \mathfrak{l} \) be the Lie algebra of \( K \). The group \( K \) has a complexification \( G \), with Lie algebra \( \mathfrak{g} \oplus it \mathfrak{l} \). On this Lie algebra, put the metric \( \rho \oplus \rho \) where \( \rho \) is the \( Ad \ K \) invariant metric on \( \mathfrak{l} \), and consider the associated Laplace operator \( \Lambda_t \). Let \( q_t(x) \) be the density of the integral operator \( e^{-t\Lambda_t} \).

Theorem 4. Let \( f \in L^2(\mu_t) \), then the integral

\[
\int_K f(y) p_t(xy^{-1}) \, dy
\]

converges for every \( x \in K \), and has a unique analytic continuation \( \mathcal{H} f \) to the group \( G \), which belongs of the space \( L^2(G, q_t(x) \, dx) \). The map \( \mathcal{H} \) is a unitary
isomorphism between the two spaces $L^2(K, p_x(x) \, dx)$ and $L^2_{hol}(G, q_x(x) \, dx)$, which is the subspace of $L^2(G, q_x(x) \, dx)$ consisting of holomorphic functions.

This result was first proved by Hall [Ha] for compact Lie groups, and then extended by Driver [Dr] to Lie groups of compact type. When $K = \mathbb{R}^n$, this result is just Theorem 1 above, thus the transform $\mathcal{F}$ is an extension of the Segal–Bargmann isomorphism to more general compact type Lie groups.

2.2. Infinite Dimensional Segal–Bargmann Transform and the Theorem of Gross and Malliavin

2.2.1. Gross and Malliavin [GM], have shown that the Hall transform can in fact be obtained from the Segal–Bargmann transform in infinite dimension, using probabilistic ideas. Let $\mathfrak{k}$ be as above, and consider a $\mathfrak{f}$ valued brownian motion $X$. This is a gaussian process based on the infinite dimensional Hilbert space $L^2(\mathbb{R}_+) \otimes \mathfrak{f}$, where $\mathfrak{f}$ has its $Ad-K$ invariant inner product.

Let $L^2(X)$ be the $L^2$ space of random variables which are measurable with respect to the $\sigma$-field generated by the brownian motion $X$. We shall consider also a brownian motion $Z$ with values in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{f}$, whose real and complex parts are brownian motion on $\mathfrak{l}$, but whose variance is half the variance of the brownian motion $X$. We let $L^2_{hol}(Z)$ be the space of holomorphic, square integrable functions of this brownian motion (see [GM] for that notion).

The Segal–Bargmann isomorphism can be extended to the setting of infinite dimensional gaussian spaces, and yields in this case a unitary isomorphism $\mathcal{F}$ of $L^2(X)$ with $L^2_{hol}(Z)$.

We refer to the paper [GM] for the exact definitions and the proofs of the statements above.

2.2.2. Using the brownian motions $X$ and $Z$, we can construct brownian motions on $K$ and $G$ respectively, namely, let us consider the stochastic differential equation, in Stratonovitch form,

\[ dU_t = U_t \cdot dB_t, \quad U_0 = I \]  

The solution to this equation is a diffusion process on $K$, with infinitesimal generator $A/2$, defined on the same probability space as $X$, measurable with respect to the $\sigma$-field of $X$, and furthermore the map $f \mapsto f(U_t)$ defines an isometric embedding of the space $L^2(K, p_x) \mapsto L^2(X)$, the space of square integrable functions, measurable with respect to the $\sigma$-field of $X$.

Analogously, the stochastic differential equation, in Stratonovitch form,

\[ dA_t = A_t \cdot dZ_t, \quad A_0 = I \]  

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has a unique solution which is a diffusion process on $G$, with infinitesimal generator $\mathcal{A}$, defined on the same probability space as $Z$, measurable with respect to the $\sigma$-field of $Z$, and furthermore the map $f \mapsto f \cdot \mathcal{A}$, defines an isometric embedding of the space $L^2(G, q_\lambda)$ into $L^2(Z)$, the space of square integrable functions, measurable with respect to the $\sigma$-field of $Z$. In this embedding, the holomorphic functions on $G$ are sent into the space $L^2_{\text{hol}}(Z)$, which is the range of the infinite dimensional Segal-Bargmann transform.

2.2.3. Theorem 5. The following diagram is commutative

$$
\begin{array}{ccc}
L^2(K, p_\lambda) & \xrightarrow{\mathcal{H}^t} & L^2_{\text{hol}}(G, q_\lambda) \\
\downarrow U_t & & \downarrow A_t \\
L^2(\mathbb{X}) & \xrightarrow{\mathcal{S}} & L^2_{\text{hol}}(Z)
\end{array}
$$

In this diagram the horizontal arrows are the Hall transform $\mathcal{H}^t$ and the infinite dimensional Segal-Bargmann transform $\mathcal{S}$, respectively, while the vertical arrows denote the maps $f \mapsto f \cdot U_t$ and $f \mapsto f \cdot A_t$, respectively. Since the three other arrows are isometric, it follows that $\mathcal{H}^t$ is isometric. Theorem 5 tells us that the Hall transform can be recovered from the infinite dimensional Segal-Bargmann transform, through the embedding into the $L^2$-space of brownian motion on the Lie algebra of the group.

This theorem is proved in [GM].

2.3. Some Additional Remarks

2.3.1. Let $U(d)$ be the group of unitary $d \times d$ matrices. The Lie algebra of $U(d)$ is the space of skew adjoint $d \times d$ matrices. We shall endow it with the inner product $\text{tr}(AB^*)$, and consider the corresponding Hall transform. It maps $L^2(U(d), p_\lambda)$ onto $L^2_{\text{hol}}(GL(d, \mathbb{C}), q_\lambda)$. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. Since the unitary matrices are normal, for any Borel function on $\mathbb{T}$, one can define a map $\theta^d_\lambda : U(d) \to M_d(\mathbb{C})$, by functional calculus. As in 1.1.3, there exists a unique probability measure $\nu^d_\lambda$ on $\mathbb{T}$ such that, for every bounded Borel function $f$ on $\mathbb{T}$, one has

$$
\int_{\mathbb{T}} f(x) \, d\nu^d_\lambda(x) = \int_{U(d)} \text{tr}(\theta^d_\lambda(g)) \, p_\lambda(g) \, dg
$$

where $dg$ is the Haar measure on $U(d)$. The map $f \mapsto \theta^d_\lambda$ is an isometry from $L^2(\nu^d_\lambda)$ into $L^2(U(d), p_\lambda, M_d(\mathbb{C}))$. It is proved in [Bi1] that the measures $\nu^d_\lambda$ converge weakly, as $d \to \infty$, to a probability measure $\nu$ on the
Let $F$ be a function holomorphic on $\mathbb{C}\setminus\{0\}$, by holomorphic functional calculus we have a map $\Theta^d_F: A \mapsto F(A)$ from $GL(d, \mathbb{C})$ to $M_d(\mathbb{C})$. Consider the inner product

$$\langle F, G \rangle_d = \int_{GL(d, \mathbb{C})} tr_d(\Theta_F(w) \Theta_G(w)^*) q(w) \, dw$$

on the space $\mathcal{A}_d$ of functions, holomorphic on $\mathbb{C}\setminus\{0\}$, such that

$$\int_{GL(d, \mathbb{C})} tr_d(\Theta_F(w) \Theta_F(w)^*) q(w) \, dw < \infty$$

We shall investigate in section 4 the limit of this inner product, as well as the corresponding Hilbert space, obtained by completion.

2.3.2. Having defined the spaces $L^2(v_d')$ and $\mathcal{A}_d'$, it is natural to investigate whether similar results as Theorems 2 and 3 hold, when the Segal-Bargmann transform is replaced by the Hall transform on unitary groups. One can show that indeed a result like Theorem 3 holds for functions which are Laurent polynomials on $T$. Since the analysis is rather complicated, we shall not do it here, (see however the remark at the end of 4.1.4), and the rest of this paper is devoted to finding the integral transform which plays the role of the map $\mathcal{F}'$ in Theorem 3. Our strategy to find this transform is the following, we shall consider semi-circular and circular systems, and the associated free brownian motions, which are in some sense the limits of brownian motion on matrices (hermitian matrices for semi-circular brownian motion, and arbitrary complex matrices for the circular brownian motion), considered as non-commutative stochastic processes, see [Bi1]. There is an analogue of the Segal-Bargmann transform which maps “non-commutative function” of the semi-circular brownian motion to “non-commutative holomorphic functions” of the circular brownian motion, which can be seen as a higher dimensional generalization of the transform $\mathcal{F}'$ of Theorem 3. Using these free brownian motions we shall consider the stochastic differential equations (2.2.2a) and (2.2.2b), where $X$ and $Z$ are now the semi-circular and the circular brownian motions, and whose solutions are non-commutative processes of operators on a free Fock space. Then we shall prove the analogue of the Gross-Malliavin result, that the free Segal-Bargmann transform maps functions of $U_t$ to holomorphic functions of $A_t$. Furthermore, we shall give an explicit integral formula for this transform.
3. THE FREE SEGAL-BARGMANN TRANSFORM

In this section we shall explain how the transform $F_t$ defined in Proposition 1 is the natural analogue of the Segal–Bargmann transform in the theory of semi-circular and circular systems due to Voiculescu. Indeed $F_t$ is the one dimensional instance of a family of isomorphisms between circular and semi-circular systems of arbitrary dimensions.

3.1. Circular and Semi-circular Systems

3.1.1. We shall first recall some notions from the free probability theory of Voiculescu. Our main reference is [VDN].

Let $A$ be a von Neumann algebra and $\tau$ a normal faithful trace on $A$. We call $W^*$-probability space such a couple $(A, \tau)$. Let $T$ be a normal element in $A$, the distribution of $T$ in the state $\tau$ is the unique probability measure $\mu_T$ on $\mathbb{C}$ such that $\tau(f(T)) = \int f(\lambda) d\mu_T(\lambda)$ for any bounded Borel function $f$ on $\mathbb{C}$, actually, the measure $\mu_T$ is supported by the spectrum of $T$.

3.1.2. We now come to the definition of freeness, which is a non-commutative analogue of the notion of independence in probability theory.

**Definition 1.** Let $(A, \tau)$ be a $W^*$-probability space, let $I$ be a set of indices, and $B_i$, for $i \in I$, be a family of von Neumann subalgebras of $A$. The family $(B_i)_{i \in I}$ is called free if for all $a_1, \ldots, a_n \in A$ such that $\tau(a_j) = 0$ for $j = 1, \ldots, n$ and $a_j \in B_{i_j}$ for some indices $i_1 \neq i_2 \neq \cdots \neq i_n$, one has $\tau(a_1 \cdots a_n) = 0$.

A family $(B_i)_{i \in I}$ of subsets of $A$ is called free if the family of algebras $(B_i)_{i \in I}$, where $B_i$ is the von Neumann algebra generated by $B_i$, form a free family.

A family $(a_i)_{i \in I}$ of elements of $A$ is called free if the family of subsets $(\{a_i\})_{i \in I}$ is free.

3.1.3. Let $(A, \tau)$ be a $W^*$-probability space and $\mathcal{H}$ a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$.

**Definition 2.** A linear map $h \mapsto s(h)$ from $\mathcal{H}$ to $A$ is called a semi-circular system if

1. For each $h \in \mathcal{H}$, the element $s(h)$ is self-adjoint and has semi-circular distribution of variance $\langle h, h \rangle$.
2. For each orthogonal system $h_1, \ldots, h_n$ in $\mathcal{H}$, the family $(s(h_j))_{j=1}^n$ is free.

**Definition 3.** A linear map $h \mapsto c(h)$ from $\mathcal{H}$ to $A$ is called a circular system if

1. For each $h \in \mathcal{H}$, the maps $h \mapsto (1/\sqrt{2})(c(h) + c^*(h))$ and $h \mapsto (1/\sqrt{2}i)(c(h) - c^*(h))$ are semi-circular systems.
(2) The subsets \( \{(1/\sqrt{2})(c_h + c_h^*) \mid h \in \mathcal{H} \} \) and \( \{(1/\sqrt{2} i)(c_h - c_h^*) \mid h \in \mathcal{H} \} \) are free in \((A, \tau)\).

Note that replacing freeness by independence, and semi-circular distribution by gaussian, we recover the classical notion of a gaussian family.

3.1.4. We shall review now a canonical construction of semi-circular and circular systems on a free Fock space, parallel to the classical construction of gaussian variables on boson Fock spaces. Let \( \mathcal{F} \) be a complex Hilbert space, and denote by \( F(\mathcal{T}) \) the free Fock space built on \( \mathcal{T} \), namely \( F(\mathcal{T}) \) is the Hilbert space orthogonal direct sum

\[
F(\mathcal{T}) = \bigoplus_{n=0}^{\infty} \mathcal{T}^\otimes n
\]

where \( \mathcal{T}^\otimes 0 \) is by definition the one dimensional Hilbert space generated by a unit vector \( \Omega \). For each \( h \in \mathcal{T} \), we let \( l_h \) and \( l_h^* \) be the left annihilation and creation operators defined as

\[
l_h(0) = 0 \quad l_h(h_1 \otimes \cdots \otimes h_n) = \langle h_1, h \rangle h_2 \otimes \cdots \otimes h_n \quad l_h^*(h_1 \otimes \cdots \otimes h_n) = h \otimes h_1 \otimes \cdots \otimes h_n
\]

For each \( h \in \mathcal{T} \), \( l_h \) and \( l_h^* \) are bounded operators adjoint of each other on \( F(\mathcal{T}) \).

3.1.5. Let now \( \mathcal{H} \) be a real Hilbert space with complexification \( \mathcal{H}^C \), and \( F(\mathcal{H}^C) \) the associated free Fock space. For each \( h \in \mathcal{H} \) we let \( X_h = l_h + l_h^* \). Let \( \mathcal{F}(\mathcal{H}) \) be the von Neumann algebra of operators on \( F(\mathcal{H}^C) \) generated by \( X_h \mid h \in \mathcal{H} \), and let \( \tau \) be the restriction to \( \mathcal{F}(\mathcal{H}) \) of the pure state associated to the vector \( \Omega \), i.e. \( \tau(T) = \langle T\Omega, \Omega \rangle \) for \( T \in \mathcal{F}(\mathcal{H}) \).

**Proposition 2.** The state \( \tau \) is a faithful normal trace on \( \mathcal{F}(\mathcal{H}) \), and the map \( h \mapsto X_h \), from \( \mathcal{H} \) to the \( W^* \)-probability space \( (\mathcal{F}(\mathcal{H}), \tau) \) is a semi-circular system.

**Proof.** See, e.g., [VDN].

3.1.6. Let \( L^2(\mathcal{F}(\mathcal{H}), \tau) \) be the Hilbert space completion of \( \mathcal{F}(\mathcal{H}) \) for the inner product \( \langle A, B \rangle = \tau(AB^*) \).
1. Let \((T_k)_{k=0}^\infty\) be the Tchebycheff polynomials, and let \((e_j)_{j=1}^\infty\) be an orthonormal basis of \(\mathcal{H}\), for any choice of integers \(k_1, \ldots, k_n\) and \(j_1, \ldots, j_n\) such that \(j_1 \neq j_2 \neq j_3 \neq \cdots \neq j_{n-1} \neq j_n\), one has

\[
T_{k_1}(X(e_{j_1})) T_{k_2}(X(e_{j_2})) \cdots T_{k_n}(X(e_{j_n})) \Omega = e_{j_1}^{\otimes k_1} \otimes \cdots \otimes e_{j_n}^{\otimes k_n}.
\]

The map \(X \mapsto X(\Omega)\) extends to a unitary isomorphism from \(L^2(\mathcal{H}(\mathcal{H}), \tau)\) to \(F(\mathcal{H}^c)\).

\textbf{Proof.} See Voiculescu [V1].

3. Let \(\mathcal{H} = \mathcal{H} \oplus \mathcal{H}\), be the double of \(\mathcal{H}\), with complexification \(\mathcal{H}^c\). For each \(h \in \mathcal{H}\), let \(Z(h) = l(h, 0) + l(0, h)\). The operators \(Z(h)_{h \in \mathcal{H}}\) act on \(F(\mathcal{H}^c)\). Let \(\mathcal{C}(\mathcal{H})\) be the von Neumann algebra generated by the operators \(Z(h)_{h \in \mathcal{H}}\), and we denote again by \(\tau\) the state on \(\mathcal{C}(\mathcal{H})\) obtained by restriction of the pure state associated with the vector \(\Omega\).

\textbf{Proposition 3.} The state \(\tau\) is a faithful normal trace on \(\mathcal{C}(\mathcal{H})\), and the map \(h \mapsto Z(h)\) from \(\mathcal{H}\) to the \(W^*\)-probability space \((\mathcal{C}(\mathcal{H}), \tau)\) is a circular system.

\textbf{Proof.} See [VDN].

We shall denote \(\mathcal{C}_{hol}(\mathcal{H})\) the Banach algebra in \(\mathcal{C}(\mathcal{H})\) generated by the elements \(Z(h)_{h \in \mathcal{H}}\), and we let \(L^2(\mathcal{C}_{hol}(\mathcal{H}), \tau)\) be the completion of this space under the Hilbert space inner product \(\langle A, B \rangle = \tau(AB^*)\). As a Banach algebra, \(\mathcal{C}_{hol}(\mathcal{H})\) can be endowed with holomorphic functional calculus. We do not know wether it is stable by holomorphic functional calculus inside \(\mathcal{C}(\mathcal{H})\) (this would mean that every element in \(\mathcal{C}_{hol}(\mathcal{H})\), invertible in \(\mathcal{C}(\mathcal{H})\), has its inverse in \(\mathcal{C}_{hol}(\mathcal{H})\)).

Using the embedding \(h \mapsto (0, h)\) of \(\mathcal{H}\) into \(\mathcal{H}\) we have an isometric embedding of \(\mathcal{C}_{hol}(\mathcal{H})\) into \(F(\mathcal{H}^c)\).

\textbf{Proposition 4.} Let \((e_j)_{j=1}^\infty\) be an orthonormal basis of \(\mathcal{H}\), for any choice of integers \(k_1, \ldots, k_n\) and \(j_1, \ldots, j_n\) such that \(j_1 \neq j_2 \neq j_3 \neq \cdots \neq j_{n-1} \neq j_n\), one has

\[
(Z(e_{j_1}))^{k_1} \cdots (Z(e_{j_n}))^{k_n} \Omega = e_{j_1}^{\otimes k_1} \otimes \cdots \otimes e_{j_n}^{\otimes k_n}.
\]

The map \(X \mapsto X(\Omega)\) extends to a unitary isomorphism from \(L^2(\mathcal{C}_{hol}(\mathcal{H}), \tau)\) to \(F(\mathcal{H}^c)\) (where \(F(\mathcal{H}^c)\) is considered as a subspace of \(F(\mathcal{H}^c)\)).

\textbf{Proof.} That \(X \mapsto X\Omega\) is an isometry follows from the definition of the inner product. The formula is easy to check, and the result follows.
3.1.8. Definition 4. The free Segal–Bargmann transform is the unitary isomorphism \( F^\circ \), obtained by composition of the isometries of Propositions 3 and 5.

\[
\begin{align*}
L^2(\mathscr{C}(\mathcal{H}), \tau) & \xrightarrow{\mathscr{F}} L^2(\mathcal{C}_{hol}(\mathcal{H}), \tau).
\end{align*}
\]

Remark. A similar construction yields the Segal–Bargmann transform, namely, replacing the free Fock space by a Boson Fock space and the left creation and annihilation operators by Boson creation and annihilation operators one obtains a construction of a gaussian family indexed by \( \mathcal{H} \). Proposition 3 and 5 have analogues, where the Tchebycheff polynomials are replaced by Hermite polynomials, and the Segal–Bargmann isomorphism is given by the same token as in the definition above.

3.2. The One-Dimensional Case

3.2.1. We investigate the isometry \( \mathscr{F}^\circ \) when \( \mathcal{H} \) is one-dimensional.

Let \( \mathcal{H} = \mathbb{R} e \) where \( e \) is a unit vector, then \( X(e) \) is a semi-circular element and thus \( \mathscr{C}(\mathcal{H}) \) is isomorphic to \( L^2(\mu^1) \) by the map \( f \mapsto f(X(e)) \), where functional calculus of self-adjoint operators is used. This map extends to an isometry between \( L^2(\mu^1) \) and \( F(\mathscr{C}(\mathcal{H})) \), in which the Tchebycheff polynomial \( T_k \) corresponds to \( e^k \), by Proposition 3.

On the other hand, the norm of the element \( Z(e) \) in \( \mathcal{C}_{hol}(\mathcal{H}) \) is equal to 2, hence for every function \( F \) holomorphic in a neighborhood of the disk of radius 2, there is a well defined element \( F(Z(e)) \) of \( \mathcal{C}_{hol}(\mathcal{H}) \), obtained by holomorphic functional calculus. Since the vectors \( (Z(e)^k)_{k=0}^{\infty} \) form an orthonormal basis of the space \( L^2(\mathcal{C}_{hol}(\mathcal{H}), \tau) \), by Proposition 5, the map \( F \mapsto F(Z(e)) \) extends to a unitary isomorphism of the Hardy space of the disk of radius 1 onto \( L^2(\mathcal{C}_{hol}(\mathcal{H}), \tau) \). Observe that, since the norm of \( Z(e) \) is 2, not every element in the Hardy space corresponds to an unbounded operator affiliated to \( \mathcal{C}_{hol}(\mathcal{H}) \). Clearly, with the identifications that we have made, the free Segal–Bargmann transform coincides with the isometry \( \mathscr{F}^1 \) of section 1. The case of \( \mathscr{F}^t \) for an arbitrary \( t > 0 \) can be treated by a scaling transformation.

3.2.2. We shall now explain how the formula (1.1.7) for the \( \mathscr{F}^t \)-transform can be interpreted as a free analogue of the formula (1.1.1) for the Segal–Bargmann transform.

Recall first that the free convolution \( \mu \boxplus \nu \) of two probability measures \( \mu \) and \( \nu \) on the real line is defined as the distribution of the sum \( X + Y \),
where $X$ and $Y$ are free self-adjoint elements in some $W^*$-probability space, with respective distributions $\mu$ and $\nu$. The free convolution of measures is not a linear operation, and so the free convolution of a function and a measure has no meaning, contrary to the case of classical convolution of probability measures on the line, nevertheless, it is possible to define a convenient substitute for this in the following way. We first quote a result of [Bi2].

**Theorem 6.** Let $(A, \tau)$ be a $W^*$-probability space, $B$ be a von Neumann subalgebra of $A$, and denote by $\tau(\cdot|B)$ the conditional expectation on $B$. Let $X, Y$ be selfadjoint elements in $A$, with $X \in B$, and $Y$ free with $B$. Denote by $\mu$ and $\nu$ the distributions of $X$ and $Y$, then there exists a Feller Markov kernel $\mathcal{K} = k(y, dx)$ on $\mathbb{R} \times \mathbb{R}$ and an analytic function $F$ on $\mathbb{C}\setminus\mathbb{R}$ such that

1. for any Borel bounded function $f$ on $\mathbb{R}$,
   $$\tau(f(X + Y)|B) = \mathcal{K}f(X)$$
   (where $\mathcal{K}f(y) = \int_{\mathbb{R}} f(x) k(y, dx)$)
2. for all $\zeta \in \mathbb{C}\setminus\mathbb{R}$ one has
   $$\int_{\mathbb{R}} (\zeta - x)^{-1} k(y, dx) = (F(\zeta) - y)^{-1}.$$

By Theorem 6, we can association a Feller Markov kernel to any pair of probability measures $\mu$ and $\nu$ on $\mathbb{R}$. Although the kernel $k(y, dx)$ is used only for $y$ in the support of $\mu$, it can be defined through (2) for all $y \in \mathbb{R}$, and gives a natural candidate for the "free convolution operator by the measure $\nu"$, which is linear. In general this kernel depends also on the starting measure $\mu$, and not only on $\nu$. If we choose $\mu = \delta_0$ and $\nu = \mu'$, the semicircular distribution, then we have (see [Bi2]) $F(\zeta) = \frac{1}{2}(\zeta + \sqrt{\zeta^2 - 4t})$ and

$$k(y, dx) = \frac{1}{2\pi} \frac{t \sqrt{4t - x^2}}{y^2 - xy + t} \frac{1}{e^{2\sqrt{4t - x^2} - t} - e^{2\sqrt{4t - y^2} - t}} dx.$$

We see that for all $f \in \mathcal{L}_2(\mu')$, the integral $\int_{\mathbb{R}} \sqrt{4t - y^2} f(y) k(x, dy)$ converges for all $x \in \mathcal{L}_1(\mathbb{R})$, and extends to an analytic function in the disk of radius $\sqrt{t}$, which is equal to $Ff$.

### 3.3. Polynomial Calculus of Matrices and of Circular and Semi-circular Systems

**3.3.1.** We shall extend to the multidimensional case of the free Segal–Bargmann transform some of the results of section 1. Indeed, we
shall prove an extension of Theorem 3 but only for polynomial functional calculus.

Let $\mathcal{H}$ be a finite dimensional real Hilbert space, and $(e_1, ..., e_n)$ an orthonormal basis. For any polynomial $P$ in $n$ non-commuting indeterminates, there is another polynomial $\Psi(P)$, in $n$ non-commuting indeterminates such that

\[
\mathcal{F}(P(X(e_1), ..., X(e_n))) = \Psi(P)(Z(1), ..., Z(n))
\]

Indeed, by Propositions 3 and 5, if $P(x_1, ..., x_n) = T_{k_1}(x_{j_1}) \cdots T_{k_r}(x_{j_r})$, where $j_1 \neq j_2 \neq \cdots \neq j_r$ and $k_1, ..., k_r \geq 1$, then $\Psi(P)(z_1, ..., z_n) = z_1^{k_1} z_2^{k_2} \cdots z_r^{k_r}$, hence $\mathcal{F}$ induces a bijective linear transformation on the space of polynomials in $n$ non-commuting variables.

3.3.2. On the space $X_d$, let us consider the $n$ maps with values in $M_d(C)$ given by $X(j) = I \otimes \langle \cdot, e_j \rangle$. Similarly, on the space $M_d(C) \otimes \mathcal{H}$, let us consider the $n$ maps with values in $M_d(C)$ given by $Z(j) = I \otimes \langle \cdot, e_j \rangle$, and let $\mathcal{S}^1$ be the Segal–Bargmann transform from $L^2(X_d \otimes \mathcal{H}, p_1, M_d(C))$ to $L^2(M_d(C) \otimes \mathcal{H}, q_1, M_d(C))$. The maps $X^{(j)}$ and $Z^{(j)}$, considered as families of non-commutative random variables, give approximations to semi-circular and circular systems, when $d \to \infty$, indeed this is the content of the main result of [V2]. The following result states that the Segal–Bargmann transform also goes to the limit as $d \to \infty$.

**Theorem 7.** For every polynomial $P$ in $n$ non-commuting indeterminates, one has

\[
\|\mathcal{S}^1(P(X^{(1)}, ..., X^{(n)})) - \Psi(P)(Z^{(1)}, ..., Z^{(n)})\|_{L^2(M_d(C) \otimes \mathcal{H}, q_1, M_d(C))} \to 0
\]

as $d \to \infty$.

**Proof.** The proof is very similar to the computations done in Subsections 1.2.5–1.2.8, so we shall only give the mainlines. It is enough to consider the case when the polynomial $P$ has the form $P(x_1, ..., x_n) = T_{k_1}(x_{j_1}) \cdots T_{k_r}(x_{j_r})$, in which case $\Psi(P)(z_1, ..., z_n) = z_1^{k_1} z_2^{k_2} \cdots z_r^{k_r}$.

So let $(Z^{(j)})^{k_1} (Z^{(j)})^{k_2} \cdots (Z^{(j)})^{k_r} \in L^2(M_d(C) \otimes \mathcal{H}, q_1, M_d(C))$, where $j_1 \neq j_2 \neq \cdots \neq j_r$ and $k_1, ..., k_r \geq 1$. We can, without loss of generality, assume that $j_1 = 1$, and we let

\[
(Z^{(j)})^{k_1} (Z^{(j)})^{k_2} \cdots (Z^{(j)})^{k_r} = (Z^{(1)})^{w_1} Q_1 (Z^{(1)})^{w_2} Q_2 \cdots (Z^{(1)})^{w_r} Q_r,
\]

where $Q_1, ..., Q_r$ are monomials in the $Z^{(j)}$ for $j \neq 1$, these monomials being nontrivial except possibly $Q_r$. Using Lemma 6, as in the proof of Lemma 7 we can see that
A similar, more complicated formula holds for $(\mathcal{G}')^{-1}$ evaluated on the product of $(Z_{j_1}^{(1)})^{k_1} (Z_{j_2}^{(1)})^{k_2} \cdots (Z_{j_r}^{(1)})^{k_r})$ with terms of the form $tr_d(R(Z^{(1)}, Z^{(2)}, \ldots, Z^{(n)}))$ for polynomial $R$. Using induction on the total degree of the polynomial $P$, and estimates analogue to Lemma 8 one can then check that Theorem 7 holds. Details are left to the reader.

4. Hall's Transform and Gross–Malliavin Theorem on the Free Fock Space

In this last part of the paper we shall describe the analogue of the Gross–Malliavin Theorem, and of the Hall transform, which will be the subject of the main result of this part, namely Theorem 7 in Section 4.3. But first we shall review stochastic calculus in the free Fock space.

4.1. Free Stochastic Calculus

4.1.1. With the notations of 3.1, let $\mathcal{H} = L^2(\mathbb{R}_+)$, and let $X_t = \mathcal{X}(1_{[0, t]})$ for $t \in \mathbb{R}_+$. The family of operators $(X_t)_{t \in \mathbb{R}_+}$ is a free brownian motion in the noncommutative probability space $(\mathcal{F}(L^2(\mathbb{R}_+)), \tau)$, which means that its time ordered increments form free families, distributed according to semi-circular distributions (see [Sp1, Bi1, Bi2]). We shall also consider the circular brownian motion $Z_t = Z(1_{[0, t]})$ in $(\mathcal{C}(L^2(\mathbb{R}_+)), \tau)$. A stochastic calculus parallel to stochastic calculus for brownian motion can be developed for circular and semi-circular brownian motions. This stochastic calculus was considered in the paper [KS, Sp2], where in fact it was constructed with respect to the creation and annihilation processes. We shall follow a slightly different approach, since we will need to integrate $L^2$ processes. We shall need however some results of [KS] which will be recalled below.

4.1.2. Let us denote by $\mathcal{F}(t) = \mathcal{F}(L^2([0, t])) \subset \mathcal{F}(L^2(\mathbb{R}_+))$ the von Neumann algebra generated by the operators $(X_t)_{t \leq 1}$, and similarly $\mathcal{H}(t) = \mathcal{H}(L^2([0, t])) \subset \mathcal{H}(L^2(\mathbb{R}_+))$ and $\mathcal{H}_{sa}(t) = \mathcal{H}_{sa}(L^2([0, t])) \subset \mathcal{H}_{sa}(L^2(\mathbb{R}_+))$. 

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Definition 5. A semi-circular biprocess on \([0, t]\), is a weakly measurable map \(s \mapsto W_s\) from \([0, t]\) to \(L^2(\mathscr{E}(t), \tau) \otimes L^2(\mathscr{E}(t), \tau)\). It is called square integrable if
\[
\|W\|^2 = \int_0^t \|W_s\|^2_{L^2(\mathscr{E}(t), \tau) \otimes L^2(\mathscr{E}(t), \tau)} ds < \infty.
\]
A circular biprocess on \([0, t]\) is a weakly measurable map \(s \mapsto W_s\) from \([0, t]\) to \(L^2(\mathscr{G}(t), \tau) \otimes L^2(\mathscr{G}(t), \tau)\). It is called square integrable if
\[
\|W\|^2 = \int_0^t \|W_s\|^2_{L^2(\mathscr{G}(t), \tau) \otimes L^2(\mathscr{G}(t), \tau)} ds < \infty.
\]

The norm \(\|W\|\) defines a Hilbert space norm on the space of equivalence classes of semi-circular (or circular) biprocesses, modulo biprocesses which are almost surely zero. This Hilbert space is naturally isomorphic to \(L^2(\{0, t\}) \otimes L^2(\mathscr{E}(t), \tau) \otimes L^2(\mathscr{E}(t), \tau)\) (or \(L^2(\{0, t\}) \otimes L^2(\mathscr{G}(t), \tau) \otimes L^2(\mathscr{G}(t), \tau)\) for circular biprocesses). We shall denote it by \(SCI(t)\) for semi-circular biprocesses and \(CI(t)\) for circular biprocesses.

Definition 6. A biprocess is simple if it is a linear combination of biprocesses of the form \(s \mapsto W_s = F \otimes G_1\) for some interval \([s_1, s_2] \subseteq [0, t]\), and \(F, G \in \mathcal{F}(t)\) for semi-circular biprocesses, \(F, G \in \mathcal{G}(t)\) for circular biprocesses.

Definition 7. A semi-circular (resp. circular) biprocess \(s \mapsto W_s\) is adapted if \(W_s \in L^2(\mathcal{F}(s), \tau)\) (resp. \(L^2(\mathcal{G}(s), \tau)\)), for all \(s \in [0, t]\).

The simple adapted biprocesses (either semi-circular or circular) form a dense subspace of the space of all square integrable adapted biprocesses.

Definition 8 (Stochastic integral of simple biprocesses). For a simple adapted semi-circular biprocess \(s \mapsto W_s = F \otimes G_1\), let
\[
\int_0^t W_s \circ dX_s = F(X_{s_2} - X_{s_1}) G.
\]
For a simple adapted circular biprocess \(s \mapsto W_s = F \otimes G_1\), let
\[
\int_0^t W_s \circ dZ_s = F(Z_{s_2} - Z_{s_1}) G
\]
and
\[
\int_0^t W_s \circ dZ^*_s = F(Z^*_{s_2} - Z^*_{s_1}) G.
\]
Note that the composition of operators is well defined since we have taken $F$ and $G$ bounded. We extend these definitions by linearity to all simple adapted biprocesses, it is easy to check that this extension is well defined.

**Lemma 9.** For any simple adapted biprocesses $W$ and $W'$, one has

$$\tau \left[ \int_0^t W_s \, dX_s \left( \int_0^t W'_s \, dX_s \right)^* \right] = \int_0^t \langle W_s, W'_s \rangle_{L^2(\mathcal{P}(t), \Omega)} \, ds$$

$$\tau = \langle W, W' \rangle_{SC(t)}$$

for semi-circular biprocesses, and

$$\tau \left[ \int_0^t W_s \, dZ_s \left( \int_0^t W'_s \, dZ_s \right)^* \right] = \int_0^t \langle W_s, W'_s \rangle_{L^2(\mathcal{P}(t), \Omega)} \, ds$$

$$\tau = \langle W, W' \rangle_{C(t)}$$

for circular biprocesses.

**Proof.** The verification of the lemma is straightforward, using the freeness of the increments of the process $X$ or $Z$, and the adaptedness condition.

Lemma 9 shows that the stochastic integral map defined on the space of simple adapted semi-circular biprocesses has a unique isometric extension to the space of square integrable adapted semi-circular biprocesses, with values in $L^2(\mathcal{P}(t), \tau)$. Also a similar result holds for circular biprocesses. In the sequel we shall use the notations $\int_0^t W_s \, dX_s$, $\int_0^t W_s \, dZ_s$, and $\int_0^t W_s \, dZ_s^*$ to denote these extensions.

If $W$ is a square integrable circular biprocess such that $W_s \in L^2(\mathcal{H}(s), \tau)$ for all $s \in [0, t]$, then $\int_0^t W_s \, dZ_s \in L^2(\mathcal{H}(s), \tau)$. We shall be particularly interested in stochastic integrals of biprocesses of the forms $s \mapsto F_s \otimes G_s$ where $F_s, G_s \in L^2(\mathcal{P}(s), \tau)$ or $L^2(\mathcal{H}(s), \tau)$ for

**Lemma 10.** Let $W$ be a square integrable biprocess then $\tau(\int_0^t W_s \, dX_s) = 0$, in the semi-circular case, $\tau(\int_0^t W_s \, dZ_s) = 0$ and $\tau(\int_0^t W_s \, dZ_s^*) = 0$ in the circular case.

**Proof.** This is easily verified on simple biprocesses, and this passes to the limit for arbitrary square integrable one.
4.1.3. We shall now state some results of [KS] concerning stochastic integrals of bounded biprocesses.

For an integer $n$, let $\Pi_n$ be the orthogonal projection onto $L^2(\mathbb{R}_+) \otimes^n$, for $n \geq 0$, and $\Pi_n = 0$ for $n < 0$. For $A \in \mathcal{F}(t)$, $n \in \mathbb{Z}$, let $A^{(n)} = \sum_{k=-\infty}^{n} \Pi_k A \Pi_{k+n}$, and for a measurable map $s \mapsto F_s$ from $[0, t]$ to $\mathcal{F}(t)$ let

$$\|F\|_{L,\infty} = \sum_{n \in \mathbb{Z}} \sup_{s \leq t} \|F^{(n)}_s\|_{\mathcal{F}(t)}$$

for all $t \geq 0$. The space of processes with $\|F\|_{L,\infty} < \infty$ is denoted by $e_{\infty}$ in [KS]. One has $\sup_{s \leq t} \|F_s\|_{\mathcal{F}(t)} \leq \|F\|_{L,\infty}$.

We define similarly $\|F\|_{L,\infty}$ for a map from $[0, t]$ to $\mathcal{C}(t)$ using the orthogonal projections on $(L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+))^\otimes n$ in the free Fock space $F(L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+))$.

Definition 9. A strongly bounded adapted semi-circular (resp. circular) process on $[0, t]$ is a measurable map $s \mapsto F_s$ from $[0, t]$ to $\mathcal{C}(t)$ (resp. $\mathcal{F}(t)$) such that $F_s \in \mathcal{F}(s)$ (resp. $\mathcal{C}(s)$) for all $s \in [0, t]$, and $\|F\|_{L,\infty} < \infty$.

Given $F$ and $G$ two adapted strongly bounded processes, stochastic integrals with respect to the creation and annihilation processes are constructed in [KS], denoted by $\int_0^t F_s \, dX_s \, G_s$ and $\int_0^t F_s \, dZ_s \, G_s$. It is easy to see that the stochastic integrals that we have constructed are given by

$$\int_0^t F_s \, dX_s \, G_s = \int_0^t F_s \, dl_s \, G_s + \int_0^t F_s \, dl_s^* \, G_s.$$

Proposition 6. The strongly bounded processes form a *-algebra for pointwise multiplication, and pointwise adjoint. Furthermore, one has the following inequality for any pair of strongly bounded processes $F$ and $G$

$$\left\| \left( \int_0^t F_s \, dX_s \, G_s \right)^{(n)} \right\|_{\mathcal{F}(t)} \leq 2(t-s)^{1/2} \sum_{k=-\infty}^{\infty} \sup_{s \leq t} \|F^{(k)}_s\| \sup_{s \leq t} \|G^{(k-1)}_s\|.$$

Proof. This follows from the estimates in Section 2 of [KS].

For any pair of strongly bounded, adapted processes, one has

$$\left\| \int_0^t F_s \, dX_s \, G_s \right\|_{\mathcal{F}(t)} \leq 2(t-s)^{1/2} \|F\|_{L,\infty} \|G\|_{L,\infty}$$

and

$$\left\| \int_0^t F_s \, dZ_s \, G_s \right\|_{\mathcal{F}(t)} \leq 2(t-s)^{1/2} \|F\|_{L,\infty} \|G\|_{L,\infty}.$$
and
\[ \left\| s \mapsto \int_0^t F_u dX_u G_u \right\|_{L^\infty} \leq 2 \sqrt{t} \|F\|_{L^\infty} \|G\|_{L^\infty}. \]

It follows from this that the process \( s \mapsto \int_0^t F_u dX_u G_u \) is continuous in norm, and is again a strongly bounded process. This allows one to consider multiple stochastic integrals.

Analogous results hold also for pairs of circular strongly bounded adapted processes.

**Proposition 7.** Let \( F, G \) be strongly bounded adapted processes then
\[ \left( \int_0^t F_s dX_s G_s \right)^* = \int_0^t G_s^* dX_s F_s^* \]
in the semi-circular case, and
\[ \left( \int_0^t F_s dZ_s G_s \right)^* = \int_0^t G_s^* dZ_s F_s^* \]
in the circular case.

**Proof.** This is easily checked for simple processes and then extended to arbitrary processes (see [KS]).

4.1.4. An Ito formula can be stated for stochastic integrals with respect to free brownian motions.

**Proposition 8.** Let \( F^1, G^1 \) be strongly bounded adapted processes, and let \( s \mapsto F^2_s \otimes G^2_s \) be a square integrable adapted biprocess, then one has
\[
\int_0^t F^1_s dX_s G^1_s = \int_0^t F^2_s dX_s G^2_s
= \int_0^t F^1_s dX_s \left( G^1_s \int_0^t F^2_u dX_u G^2_u \right)
+ \int_0^t \left[ \int_0^t F^1_u dX_u G^1_u \right] F^2_s dX_s G^2_s + \int_0^t \left[ \int_0^t F^1_u dX_u G^2_u \right] G^2_s dX_s
\]
in the semi-circular case
\[
\int_0^t F^1_s dZ_s G^1_s = \int_0^t F^2_s dZ_s G^2_s
= \int_0^t F^1_s dZ_s \left( G^1_s \int_0^t F^2_u dZ_u G^2_u \right)
+ \int_0^t \left[ \int_0^t F^1_u dZ_u G^1_u \right] F^2_s dZ_s G^2_s
\]
in the circular case.

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and

\[
\int_0^t F^1_s dZ_s G^1_s \int_0^s F^2_s dZ_s^* G^2_s \\
= \int_0^t F^1_s dZ_s \left[ G^1_s \int_0^s F^2_s dZ_s^* G^2_s \right] \\
+ \int_0^t \left[ \int_0^s F^2_s dZ_s G^1_s \right] F^2_s dZ_s^* G^2_s + \int_0^t F^1_s \tau(G^1_s F^2_s) G^2_s \, ds
\]

in the circular case.

Similar statements hold also if one exchanges the roles of \((F^1, G^1)\) and \((F^2, G^2)\), or \(Z\) and \(Z^*\).

**Proof.** The case where \(F^2_s, G^2_s\) are also strongly bounded processes is treated in Section 4 of [KS]. The general case follows from an approximation argument. Since the product of an element in \(L^2(\mathcal{H}(t), \tau)\) by an element in \(\mathcal{H}(t)\) makes sense as an element of \(L^2(\mathcal{H}(t), \tau)\), all terms in the formula have a meaning as stochastic integrals of square integrable biprocesses.

**Remark.** The above Ito formula can be seen as the limit of Ito formula for matrix valued brownian motion, namely let \((X_t)_{t \in \mathbb{R}}\) be a brownian motion with values in \(\mathcal{A}\), and let \(F^1, G^1, F^2, G^2\) be adapted stochastic processes with values in \(d \times d\) matrices, then one can consider the matrix stochastic integrals \(\int_0^t F^1_s dX_s G^1_s\) and \(\int_0^t F^2_s dX_s G^2_s\), and the corresponding Ito formula reads

\[
\int_0^t F^1_s dX_s G^1_s \int_0^s F^2_s dX_s G^2_s \\
= \int_0^t F^1_s dX_s \left[ G^1_s \int_0^s F^2_s dX_s G^2_s \right] \\
+ \int_0^t \left[ \int_0^s F^2_s dX_s G^1_s \right] F^2_s dX_s G^2_s + \int_0^t F^1_s \tau(G^1_s F^2_s) G^2_s \, ds.
\]

Replacing the normalized trace \(\tau_d\) by \(\tau\) yields the Ito formula for free brownian motion. Using this Ito formula as well as some concentration of measure estimates, which allow one to replace \(\tau_d\) by its expectation, when \(d \to \infty\), it is possible to show that the exact computations done below for the free brownian motion (namely Proposition 9, Lemmas 19 and 20), can indeed be done asymptotically for matrix valued brownian motion, and this yields an analogue of Theorem 3 for Hall’s transform. Since the analysis is rather long, we shall not carry it out here.
The free Segal–Bargmann transform behaves nicely with respect to the stochastic integral.

**Proposition 9.** Let \( W_s \in L^2(\mathcal{F}(s), \tau) \otimes L^2(\mathcal{F}(s), \tau) \) be an adapted square integrable semi-circular biprocess, then one has

\[
\mathcal{F} \left( \int_0^t W_s \, dX_s \right) = \int_0^t \mathcal{F} \otimes \mathcal{F}(W_s) \, dZ_s
\]

where \( \mathcal{F} : L^2(\mathcal{F}(t), \tau) \rightarrow L^2(\mathcal{F}(t), \tau) \) is the free Segal–Bargmann transform (see Section 3.1.8).

**Proof.** This is easily checked for simple adapted biprocesses, and this extends to arbitrary biprocesses by the isometry property.

It follows in particular that, for a pair of adapted strongly bounded semi-circular processes one has

\[
\mathcal{F} \left( \int_0^t F_s \, dX_s G_s \right) = \int_0^t \mathcal{F}(F_s) \, dZ_s \mathcal{F}(G_s)
\]

and for adapted strongly bounded circular processes

\[
\int_0^t \mathcal{F}^{-1}(F_s) \, dX_s \mathcal{F}^{-1}(G_s) = \mathcal{F}^{-1} \left( \int_0^t F_s \, dZ_s G_s \right).
\]

Observe that in this equation, the operators on the right hand side are not necessarily bounded, this is why we need to consider integrals of square integrable biprocesses.

**4.2. Free Multiplicative Brownian Motions**

**4.2.1.** We use the notations of Section 4.1. According to Section 5 of [KS] there exists a unique family of unitary operators \( (U_t)_{t \in \mathbb{R}_+} \), solution to the stochastic differential equation

\[
dU_t = i \, dX_t \, U_t - \frac{1}{2} U_t \, dt \quad U_0 = 1.
\]

This means that \( t \mapsto U_t \) is a strongly bounded semi-circular adapted process and

\[
U_t = 1 + i \int_0^t dX_s U_s - \frac{1}{2} \int_0^t U_s \, ds \tag{4.2.1}
\]
for all $t \geq 0$. The expression $\int_0^t dX_s U_s$ denotes the stochastic integral of the biprocess $1 \otimes U_s$, we shall use similar notations in the sequel.

The process $(U_t)_{t \in \mathbb{R}}$ is a multiplicative free brownian motion (see Theorem 2.3 in [Bi1]), and can be seen indeed, as the limit of brownian motion on the unitary group $U(d)$, as $d \to \infty$. Observe that Eq. (4.2.1) is the Ito form of the Stratonovitch equation (2.2.a), and also that we have exchanged right and left (this is only a matter of convenience, to keep with the notations we used in [Bi1]).

By Proposition 7, the adjoint of $U_t$ satisfies the equation

$$dU_t^* = -i U_t^* dX_t - \frac{1}{2} U_t^* dt.$$ 

Let $L^2(U_t, \tau)$ denote the closure of the space of operators $f(U_t)$ for $f$ bounded Borel function on $T$, in $L^2(\mathcal{F}(t), \tau)$, then the map $f \mapsto f(U_t)$ extends to a unitary isomorphism between $L^2(V^v)$ and $L^2(U_t, \tau)$, where $V^v$ is the distribution of $U_t$.

Let $(A_t)_{t \in \mathbb{R}}$ be the solution to the free stochastic differential equations

$$dA_t = i dZ_t A_t, \quad A_0 = 1.$$ 

Existence and uniqueness of the solution to this equation follows from the same techniques as in Section 3 of [KS]. In particular, the $A_t$ are bounded operators, but they are not normal. Using Ito’s formula one can check that $A_t$ has a bounded inverse and that this inverse is solution to the free stochastic differential equation

$$dA_t^{-1} = -i A_t^{-1} dZ_t, \quad A_0^{-1} = 1,$$

see [Bi1]. Also the adjoints $A_t^*$ and $(A_t^{-1})^*$ satisfy

$$dA_t^* = -i A_t^* dZ_t^*, \quad (A_t^{-1})^* = i dZ_t^*(A_t^{-1})^*.$$ 

Since these equations are solved using Picard iteration scheme, which converges in the norm topology, it follows that $A_t$ and $A_t^{-1}$ belong to $\mathcal{C}_{\text{hol}}(t)$ for all $t > 0$.

Let $F$ be a holomorphic function in $\mathbb{C} \setminus \{0\}$, then an operator $F(A_t) \in \mathcal{C}_{\text{hol}}(t)$ is defined by holomorphic functional calculus, and we let $L^2_{\text{hol}}(A_t, \tau)$ be the completion of the space of operators of the form $F(A_t)$ in $L^2(\mathcal{C}_{\text{hol}}(t), \tau)$. We shall see that the free Segal–Bargmann transform from $L^2(\mathcal{C}_{\text{hol}}(t), \tau)$ to $L^2(\mathcal{C}_{\text{hol}}(t), \tau)$ sends $L^2(U_t, \tau)$ to $L^2_{\text{hol}}(A_t, \tau)$, isomorphically, and can be expressed, between these two spaces, as an integral operator, with a formula similar to (1.1.7). But first we shall give a more precise description of the spaces $L^2(U_t, \tau)$ and $L^2_{\text{hol}}(A_t, \tau)$. 
4.2.2. We shall compute the distribution of $U_t$, i.e. the probability distribution $\nu^t$ on $\mathbb{T}$ such that $\int_{\gamma} \phi^\nu(\omega) = \tau(U^\nu_t)$ for all $n \geq 0$. Let

$$\kappa(t, z) = \int_{\gamma} \frac{\omega + z}{\omega - z} d\nu(\omega).$$

The function $\kappa(t, \cdot)$ is analytic in $D$, the open disk or radius 1 in $\mathbb{C}$, and has positive real part there.

**Lemma 11.** One has

$$\frac{\kappa(t, z) - 1}{\kappa(t, z) + 1} e^{\kappa(t, z)} = z$$

(4.2.2.a)

for all $z \in D$.

**Proof.** This lemma is an immediate consequence of the computations in [Bi1]. We give the main line of these computations for the reader’s convenience.

Let $V_t = e^{tU^*_t}$, then $V_t$ is solution of the equation

$$V_t = 1 - i \int_0^t V_s dX_s.$$

(4.2.2.b)

Using Ito’s formula and Eq. (4.2.2.b) it follows, by induction on $n$, that for all $n \geq 0$ and $t > 0$ one has

$$V^n_t = 1 - i \sum_{k=1}^n \int_0^t V^k_s dX_s V^{n-k}_s - \sum_{k=1}^{n-1} \int_0^t k V^k_s \tau(V^{n-k}_s) ds.$$

Evaluating the trace $\tau$ on both sides of this equation, and using the fact that the trace of a stochastic integral is zero (see Lemma 10), we get

$$\tau(V^n_t) = 1 - \sum_{k=1}^{n-1} k \int_0^t \tau(V^k_s) \tau(V^{n-k}_s) ds.$$

Let us introduce the generating function

$$\eta(t, z) = \sum_{n=1}^\infty z^n \tau(V^n_t).$$

We have

$$\eta(t, z) = \frac{z}{1 - z} - \int_0^t \eta(s, z) \frac{\partial \eta(s, z)}{\partial z} ds.$$
for \( z \) in a neighbourhood of zero. This implies that

\[
\eta(t, z)
\]

\[
\eta(t, z)
\]

\[
\text{in a neighbourhood of zero. Since}
\]

\[
\kappa(t, z) = 1 + 2 \sum_{n=1}^{\infty} z^n \int_T \omega^{-n} \omega'(d\omega) = 2\eta(t, e^{-\frac{1}{2}z}) + 1,
\]

Eq. (4.2.2.a) follows for \( z \) in a neighbourhood of zero. It holds in all \( D \), by analytic continuation, since the function \( \kappa \) takes values with positive real part in \( D \), and thus cannot take the value \(-1\). 

Remark. The moments of the distribution \( \nu' \) were computed in [Bi1], one has

\[
\int_T \omega^n \omega'(d\omega) = Q_n(t) e^{-\frac{n}{2}}
\]

where

\[
Q_n(t) = \sum_{k=0}^{n-1} (-1)^k \frac{k^k}{k!} n^{k-1} \binom{n}{k+1}
\]

The polynomials \( Q_n \) can be expressed in terms of Laguerre polynomials, namely one has

\[
Q_n(t) = \frac{1}{n} L_n^{(1)}(nt)
\]

where \( L_n^{(1)} \) are the Laguerre polynomials (see [Sz]). The probability distributions \( \nu' \) form a semi-group for the free multiplicative convolution of measures on \( \mathbb{T} \) (see [VDN] and [Bi1]).

4.2.3. Let us define the following regions in \( \mathbb{C} \), for \( t > 0 \),

\[
\Gamma_t = \left\{ z \in \mathbb{C} \mid z + \bar{z} > 0; \left| \frac{z-1}{z+1} e^{\frac{t}{2}z} \right| < 1 \right\}
\]

In order to describe more precisely the regions \( \Gamma_t \), we shall consider the cases \( t \leq 4 \) and \( t > 4 \) separately.

Suppose first that \( t \leq 4 \), and let

\[
\phi_t(z) = \left| \frac{z-1}{z+1} e^{\frac{t}{2}z} \right|
\]
for \( x > 0 \), then there exists \( x(t) > 1 \) such that \( \phi_*(x) < 1 \) for \( 0 < x < x(t) \) and \( \phi_*(x) > 1 \) for \( x > x(t) \). For any complex number \( z = x + iy \) with \( x > 0 \), one has

\[
\left| \frac{z - 1}{z + 1} e^{iz} \right|^2 = \frac{(x - 1)^2 + y^2}{(x + 1)^2 + y^2} e^{i\pi x} \geq \phi_*(x)^2;
\]

hence the points in the region \( \Gamma_* \) have real part in the interval \( ]0, x(t)[ \). For any \( x \in ]0, x(t)[ \) the inequality

\[
\frac{(x - 1)^2 + y^2}{(x + 1)^2 + y^2} e^{i\pi x} < 1
\]

is satisfied by \( y \in ]-k_*(x), +k_*(x)[ \), where

\[
k_*(x) = \sqrt{\frac{(x + 1)^2 - (x - 1)^2 e^{i\pi x}}{e^{i\pi x} - 1}}.
\]

Observe that \( k_*(x) \) is a well-defined positive real number since \( x \in ]0, x(t)[ \), and

\[
k_*(x) \to \frac{4}{\sqrt{t} - 1}.
\]

as \( t \to 0+ \). The conclusion of this discussion is that \( \Gamma_* \) is a Jordan domain in \( \mathbb{C} \), whose boundary is the closed Jordan curve which is the union of the curves \( y = k_*(x) \) for \( x \in ]0, x(t)[ \), \( y = -k_*(x) \) for \( x \in ]0, x(t)[ \), and \( x = 0 \),

\[
y \in \left[ -\frac{4}{\sqrt{t} - 1}, \frac{4}{\sqrt{t} - 1} \right].
\]

When \( t > 4 \), the function \( \phi_* \) is \( < 1 \) on some interval \( ]x_-(t), x_+(t)[ \), with \( 0 < x_-(t) < 1 < x_+(t) \), and similar considerations show that \( \Gamma_* \) is the region bounded by the curves \( y = \pm k_*(x) \) for \( x \in [x_-(t), x_+(t)] \), whose union is a closed Jordan curve in \( \mathbb{C} \).

The regions \( \Gamma_* \), for several values of \( t \), are depicted in the figures at the end of the paper.

**Lemma 12.** For every \( t > 0 \), the map \( \kappa(t, \cdot) \) defines a conformal one to one map from the disk \( D \) to the region \( \Gamma_* \). Furthermore, the function \( \kappa(t, \cdot) \) extends continuously to the boundary of \( D \) and defines a homeomorphism of \( D \) with the closure of \( \Gamma_* \).
Proof. The map $\kappa(t, \cdot)$ is clearly injective on $D$, since it has a left inverse by equation (4.4.2.a), hence it is a conformal one to one mapping from $D$ to its image. Let us prove that this image is $\Gamma_t$. For any $r < 1$, it is easy to see that the set of $z \in \mathbb{C}$, with positive imaginary part, such that

$$\left| \frac{z - 1}{z + 1} e^{\frac{t}{2} z} \right| = r$$

is a closed Jordan curve $\gamma_{r,t}$ of the form $y = \pm k_{r,t}(x)$ for $x$ in some interval around 1. The image of the circle of radius $r$ by $\kappa(t, \cdot)$ is a curve included in $\gamma_{r,t}$, and since $\kappa(t, \cdot)$ is injective, it follows that the image of the circle of radius $r$ is exactly $\gamma_{r,t}$. Since the domain $\Gamma_t$ is the union of the point 1 and all the curves $\gamma_{r,t}$ for $r > 0$, we see that the image of $D$ is $\Gamma_t$. The second part of the lemma follows from Caratheodory theorem since $\Gamma_t$ is a Jordan domain (see, e.g., Pommenke [P]).

Remark that one has

$$\frac{(z - 1)}{(z + 1) e^{\frac{t}{2} z}} = \frac{1}{z + 1} e^{-\frac{t}{2} z}$$

hence the map

$$z \mapsto \frac{z - 1}{z + 1} e^{\frac{t}{2} z}$$

is a conformal one to one map from the region $-\Gamma_t$ onto the complement of $D$ in the Riemann sphere, and this maps extends homeomorphically to the boundary of $\Gamma_t$.

If $t < 4$, then the boundaries of the regions $\Gamma_t$ and $-\Gamma_t$ meet on the imaginary axis. The map $\kappa(t, \cdot)$ extends to a one to one conformal mapping of the interior of $\Gamma_t \cup (-\Gamma_t)$ onto the complement in $\mathbb{C}$ of the interval

$$\left\{ e^{\theta} \mid -\frac{1}{2} \sqrt{(4-t) t - \arccos \left( 1 - \frac{t}{2} \right)} \leq \theta \leq \frac{1}{2} \sqrt{(4-t) t + \arccos \left( 1 - \frac{t}{2} \right)} \right\}.$$

Note also that the points $\pm i \sqrt{4/4-t-1}$ are the only points where the derivative of

$$z \mapsto \frac{z - 1}{z + 1} e^{\frac{t}{2} z}$$

vanishes, so that the map $\kappa$ has an analytic continuation in a neighbourhood of $D \setminus \{ a_t, \tilde{a}_t \}$, where $a_t$, $\tilde{a}_t$ are the endpoints of the interval above.
When $t > 4$, the map $\kappa(t, \cdot)$ has an analytic continuation in a neighbourhood of $D$.

4.2.4. We can now describe precisely the measure $\nu'$.

**Proposition 10.** For every $t > 0$, the measure $\nu'$ is absolutely continuous with respect to the Haar measure on $\mathbb{T}$. Its support $I_t$ is equal to $\mathbb{T}$ for $t > 4$, and is equal to the interval

$$\{ e^{i\theta} \left| -\frac{1}{2} \sqrt{(4-t)t - \arccos \left( 1 - \frac{t}{2} \right)} \leq \theta \leq \frac{1}{2} \sqrt{(4-t)t + \arccos \left( 1 - \frac{t}{2} \right)} \right. \}$$

for $t \leq 4$, where $\arccos \in [0, \pi]$. The density is positive on the interior of the interval $I_t$ (except at $-1$ for $t = 4$) and is equal, at the point $\omega \in I_t$, to the real part of $\kappa(t, \omega)$, where $\kappa(t, \omega)$ is the only solution, with positive real part, of the equation

$$\frac{z - 1}{z + 1} e^{iz} = \omega.$$

**Proof.** We have seen in Lemma 12 that the function $\kappa(t, \cdot)$ extends continuously to the boundary of $D$, it follows that

$$\kappa(t, z) = \int_\mathbb{T} \frac{\omega + z}{\omega - z} \Re(\kappa(t, \omega)) \, d\omega,$$

where $d\omega$ is the normalized Haar measure on $\mathbb{T}$, and $\Re$ denotes the real part. The uniqueness of Herglotz representation shows that the measure $\nu'$ is equal to $\Re(\kappa(t, \omega)) \, d\omega$. The rest of the proposition follows easily from this. In particular, the assertions concerning the support of $\nu'$ are derived by looking at the image by

$$z \mapsto \frac{z - 1}{z + 1} e^{iz}$$

of the part of the boundary of $I_t$, which is on the imaginary axis. $lacksquare$

4.2.5. Let us now show similar considerations allow us to describe the distribution of the self-adjoint positive element $A_t^* A_t \in \mathcal{E}(t)$. We shall not need the result in the sequel, but it will yield the exact expression of the norm of $A_t$.

Let $\pi'$ be this distribution, this is a probability measure with compact support on $]0, +\infty[$. Let

$$\rho(t, z) = \int_{\mathbb{R}_+} \frac{x + z}{x - z} \pi'(dx)$$
for $z$ complex, outside the support of $\pi'$, then $\rho(t, z)$ has positive imaginary part for $z \in \mathbb{C}^+$, the open upper half plane in $\mathbb{C}$.

**Lemma 13.** For all $t > 0$ and $z \in \mathbb{C}^+$, one has

$$\rho(t, z) - \frac{1}{\rho(t, z)} + 1 = e^{-t\rho(t, z)} = z.$$ 

**Proof.** Let us define the process $R_t = e^{-tA_t^{-1}}(A_t^{-1})^*$. From the equations verified by $A$ and $A^{-1}$ we deduce, using Ito's formula, that $R$ satisfies

$$dR_t = ie^{-tA_t^{-1}}(dZ_t^* - dZ_t)(A_t^{-1})^*.$$ 

By Ito's formula again, it follows by induction on $n$ that

$$dR_t^n = \sum_{k=0}^{n-1} R_t^k dR_t R_t^{n-1-k} + 2 \sum_{k=1}^{n-1} k R_t^k \tau(R_t^{n-k}) \ dt.$$ 

Taking the trace on both sides of this equation, we obtain that the generating function $\gamma(t, z) = \sum_{n=1}^\infty z^n \tau(R_t^n)$ satisfies the equation

$$\frac{\partial \gamma(t, z)}{\partial t} = 2z \gamma(t, z) \frac{\partial \gamma(t, z)}{\partial z},$$ 

and this implies that

$$\frac{\gamma}{1 + \gamma} e^{-2\gamma} = z.$$ 

Hence the result since

$$\rho(t, z) = \frac{\gamma(t, e^{-t}z) - \frac{1}{2}}{2}.$$ 

For $t > 0$, let $y(t)$ be the smallest positive zero of the function $y \mapsto 1 - y^2 + 2y \cot(2ty)$. Let $\Omega_t$ be the region

$$\Omega_t = \{ x + iy \in \mathbb{C} \mid 0 < y < y(t); x^2 \leq 1 - y^2 + 2y \cot(2ty) \}.$$ 

The boundary of the region $\Omega_t$ is a closed Jordan curve whose intersection with $R$ is the interval

$$\left[ -\sqrt{1 + \frac{1}{t}}, \sqrt{1 + \frac{1}{t}} \right].$$ 

We have the analogue of Lemma 12.
Lemma 14. The map $\rho(t, \cdot)$ is a conformal one to one map from $\mathbb{C}^+$ to the region $\Omega_i$, which extends homeomorphically to the boundary $\mathbb{R} \cup \{ \infty \}$ of $\mathbb{C}^+$ in the Riemann sphere.

Proof. The proof of this lemma is similar to Lemma 12 and is left to the reader.

We can now describe the measure $\pi'$ explicitly.

Proposition 11. For every $t > 0$, the measure $\pi'$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$. Its support $J$ is equal to the interval

$$[(2t + 1 - 2 \sqrt{t(1+t)}), (2t + 1 + 2 \sqrt{t(1+t)}) e^{-t/2}]$$

The density is positive on the interior of the interval $J$, and is equal, at the point $x \in J$, to $(1/2\pi x) \Im(\rho(t, x))$ where $\rho(t, x)$ the only solution on $\partial \Omega_i$ of the equation

$$z - 1 z + 1 e^{-iz} = x$$

$(\Im$ denotes the imaginary part).

Proof. The proof uses the inversion formula for the Cauchy transform of a probability measure on the line, and is similar to the proof of Proposition 10.

From this we deduce that the norms of $A_t$ and $A_t^{-1}$ are both equal to

$$\varphi(t) = \sqrt{(2t + 1 + 2 \sqrt{t(1+t)}) e^{\sqrt{t(1+t)}}}$$

so that the spectrum of $A_t$ is included in the annulus $1/\varphi(t) \leq |z| \leq \varphi(t)$.

4.2.6. We shall now describe the integral transform $\mathcal{F}'$ which plays the role of the transform $\mathcal{F}$ in part 1. In order to motivate the formula for the integral transform we shall follow the remark at the end of Section 3.2.2. First we quote the following result from [Bi2], which is the multiplicative analogue of Theorem 6.

Theorem 8. Let $(A, \tau)$ be a $W^*$-probability space, $B \subset A$ be a von Neumann sub-algebra, and $U, V \in A$ such that $U$ and $V$ are unitary, with respective distributions $\mu$ and $\nu$, one has $U \in B$, and $V$ is free with $B$, then there exists a Feller Markov kernel $\mathcal{F} = k(\xi, dw)$ on $\mathbb{T} \times \mathbb{T}$ and an analytic function $F$, defined on $\mathbf{D}$, such that
(1) for any bounded Borel function \( f \) on \( T \),
\[
\tau(f(UV) \mid B) = \mathcal{N}(f(U))
\]
(2) for all \( z \in D \)
\[
\int_{\mathbb{T}} \frac{z \omega}{1 - z \omega} k(\xi, d\omega) = \frac{F(z) \xi}{1 - F(z) \xi}.
\]
Let us take for \( \mu \) the measure \( \delta_1 \) on \( \mathbb{T} \) and for \( \nu \) the measure \( \nu' \), then we obtain by Theorem 8 a kernel \( k'(\xi, d\omega) \) of probability measures on \( \mathbb{T} \). Let
\[
\chi(t, z) = \frac{\kappa(t, z) - 1}{\kappa(t, z) + 1} \quad \text{for} \quad z \in D.
\]
One has \( k'(1, d\omega) = \nu'(d\omega) \) by (1), hence
\[
\int_{\mathbb{T}} \frac{z \omega}{1 - z \omega} \nu'(d\omega) = \frac{F(z)}{1 - F(z)}
\]
and thus \( F(z) = \chi(t, z) \). It follows from (2) that the kernels satisfy
\[
\int_{\mathbb{T}} \frac{\omega + z}{\omega - z} k'(\xi, d\omega) = \frac{\xi + \chi(t, z)}{\xi - \chi(t, z)}
\]
for \( z \in D \).

We shall now define a new family of regions in \( \mathbb{C} \). For \( t > 0 \), let \( \Sigma_t \), be the image, by the Möbius transformation
\[
z \mapsto \frac{z - 1}{z + 1}
\]
of the complement of \( \mathbb{F} \) in the Riemann sphere. Let us recall the discussion of the regions \( I_t \) in 4.2.3. Let \( t < 4 \), then the boundary of the compact set \( \mathbb{F} \) is the union of the curves \( y = \pm k(x); 0 \leq x \leq x(t) \), and their symmetric with respect to the imaginary axis. These curves are exactly the images of the interval \( I_t \), support of \( \nu' \), by the maps \( \kappa(t, \cdot) \) and \( -\kappa(t, \cdot) \), so they are analytic except perhaps at the intersection with the imaginary axis. It follows that for \( t < 4 \), the region \( \Sigma_t \) is bounded, simply connected, with a piecewise analytic, \( C^1 \) boundary. Furthermore, this region is invariant under conjugation and under the map \( z \mapsto 1/z \), and its boundary consists in the images of \( I_t \) by the maps \( \chi(t, \cdot) \) and \( 1/\chi(t, \cdot) \). The map \( \chi(t, \cdot) \) sends the interior of \( I_t \) in the disk \( D \). The intersection of \( \Sigma_t \), with \( \mathbb{T} \) consists in the two points in \( \mathbb{T} \) with real part \( 1 - t/2 \).
Similar considerations show that for \( t > 4 \), the region \( \Sigma_t \) is bounded by two analytic disjoint curves, which are the images of \( T \) by the maps \( \gamma(t, \cdot) \) and \( 1/\gamma(t, \cdot) \), respectively. These curves are simple, and \( |\gamma(t, \omega)| < 1 \) for \( \omega \in T \). In particular, the region \( \Sigma_t \) has the conformal type of an annulus.

Finally, for \( t = 4 \), the images of \( I_t \) by the two maps \( \gamma(t, \cdot) \) and \( 1/\gamma(t, \cdot) \) have a common point \(-1\). The region \( \Sigma_t \) is the region bounded by these two curves, so its boundary is a curve with one double point. The region \( \Sigma_4 \) is simply connected, but the complement of \( \Sigma_4 \) has two connected components.

The regions \( \Sigma_t \), for various values of \( t \), are pictured at the end of the paper.

**Proposition 12.** For every \( t > 0 \) and \( \xi \in \Sigma_t \cap \mathbb{T} \), the measure \( k'(\xi, d\omega) \) is absolutely continuous with respect to the measure \( v' \), with density

\[
\frac{|1 - \gamma(t, \omega)|^2}{(\xi - \gamma(t, \omega))(\xi^{-1} - \bar{\gamma}(t, \omega))}
\]

**Proof.** One has, for all \( z \in \mathbb{D} \), and \( \xi \in T \),

\[
\frac{\xi + \gamma(t, z)}{\xi - \gamma(t, z)} = \frac{\xi - 1 + \nu(t, z)}{1 + \nu(t, z)} \frac{1}{\xi + 1} \quad \text{for} \quad \xi \neq -1
\]

\[
= \frac{1}{\nu(t, z)} \quad \text{for} \quad \xi = -1
\]

and \((\xi - 1)/(\xi + 1)\) is a purely imaginary number. The Möbius transformation

\[
\zeta \mapsto \frac{\zeta - 1 + \nu}{1 + \zeta \frac{\zeta - 1}{\zeta + 1}}
\]

maps the right half plane into itself. If \( \xi \in \Sigma_t \cap \mathbb{T} \), the functions

\[
z \mapsto \frac{\xi + \gamma(t, z)}{\xi - \gamma(t, z)}
\]

remains bounded in \( \mathbb{D} \), and is a homeomorphism of \( \mathbb{D} \) with a bounded region in the half plane \( \Re(z) > 0 \), hence by the same argument as in the
proof of Proposition 10, we have that $k(\zeta, d\omega)$ is absolutely continuous, with respect to $d\omega$, with density
\[ R \frac{1 - |\zeta(t, \omega)|}{(\zeta - \zeta(t, \omega))(\zeta^{-1} - \zeta(t, \omega))}. \]
Since the density of $v = k(1, d\omega)$, is
\[ \frac{1 - |\zeta(t, \omega)|^2}{|1 - \zeta(t, \omega)|^2}, \]
we see that $k(\zeta, d\omega)$ is absolutely continuous with respect to $v$, with the required density.

By analogy with Section 3.2.2 we now define an integral transform. We shall show, in Section 4.4, that it plays the role of the transform $\mathcal{F}$, in the free analogue of the Hall transform.

**Proposition 13.** Let $f \in L^2(v)$, then the integral
\[ \mathcal{G}f(\zeta) = \int f(\omega) \frac{1 - |\zeta(t, \omega)|}{(\zeta - \zeta(t, \omega))(\zeta^{-1} - \zeta(t, \omega))} v(d\omega) \]
converges for all $\zeta \in \Sigma$, and defines an analytic function there.

**Proof.** The boundary of $\Sigma$, is exactly the set of values taken by the functions $\zeta(t, \cdot)$ and $1/\zeta(t, \cdot)$ in the interval $I$, support of the measure $v$, it follows that the function
\[ \omega \mapsto \frac{1 - |\zeta(t, \omega)|^2}{(\zeta - \zeta(t, \omega))(\zeta^{-1} - \zeta(t, z))} \]
is bounded on $I$, if $\zeta \in \Sigma$, and the result follows by standard arguments.

**4.2.7.** We shall compare the range of the integral transform $\mathcal{G}$ with the Hardy space of the region $\Sigma$. Since the boundary $\partial \Sigma$, is piecewise analytic, one can consider $\sigma$, the arc length measure on $\partial \Sigma$, and for any $f \in L^2(\sigma)$, its Cauchy integral
\[ \mathcal{C}f(\zeta) = \frac{1}{2\pi i} \int_{\partial \Sigma} \frac{f(z)}{z - \zeta} dz \]
defines an analytic function of $\zeta$ in $\Sigma$. It is well known that the map $f \mapsto \mathcal{C}f$ is bounded from $L^2(\sigma)$ to the Hardy space $H^2(\Sigma)$ (see, e.g., [CM] or [Da]).
Lemma 15. Let $f \in L^2(\nu)$, then for any $\zeta \in \Sigma$, one has
\[ G f(\zeta) = \frac{1}{2\pi i} \int_{\gamma_{\zeta}} f(z e^{\frac{t}{z-1} + 1}) \left( \frac{zt}{(z-1)^2} + 1 \right) \frac{dz}{z-\zeta} \]
where the orientation of the curve $\gamma_{\zeta}$ is such that $\Sigma_{\zeta}$ is on the left.

Proof. Let $h_t = \frac{1}{2} \sqrt{(t-4) + \arccos(1-(t/2))}$, if $t \leq 4$, $h_t = \pi$ if $t > 4$, then we have
\[ G f(\zeta) = \frac{1}{2\pi} \int_{-h_t}^{h_t} f(e^{i\theta}) \left( \frac{1}{2} \left( \frac{\zeta + \chi(t, e^{i\theta})}{\zeta - \chi(t, e^{i\theta})} + \frac{\zeta^{-1} + \chi(t, e^{i\theta})^{-1}}{\zeta^{-1} - \chi(t, e^{i\theta})^{-1}} \right) \right) d\theta \]
\[ = \frac{1}{2\pi} \int_{-h_t}^{h_t} f(e^{i\theta}) \frac{\chi(t, e^{i\theta})}{\zeta - \chi(t, e^{i\theta})} d\theta \]
\[ - \frac{1}{2\pi} \int_{-h_t}^{h_t} f(e^{i\theta}) \frac{\chi(t, e^{i\theta})^{-1}}{\zeta - \chi(t, e^{i\theta})^{-1}} d\theta. \]
The map $\theta \mapsto \chi(t, e^{i\theta})$, $\theta \in [-h_t, h_t]$ is a parametrisation of the part of the curve $\Sigma$, which is in $D$, while
\[ \theta \mapsto \frac{1}{\chi(t, e^{i\theta})} \theta \in [-h_t, h_t] \]
is a parametrisation of the part of the curve $\Sigma_t$ which is outside $D$. The result follows by a change of variable, using the fact that
\[ \chi(t, e^{i\theta}) e^{\frac{1}{2} \frac{t}{1 - \chi(t, e^{i\theta})}} e^{i\theta} = e^{i\theta} \quad \text{in } [-h_t, h_t] \]
to compute the Jacobian.

Lemma 16. Let $f \in L^2(\nu)$, then the function
\[ Z f(z) = f(z e^{\frac{t}{z-1} + 1}) \left( 1 + \frac{zt}{(z-1)^2} \right) \]
on $\partial \Sigma_t$ is in $L^2(\sigma)$, furthermore, if $t \neq 4$, there are constants $c$, $C$ such that
\[ c \| f \|_{L^2(\nu)} \leq \| Z f \|_{L^2(\sigma)} \leq C \| f \|_{L^2(\nu)} \]
for all $f \in L^2(\nu)$. If $t = 4$, there is a constant $C$ such that
\[ \| Z f \|_{L^2(\sigma)} \leq C \| f \|_{L^2(\nu)}, \]
for all $f \in L^2(\nu)$.
Proof. The image of the measure \( \nu' \) by the map \( \varphi(t, \cdot) \) is absolutely continuous with respect to \( \sigma \), with density

\[
\left| \frac{1-|z|^2}{|1-z|^2} \left( 1 + \frac{zt}{(z-1)^2} \right) \right|
\]

on \( \partial \Sigma_i \cap D \), and similarly the image of the measure \( \nu' \) by the map \( \varphi(t, \cdot)^{-1} \) is absolutely continuous with respect to \( \sigma \), with density

\[
\left| \frac{1-|z|^2}{|1-z|^2} \left( 1 + \frac{zt}{(z-1)^2} \right) \right|
\]

on \( \partial \Sigma_i \cap (C \setminus D) \). Let \( f \in L^2(\nu') \) then

\[
\|f\|_{L^2(\nu')} = \frac{1}{2} \int_{\partial \Sigma_i} |f(z)|^2 \left| \frac{1-|z|^2}{|1-z|^2} \left( 1 + \frac{zt}{(z-1)^2} \right)^{-1} \right| \sigma(dz)
\]

\[
= \frac{1}{2} \int_{\partial \Sigma_i} |Zf(z)|^2 \left| \frac{1-|z|^2}{|1-z|^2} \left( 1 + \frac{zt}{(z-1)^2} \right)^{-1} \right| \sigma(dz).
\]

If \( t > 4 \) the numerator and denominator of the function

\[
\left| \frac{1-|z|^2}{|1-z|^2} \left( 1 + \frac{zt}{(z-1)^2} \right)^{-1} \right|
\]

do not vanish on \( \partial \Sigma_i \), so that this quantity is bounded above and below by some constants \( 0 < c < C < \infty \), and we get the result. If \( t < 4 \), one has

\[
1 + \frac{zt}{(z-1)^2} = \frac{(z-b_i)(z-\tilde{b}_i)}{(z-1)^2},
\]

where \( b_i \) and \( \tilde{b}_i \) are the intersections of \( \partial \Sigma_i \) with \( \mathbb{T} \). Since the curve \( \partial \Sigma_i \) intersects \( \mathbb{T} \) orthogonally, as can be checked, we see that \( (1-|z|)/(|z-b_i|) \) is bounded above and below for \( z \in \partial \Sigma_i \), and similarly for \( \tilde{b}_i \), so that

\[
0 < c < \left| \frac{1-|z|^2}{|1-z|^2} \left( 1 + \frac{zt}{(z-1)^2} \right)^{-1} \right| < C < \infty
\]

for \( z \in \partial \Sigma_i \), for some constants \( c \) and \( C \). For \( t = 4 \), one only has an inequality

\[
0 < c < \left| \frac{1-|z|^2}{|1-z|^2} \left( 1 + \frac{zt}{(z-1)^2} \right)^{-1} \right|.
\]
From Lemmas 15 and 16 we infer that the map \( \mathcal{G} \) is a bounded map from \( L^2(\nu') \) to the Hardy space \( H^2(\Sigma) \).

**Lemma 17.** The map \( \mathcal{G} : L^2(\nu') \to H^2(\Sigma) \) is injective, for all \( t \neq 4 \).

**Proof.** We shall distinguish the cases \( t > 4 \) and \( t < 4 \), and give quite distinct arguments for these two cases. Unfortunately, none of these arguments seems to work for \( t = 4 \), so we shall leave this case aside. We start with the easiest case, which is \( t > 4 \). In this case, the function \( \gamma(t, \cdot) \) takes values in \( D \), so that

\[
\mathbb{R} \left( \frac{\zeta + \gamma(t, \omega)}{\zeta - \gamma(t, \omega)} \right) = 1 + \sum_{n=1}^{\infty} \zeta^{-n} \gamma(t, \omega)^n + \zeta^n \gamma''(t, \omega)
\]

and the sum is uniformly convergent for \( \zeta, \omega \in \mathbb{T} \). The measure \( \nu' \) is equivalent with the Haar measure, so that they have the same \( L^2 \) space. If \( f \in L^2(\omega) \) is such that \( \mathcal{G}f \equiv 0 \) on \( \mathbb{T} \), then in particular all its Fourier coefficients are zero, so that \( \int_{\mathbb{T}} f(\omega) \mathcal{G}(\omega) d\omega = \int_{\mathbb{T}} f(\omega) \mathcal{G}''(t, \omega) d\omega = 0 \) for all \( n \geq 0 \). The polynomials in \( \gamma(t, \cdot) \) and \( \mathcal{G}(t, \cdot) \) form a total space in \( L^2(\mathbb{T}, d\omega) \) (this follows from the fact that \( \gamma(t, \cdot) \) is continuous, separates the points of \( \mathbb{T} \) and the Stone-Weierstrass Theorem), so that necessarily \( f \equiv 0 \).

Let us now consider the case \( t < 4 \). Suppose that \( \mathcal{G}f \equiv 0 \), this means that the Cauchy transform of the function \( Zf(z) \) (cf. Lemma 16) is identically zero. By Calderon’s Theorem (see again [CM] or [Da]) this implies that \( Zf \) is the boundary value of the function

\[
\zeta \mapsto \frac{1}{2\pi i} \int_{\mathbb{T}} f(z \frac{\zeta + 1}{\zeta - 1}) \left( \frac{z^t}{(z - 1)^t} + 1 \right) \frac{dz}{z - \zeta} \quad \zeta \in \mathbb{C} \setminus \bar{\mathbb{E}},
\]

which vanishes at infinity and is in the Hardy space of the region \( \mathbb{C} \setminus \bar{\mathbb{E}} \). Let us define four maps \( h_1, h_2, h_3, h_4 \), such that

\[
\begin{align*}
(1) \quad h_1(z) &= z^{\frac{1}{2}} (1 + i), \\
(2) \quad h_2(z) &= 2 - \frac{1}{z + 1} \frac{a_i + 1}{a_i - 1}, \\
(3) \quad h_3(z) &= \frac{1}{z + \sqrt{z^2 - 4}},
\end{align*}
\]

where \( a_i = h_1(b_i) \), \( b_i \) being as in the proof of Lemma 16. This function is a one-to-one conformal map from \( (\mathbb{C} \cup \{ \infty \}) \setminus I \), onto \( (\mathbb{C} \cup \{ \infty \}) \setminus [-2, 2] \).
where the square root is chosen so that this defines a one-to-one conformal transformation of \((\mathbb{C} \cup \{\infty\}) \setminus [-2, 2]\) onto \((\mathbb{C} \cup \{\infty\}) \setminus \mathcal{D}\)

\[
(4) \quad h_4(z) = \frac{1 + z\bar{c}_4}{z - c_4},
\]

where \(c_4 = h_3 \circ h_2(\infty)\).

Then one can check that the map \(\Phi = h_4 \circ h_3 \circ h_2 \circ h_1\) defines a conformal one-to-one map from \((\mathbb{C} \cup \{\infty\}) \setminus \Sigma_t\) onto \((\mathbb{C} \cup \{\infty\}) \setminus \mathcal{D}\) such that (i) \(\Phi(\infty) = \infty\), (ii) \(\Phi'\) and \(1/\Phi'\) are bounded on \(\mathbb{C} \setminus \Sigma_t\), (iii) \(\Phi\) has an analytic continuation to a neighbourhood of \((\mathbb{C} \cup \{\infty\}) \setminus \Sigma_t\), (iv) \(\Phi(\bar{z}) = \bar{\Phi}(z)\) for all \(z\). One deduces that \(\Phi \mapsto \Phi^{-1}\) is a bounded map, with bounded inverse from \(\mathcal{H}^2((\mathbb{C} \cup \{\infty\}) \setminus \Sigma_t)\) onto \(\mathcal{H}^2((\mathbb{C} \cup \{\infty\}) \setminus \mathcal{D})\), so that \(Zf \mapsto Zf \circ \Phi^{-1}(1/z)\) is the boundary value of some function in \(\mathcal{H}^2((\mathbb{C} \cup \{\infty\}) \setminus \mathcal{D})\). The map \(z \mapsto Zf \circ \Phi^{-1}(1/z)\) is the boundary value of some function in \(\mathcal{H}^2(\mathcal{D})\) which vanishes at zero. Let \(h_3\) be the Möbius transformation which maps \(\mathcal{D}\) into itself and such that the images of \(\Phi(b_t)\) and \(\Phi(b_t)\) are 1 and \(-1\), and \(\Psi(z) = h_3(1/\Phi(z))\), then one has \(\Psi(1/z) = \Psi(z)\) for all \(z \in (\mathbb{C} \cup \{\infty\}) \setminus \Sigma_t\), and \(\Psi(b_t) = -1\), \(\Psi(b_t) = 1\). Since one has

\[
\left(1 + \frac{t\Psi^{-1}(z)}{(\Psi^{-1}(z) - 1)^2}\right) = \frac{\left(\Psi^{-1}(z) - b_t\right)(\Psi^{-1}(z) - \bar{b}_t)}{(\Psi^{-1}(z) - 1)^2},
\]

the function

\[
\left(1 + \frac{t\Psi^{-1}(z)}{(\Psi^{-1}(z) - 1)^2}\right)^{-1}(1 - z^2)
\]

is holomorphic and bounded on \(\mathcal{D}\), so that the function

\[
g(z) = Zf \circ \Psi^{-1}(z) \left(1 + \frac{t\Psi^{-1}(z)}{(\Psi^{-1}(z) - 1)^2}\right)^{-1}(1 - z^2)
\]

is the boundary value of a function in \(\mathcal{H}^2(\mathcal{D})\) which vanishes at \(\Psi(\infty)\). By the definition of this function, and the properties of \(\Psi\) we see that it satisfies \(g(e^{-\theta}) = -e^{-2\theta}g(e^{\theta})\) on \(\mathbb{T}\). Since \(g(e^{\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}\), \(a_n = 0\) for \(n \neq 0, 2\), and \(a_0 = -a_2\). Since \(g\) is the boundary value of a function which vanishes at some point in \(\mathcal{D}\) we see that \(g \equiv 0\), hence that \(f \equiv 0\).

We arrive now at the explicit description of the range of the transform \(\Psi'\).

**Proposition 14.** For all \(t \neq 4\), the function

\[
K(z, \zeta) = \frac{1 - \Re(t, \omega)^2}{(z - \Re(t, \omega))(\zeta - \Re(t, \omega))} \frac{1 - \Re(t, \omega)^2}{(\zeta - \Re(t, \omega))(\zeta - \Re(t, \omega))} v'(d\omega)
\]

where the square root is chosen so that this defines a one-to-one conformal transformation of \((\mathbb{C} \cup \{\infty\}) \setminus [-2, 2]\) onto \((\mathbb{C} \cup \{\infty\}) \setminus \mathcal{D}\). **Theorem.**
is the reproducing kernel of a Hilbert space of analytic functions on $\Sigma_t$. The map $\mathcal{G}'$ is a unitary isomorphism of Hilbert spaces between $L^2(v')$ and this Hilbert space.

Proof. The function $K(z, \zeta)$ is a sesqui-analytic kernel, which is positive definite, by construction. The first part of the claim follows from general results on such kernels (see, e.g., [Do]). The second part is a direct consequence of the fact that the map $\mathcal{G}'$ is injective.

We shall denote by $\mathcal{A}'$ the Hilbert space of analytic functions given by the reproducing kernel of Proposition 14. It plays the same role, with respect to the spaces $\mathcal{A}'$ as the Hardy space $H^2_{\nu}$ with respect to the spaces $H^2_{\nu}$. Note that, by Lemma 16, one has $\mathcal{A}' \subset H^2_{\nu}(\Sigma_t)$. We do not know whether the spaces actually coincide, but one can show that the scalar product on $\mathcal{A}'$ is not the same, indeed, there is no measure on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} F(z) G(z) \, dm(z) = \langle F, G \rangle_{\mathcal{A}'}$$

for all $F, G$ entire functions on $\mathbb{C}$.

4.3. The Analogue of Gross–Malliavin Theorem

**Theorem 9.** (1) The map $F \mapsto F\mathcal{A}_t$ from the space of holomorphic functions in $\mathbb{C}\setminus\{0\}$ to $\mathcal{E}_{\nu}(t)$ extends to an isometry from $\mathcal{A}'$ onto $L^2_{\nu}(A_t)$.

(2) The free Segal–Bargmann transform on $L^2(\mathcal{Y}\mathcal{E}(t), \tau)$ sends $L^2(U_t, \tau)$ onto $L^2_{\nu}(A_t, \tau)$.

(3) There is a commutative diagram

$$\begin{array}{ccc}
L^2(v') & \xrightarrow{U_t} & L^2(U_t, \tau) \\
\mathcal{G}' \downarrow & & \downarrow \mathcal{G} \\
\mathcal{A}' & \xrightarrow{A_t} & L^2_{\nu}(A_t, \tau)
\end{array}$$

where all maps are unitary isomorphisms.

The proof of Theorem 9 occupies the rest of this section.

Let $P_n(t, x)$ be the polynomials determined by the generating series

$$\sum_{n=0}^{\infty} u^n P_n(t, x) = \frac{1}{1 - \frac{u}{1 + u} e^{\frac{t}{1 + 2u}} x}.$$ 

**Lemma 18.** For all $n \geq 0$, $t > 0$, one has

$$\mathcal{G}'(P_n(t, x))(\zeta) = \zeta^n$$

$$\mathcal{G}'(P_n(t, \bar{x}))(\xi) = \zeta^{-n}.$$
Proof. Using computations analogous as the ones after Theorem 7, we see that the kernels $k'(\xi, d\omega)$ satisfy

$$\int_\mathbb{T} \frac{u\omega}{1 - u\omega} k'(\xi, d\omega) = \frac{\xi \tau(t, u)}{1 - \xi \tau(t, u)},$$

where $\tau(t, \cdot)$ is the inverse function of $z \mapsto (z/(1 + z)) e^{\frac{t}{2} (1 + 2z)}$ near the origin, so that

$$\int_\mathbb{T} \frac{z}{1 + z} e^{\frac{t}{2} (1 + 2z)} \frac{k'(\xi, d\omega)}{1 - \xi \omega} = \frac{z^2}{1 - z^2}.$$ 

The first result follows by expanding both sides in power series of $z$ near zero, and identifying coefficients, then making analytic continuation to $\mathcal{S}$, and the second result is similar.

Note that the preceding lemma implies that the range of the integral transform $\mathcal{G}$ contains all Laurent polynomials, hence that the space $\mathcal{O}$ is dense in $H^2(\mathcal{S})$.

**Lemma 19.** For all $n \geq 1$, one has

$$P_n(t, U_t) = P_n(0, 1) + \sum_{k=0}^{n-1} P_k(s, U_s) dX P_{n-k}(s, U_s)$$

Proof. Let $V_t = e^{t^2} U_t$. One has $P_n(t, U_t) = R_n(t, V_t)$ where the polynomials $R_n(t, \cdot)$ have the generating series

$$\sum_{n=0}^{\infty} z^n R_n(t, x) = \frac{1}{1 - \frac{z}{1 + z} e^{tx}}.$$

Let us apply Ito formula to the expression

$$\frac{z}{1 + z} e^{t\omega} V_i = \frac{h(t, z) V_i}{1 - h(t, z) V_i}.$$
We obtain

\[
\frac{h(t, z)}{1 - h(t, z)} V_t = \sum_{n=1}^{\infty} h(t, z)^n V^n_t
\]

\[
= z + i \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} h(s, z)^n V^k_s dX_s V^{n-k}_s \right)
\]

\[
- \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} h(s, z)^n k V^k_s \tau(V^{n-k}_s) ds
\]

\[
+ \sum_{n=1}^{\infty} \left[ n h(s, z)^n \frac{\partial h(s, z)}{\partial s} V^n_s ds \right]
\]

\[
\quad = z + i \int_0^t \left( \sum_{n=0}^{\infty} h(s, z)^n V^n_s \right) dX_s \left( \sum_{n=1}^{\infty} h(s, z)^n V^n_s \right)
\]

\[
- \int_0^t \sum_{n=0}^{\infty} n h(s, z)^n V^n_s \eta(s, h(s, z)) ds
\]

\[
+ \sum_{n=1}^{\infty} \int_0^t n h(s, z)^n \frac{\partial h(s, z)}{\partial s} V^n_s ds
\]

\[
\quad \text{where } \eta \text{ is as in the proof of Lemma 11}
\]

\[
= z + i \int_0^t \left( \sum_{n=0}^{\infty} h(s, z)^n V^n_s \right) dX_s \left( \sum_{n=1}^{\infty} h(s, z)^n V^n_s \right)
\]

\[
\quad \text{since } \eta(s, h(s, z)) = z \quad \text{and} \quad \frac{\partial h(s, z)}{\partial s} = zh(s, z).
\]

The formula follows by expanding in power series of \(z\) and by identifying the coefficient of \(z^n\) in the two sides of the formula.

**Lemma 20.** For all \(n \geq 0\) and \(t > 0\) one has

\[
\mathcal{F}^{-1}(A^n_t) = P(t, U_t) \quad \text{and} \quad \mathcal{F}^{-1}(A^{-n}_t) = P(t, U^*_t)
\]

**Proof.** Let \(M_n(t) = \mathcal{F}^{-1}(A^n_t)\), Ito’s formula for the circular brownian motion \(Z_t\), yields easily

\[
A^n_t = 1 + i \sum_{k=0}^{n-1} \int_0^t A^n_k dZ_s A^{n-k}_s.
\]
Fig. 1. The region \( \Omega_t \).
Fig. 2. The region $\Sigma_t$. 

\begin{align*}
\text{Fig. } 2; \quad \text{The region } \Sigma_t. 
\end{align*}
By Proposition 9 one has

\[ M_n(t) = 1 + i \sum_{k=0}^{n-1} \int_0^t M_k(s) \, dX_s M_{n-k}(s). \]

Thus, the \( M_n \) satisfy the same equations as \( P_n(t, U_\tau) \), by Lemma 19. Applying If's formula to the product \( L_n(t) = e^{-U_\tau} M_n(t) \), one obtains

\[ L_n(t) = 1 + i \sum_{k=1}^{n-1} \int_0^t L_k(s) \, dX_s e^{U_\tau} L_{n-k}(s) \]

\[ + \sum_{k=1}^{n-1} \int_0^t L_k(s) \tau(e^{U_\tau} L_{n-k}(s)) \, ds \]

so that the sequence of processes \( L_n \) can be computed by induction on \( n \).

It follows that \( M_n(t) = P_n(t, U_\tau) \) for all \( n \geq 0 \) and \( t > 0 \). The formula for \( \mathcal{F}^{-1}(A_{\gamma}^{-1}) \) is obtained by similar arguments.

**End of proof of Theorem 9.** We start with (3). By Lemma 18 and 19, we see that the two sides of the diagram coincide on polynomials in \( \zeta \) and \( \zeta^{-1} \). Since polynomials are dense in \( L^2(\nu') \) and the maps are isometries, we get the result. Parts (1) and (2) of the theorem follow from (3).

**REFERENCES**


