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Journal of Applied Logic 2 (2004) 153–168
JOURNAL OF APPLIED LOGIC
www.elsevier.com/locate/jal

Action negation and alternative reductions for dynamic deontic logics

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Abstract

Dynamic deontic logics reduce normative assertions about explicit complex actions to standard dynamic logic assertions about the relation between complex actions and violation conditions. We address two general, but related problems in this field. The first is to find a formalization of the notion of 'action negation' that (1) has an intuitive interpretation as an action forming combinator and (2) does not impose restrictions on the use of other relevant action combinators such as sequence and iteration, and (3) has a meaningful interpretation in the normative context. The second problem we address concerns the reduction from deontic assertions to dynamic logic assertions. Our first point is that we want this reduction to obey the free-choice semantics for norms. For ought-to-be deontic logics it is generally accepted that the free-choice semantics is counter-intuitive. But for dynamic deontic logics we actually consider it a viable, if not, the better alternative. Our second concern with the reduction is that we want it to be more liberal than the ones that were proposed before in the literature. For instance, Meyer's reduction does not leave room for action whose normative status is neither permitted nor forbidden. We test the logics we define in this paper against a set of minimal logic requirements.

Keywords: Deontic logic; Dynamic logic; Action theory; Action negation and refraining

1. Introduction

The central question for research on dynamic deontic logics is: what is the logical structure of ought-to-do deontic assertions in terms of the structure of complex actions? Thus, whereas all other ought-to-do deontic logics study the logic of action related deontic assertions over the *standard logic connectives* (disjunction: $\vee$, conjunction: $\wedge$, negation: $\neg$),

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1570-8683/$ – see front matter
doi:10.1016/j.jal.2004.01.007

dynamic deontic logics study the logic of deontic assertions over *action combinators* (choice: $\cup$, concurrent execution: $\cap$, action negation: $-$, sequence: ;, iteration: *, converse: $^{\leftarrow}$). Therefore, dynamic deontic logics take a unique position in the ought-to-do deontic logic landscape.[1] Before we present the problems of dynamic deontic logic we address in this paper, we briefly sketch the historical context that led to the definition of this type of logics.

Among the most well studied ought-to-do deontic logics are the STIT (Seeing To It That) logics [1–4], which have their roots in older work on BIAT (Bringing It About That) logics [5–7]. These logics study expressions of the form $XE_i\,p$, with X for either *permission*, *prohibition* or *obligation*, and $E_i\,p$ representing 'agent i sees to it that p'. Terms $E_i\,p$, are studied as operators in their own right and are referred to as 'action modalities'. This terminology might be considered slightly inaccurate, because $E_i\,p$ does not explicitly refer to an action by which agent i realizes condition p; it only refers to an agent, namely i, and the *post*-condition p of some anonymous action.[2] Therefore, deontic logics over assertions of the form $E_i\,p$ are not concerned with action combinators, but with the standard logic connectives that express the logical structure of post-conditions p.

One of the problems with STIT-type logics (and with other ought-to-do deontic logics) concerns the question of whether or not we can say that an agent fulfills its obligation if another agent (or 'nature') brings about the required condition. Of course, it is very hard to claim something general about this issue; sometimes it is required that the agent brings about the condition personally, and sometimes it is allowed that another agent or nature brings it about. But then, one would like to be able to express the difference between such cases. And this is where deontic operators that refer not so much to the post-condition of the action as to 'the action itself' can be of help. Von Wright argued more or less for the same point, on grounds of a related observation. He observed [8,9] that we sometimes need to express the distinction between 'seeing to it that p' and 'preventing that it occurs that $\neg p$'. It might for instance be obligatory to close the window, but forbidden to prevent it from opening. He argues that this distinction calls for 'a symbolism for schematic action sentences and rules for how to handle them' [9]. It is not entirely clear whether or not he alludes to a language of explicit actions here. But, obviously, such a language, in combination with an appropriate theory of deontic assertions over them, could alleviate his problem. Recently, also Sergot and Richards [3] discussed the pros (and cons) of an explicit action language as a basis for an ought-to-do deontic logic.

Observations similar to those made above, inspired Castañeda to divide ought-to-do norms into those pertaining directly to 'practitions', and those pertaining to 'action propositions', being conditions concerning the situation before an action takes place and/or after an action has taken place. A practition is thus, what we above have called 'the action itself'. Castañeda [10–12] observed that many anomalies of deontic reasoning are not inherited by logics that interpret normative assertions as taking practitions as an argument. It was explained by Hilpinen [13] that the group of dynamic deontic logics [14–16] are deontic 'practition logics' in the sense described by Castañeda. As Hilpinen argues, under

[1] Note that another interesting question is what the logics of desire, intention, goals, etc. over action combinators would be.

[2] This difference is also known as the difference between 'endogenous' and 'exogenous' logics of action.

the practition view it is most natural to see an action as a ‘state transformer’ bringing about a change in the world. This corresponds with the way actions are interpreted in dynamic logics, namely, as a directed relation from possible execution states to possible result states.

The original system of dynamic deontic logic as defined by Meyer [14], was criticized by van der Meyden [15] for the fact that it reduces deontic operators over complex action to violation conditions that can only occur in resulting states. That is, if O, P and F stand for Obligation, Permission and Prohibition (‘Forbiddeness’) respectively, for Meyer’s system it holds that any of the formulas $P(a;b) \wedge \neg P(a)$, $F(a) \wedge \neg F(a;b)$, and $O(a;b) \wedge \neg O(a)$, with a and b atomic actions, is consistent. Thus, it can be permitted to perform the action sequence $a;b$, while at the same time a permission to perform the single action a is lacking. It may even be the case that a is explicitly forbidden.[3] As in [17] we adopt the term ‘goal norms’ for this type of norms, because their violation conditions only concern the ‘goal states’ of complex actions. However, we do not, like van der Meyden, deem goal norms to be intuitively dissatisfying. They are simply a semantic alternative to, what we call, ‘process norms’ [17]. For process norms we take the alternative position, that is, that each of the formulas $P(a;b) \wedge \neg P(a)$, $F(a) \wedge \neg F(a;b)$, and $O(a;b) \wedge \neg O(a)$ is inconsistent. So, a permission to perform a sequential compound action requires permissions for all sub-actions. For this type of norms, violations may occur at any point during the process of action performance. In [16] we showed how to define a reduction that translates process norm expressions to mu-calculus formulas expressing which violation conditions hold for the states that are attained *during* execution of a complex action. These reductions are considerably more complex than the reductions for goal-type norms.

After this brief sketch of the history of dynamic deontic logics, it is time to set out the problems to be discussed in this paper. We restrict ourselves to goal type norms. Our first problem concerns the notion of ‘action negation’. We want a formalization of this notion that (1) has an intuitive interpretation as an action forming combinator and (2) does not impose restrictions on the use of other relevant action combinators such as sequence and iteration, and (3) has a meaningful interpretation in the normative context. Meyer’s action negation [14] does not meet all these requirements. In particular, Meyer’s system does not include iteration. Taking seriously the words of Segerberg [18] and von Wright [9,19] that any ought-to-do deontic logic should be preceded by and based on a sound theory of action, in this paper we will first deal with this problem of action negation. We define a series of dynamic logics with a new type of action negation that has a clear and intuitive semantics as a combinator that reflects the ‘act’ of ‘refraining’ from an action. The strongest dynamic logic includes all relevant action combinators, and the weakest includes only the action negation and non-deterministic choice. For each of these separate logics the action negation has a slightly different relational interpretation: its definition depends on what other action combinators, besides action negation, are in the action language. For this reason we call the action negation *relativized*.

The second problem we address concerns the reduction from deontic assertions to dynamic logic assertions. Our first point is that we want this reduction to obey the free-choice semantics for norms, because we consider the free-choice semantics a viable, if not, bet-

[3] Note that we do not assume that $P(\alpha) \leftrightarrow \neg F(\alpha)$, but only that $\neg(P(\alpha) \wedge F(\alpha))$.

ter alternative for dynamic deontic logics. Our second concern with the reduction is that we want it to be more liberal than the ones that were proposed before in the literature. For instance, Meyer's reduction does not leave room for action whose normative status is neither permitted nor forbidden. We define an alternative for Meyer's deontic-to-dynamic reduction that not only respects the free-choice semantics for norms, but that also avoids the strong inter-definabilities $P(\alpha) = \neg F(\alpha)$ and $O(\alpha) = F(-\alpha)$ (with '$-$' denoting action negation). $P(\alpha) = \neg F(\alpha)$ is avoided in order to leave room for actions that are neither permitted nor forbidden. And following Hintikka [20], also $O(\alpha) = F(-\alpha)$ is avoided, because $O(\alpha) \leftarrow F(-\alpha)$ is often too strong. We show that the reduction obeys a set of intuitive logic requirements that hold for deontic dynamic logics.

In Section 2 we recall the definition of dynamic logic. In Section 3 we define extensions and weakenings of dynamic logic that encompass a relativized notion of action negation. In Section 4, we discuss the problems of free-choice and inter-definability of operators. In Section 5 we mention some minimal deontic requirements for the deontic action logic we want to base on the dynamic logics defined in Section 2. In Section 6 we define a deontic dynamic reduction that meets the requirements of Section 5. Section 7 finishes the paper with some conclusions.

2. Dynamic logic

In this section we recall the definition of dynamic logic. Taking 'a' to represent arbitrary elements of a given countable set of atomic action symbols $\mathcal{A}$, and taking 'P' to represent arbitrary elements of a given countable set of proposition symbols $\mathcal{P}$, *well-formed formula's* $\phi, \psi, \ldots$ and well-formed action terms $\alpha, \beta, \ldots$ of the dynamic logic language $\mathcal{L}(\cup, ;, ^*, ^{\leftarrow})$ are defined by the following BNF:

$$\phi, \psi, \ldots := P \mid \top \mid \bot \mid \neg\phi \mid \phi \vee \psi \mid \langle\alpha\rangle\phi$$
$$\alpha, \beta, \ldots := a \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^{\leftarrow}$$

We make use of the standard syntactic abbreviations. Formulas are interpreted over modal action models. A modal action model $\mathcal{M} = (S, R^{\mathcal{A}}, V^{\mathcal{P}})$ is defined as follows:

- S is a nonempty set of possible states
- $R^{\mathcal{A}}$ is an action interpretation function $R^{\mathcal{A}} : \mathcal{A} \to 2^{(S \times S)}$, assigning a binary relation over $S \times S$ to each atomic action a in $\mathcal{A}$.
- $V^{\mathcal{P}}$ is a valuation function $V^{\mathcal{P}} : \mathcal{P} \to 2^S$ assigning to each proposition P of $\mathcal{P}$ the subset of states in S for which P is valid.

The *dynamic logic semantics* is determined by the notion of validity of a formula φ in a state s of a model $\mathcal{M}$, denoted $\mathcal{M}, s \models \varphi$ that uses an extension of the interpretation function $R^{\mathcal{A}}$ for atomic actions a, to an interpretation function R for compound actions α.

$$R(a) = R^{\mathcal{A}}(a) \quad \text{for } a \in \mathcal{A}$$
$$R(\alpha \cup \beta) = R(\alpha) \cup R(\beta)$$

$$R(\alpha;\beta) = R(\alpha) \circ R(\beta)$$
$$R(\alpha^*) = \big(R(\alpha)\big)^*$$
$$R(\alpha^{\leftarrow}) = \big\{(s,t) \mid (t,s) \in R(\alpha)\big\}$$
$$\mathcal{M}, s \models P \quad \text{iff} \quad s \in V^{\mathcal{P}}(P)$$
$$\mathcal{M}, s \models \neg\varphi \quad \text{iff} \quad \text{not } \mathcal{M}, s \models \varphi$$
$$\mathcal{M}, s \models \varphi \wedge \psi \quad \text{iff} \quad \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi$$
$$\mathcal{M}, s \models \langle\alpha\rangle\varphi \quad \text{iff} \quad \text{there is a } t \text{ such that } (s,t) \in R(\alpha) \text{ and } \mathcal{M}, t \models \varphi$$

Validity on a model $\mathcal{M}$ is defined as validity in all states of the model. General validity of a formula ϕ is defined as validity on all models. The semantics enforces that $[\alpha]\phi$ holds in a state whenever all states reachable by α obey ϕ, which interprets 'execution of α results in ϕ'. We denote the above defined dynamic logic by $DL(\cup, ; , ^*, ^{\leftarrow})$, thereby characterizing it by the set of action combinators it supports. In the next section we consider the extension to the logic $DL(\cup, ; , ^*, ^{\leftarrow}, \sim)$, where $\sim$ is the standard action (relation) complement. Also, we consider dynamic logics that both weaken and strengthen $DL(\cup, ; , ^*, ^{\leftarrow})$: action negation is added, but other connectives are possibly dropped, resulting in logics like $DL(; , \cup, \wr^{K4})$, without iteration or converse, but including the action negation $\wr^{K4}$. It turns out that dynamic logics with separate subsets of connectives from the set $\{\cup, ; , ^*, ^{\leftarrow}\}$ require separate notions of action negation, which is why the action negation is called 'relativized'.

3. Action negation

In the first of the next three sub-sections, we investigate what kind of action negation we need, and what exactly we need it for. The second sub-section reviews some known alternatives for action negation from the literature, and then in the third sub-section we define our alternative.

3.1. Action negation and the concept of 'refraining'

We need an action negation that is useful in formalizing such intuitively valid sentences as 'practition α's being obligatory implies that *alternative* practitions are forbidden'. To formalize this we need to decide when a practition counts as an 'alternative practition'. This question, in turn, can only be answered if we know when two practicions are identical. Since we decided to model practitions as actions in dynamic logic, the answer to this last question is given by the action semantics of dynamic logic: two actions are the same when they are interpreted by the same relation over the state space. So, the same action can be denoted by different action terms, for instance by '$a; b$' or 'c'. Then, other actions count as 'different' only because they are interpreted by a different relation over states: possibly they are executable from different states and/or they reach different result states.

But not all actions that count as 'different' in the dynamic logic sense, count as 'alternative' actions in the sense meant in sentences like 'practition α's being obligatory implies that *alternative* practitions are forbidden'. First of all, alternatives to an action are only possible if the primary action itself is possible. Clearly, this condition is not necessarily valid for 'different actions'. Second, an action whose relational interpretation is a strict sub-set of some action, and that is thus 'different' in the dynamic logic sense, should not be considered an 'alternative' action in the sense meant in the above deontic sentence. After all, the obligation to perform α should not imply the prohibition to do an action whose interpretation is contained in that of α; such actions actually constitute 'a way' to fulfill the obligation. This explains why in the rest of the paper we distinguish between 'different actions' and 'alternative actions'. A 'different action' is an action with another relational interpretation, and an 'alternative action', is an action whose relational interpretation is not contained in that of the relational interpretation of the primary action. So, α is alternative to β if it is not the case that all ways to perform α (which includes the concurrent performance with other actions) are also a way to perform β, that is: $R(\beta) \nsubseteq R(\alpha)$.

In general there are many actions that count as 'alternative' to a given action α. But, like for instance concurrent composition ($\cap$) or iteration (*), the notion of action negation to be defined for dynamic logic has to be an action forming combinator that returns *one specific* negative action. Then, in order to formalize the type of sentences mentioned above, there should be a clear relation between the action that is the negation of α and the set of actions that count as alternatives to α. To this end we aim to define a notion of action negation that captures the intuition of 'refraining' from an action. Thus, if α is some (primary) action, the action $-\alpha$ (with, for now, '$-$' denoting negation) is the act of *refraining* from α. We define refraining of α to be the act of 'doing anything but α'. Refraining from α is thus actively ensuring that α is not done. This means that refraining is *not* the same as performing *some* alternative action. Of course, by doing an alternative action it is *possible* that we refrain from doing the primary action. But, the act of refraining itself should express 'doing *anything* but α', rather than 'doing *something* other than α'. Thus, the semantics of the combinator '$-$' should be such that $-\alpha$ is the *strongest* of all the actions alternative to α; strongest in the sense that it does not have a non-deterministic possibility in common with α, and that any action alternative to α has a non-deterministic possibility in common with $-\alpha$.

Summarizing, we aim at a notion of action negation that renders the action $-\alpha$ as (1) *different* from α, and (2) *alternative* to α, and (3) interpretable as the act of *refraining* from α. The above mentioned deontic sentence can now be rephrased as 'being obliged α implies prohibition to perform any alternative act, i.e., any act that includes the non-deterministic possibility of refraining from α'. We will introduce a notion of action negation in dynamic logic whose semantics closely follows the above explained intuitions for refraining. But first we briefly review the literature on other notions of action negation defined for dynamic logics.

3.2. Known alternatives for dynamic logic action negation

First we look at a recent proposal by Wansing [21]. Also his aim is to give a dynamic logic formalization of the notion of 'refraining'. Although technically pleasing, Wansing's notion of action negation does not suit our purposes. Let us first explain why it is technically pleasing. His refraining operation obeys $R_{-(\alpha;\beta)} = R_{-\alpha;-\beta}$ and $R_{-(\alpha^*)} = R_{(-\alpha)^*}$

and $R_{-(\varphi?)} = R_{\neg\varphi?}$ (actions $\varphi?$ are known as 'tests'). It is easy to see that these properties, together with the standard boolean interactions, enable negations to be pushed down to the atomic action level without affecting validity of a formula. But, at this level of atomic actions, Wansing leaves the relation between for instance a and its negated companion $-a$ completely free. He argues that 'this is not an issue to be settled by the logic'. The technical consequence is that the resulting logic is not significantly more complex than the logic without the negation. However, we feel that it might be too liberal not to impose any restriction on the interpretation of $-a$ given an interpretation of a. It means for instance that $-a$ and a can be identical in the dynamic logic sense, i.e., that they are interpreted by identical relations over the state space. The same holds for arbitrary complex actions α and $-\alpha$. This means that Wansing's refraining operation does not necessarily return an action that is 'alternative' or even an action that is 'different' in the sense defined in the previous section. This makes it clear that Wansing's negation does not suit our purposes. Another reason for not adopting it is that we feel that a property like $R_{-(\alpha;\beta)} = R_{-\alpha;-\beta}$ is too strong: why should it be that refraining from a sequence requires refraining from each individual action in the sequence?

Meyer [14] introduces the action negation, denoted $\overline{\alpha}$, as part of an algebra of actions. The action algebra obeys some properties that are deemed intuitive for action negation. In particular it obeys the axiom $\overline{\alpha;\beta} = \overline{\alpha} \cup \alpha; \overline{\beta}$. The axioms for the algebra are shown to be consistent by providing an intuitive semantics. The algebra is then used as an interpretation for dynamic logic actions: syntactically, the actions of the algebra are identified with the actions within the modal box of dynamic logic, and semantically, a connection is made between the algebraic semantics and the relational semantics on dynamic logic models. However, the connection between the modal part and the algebraic part leaves some room for alternative interpretations, which makes it unclear how to axiomatize the dynamic logic. But the main reason for not following this route in the present paper, is that it is not clear how to generalize the action negation $\overline{\alpha}$ such that it encompasses iteration and converse of action.

Meyer's motivation to look for intuitive algebraic properties of the action negation was that the *standard* notion of relational complement, as studied in relation algebra [22], is not suitable for reasoning about action. Yet this standard notion of relational complement is the one studied most in the dynamic logic literature [23,24]. In standard relation algebra, the relational complement is defined with respect to the *universal relation*. It is straightforward to import this notion of complement in dynamic logics. We denote it with the symbol '$\sim$'. If *AC* is a set of action combinators, $\sim$-logics are logics $DL(AC)$, where $\sim \in AC$. The semantics of $\sim$-logics follows from the addition of the following clause to the standard dynamic logic semantics of Section 2:

$$R(\sim\alpha) = S \times S \setminus R(\alpha)$$

The intersection operator, that we use to model concurrency of action, is definable in dynamic logics extended with this universal action complement.

$$\alpha \cap \beta \underset{\text{def}}{\equiv} \sim (\sim \alpha \cup \sim \beta)$$

Introduction of the universal complement enhances expressiveness considerably but makes most dynamic logics (e.g., $DL(\cup, ;)$) undecidable. However, the standard relation algebraic complement does *not* suit our purposes. If $\sim$ is interpreted to mean 'refraining', then the expression $\sim\alpha \cup \alpha$ should reflect the concept of 'doing anything', that is, the 'most general action' whose 'reach' is determined by what can be reached by the union of any atomic and compound action. But, the problem is that $\sim\alpha \cup \alpha$ reaches much *more* than that. The modality $[\sim\alpha \cup \alpha](.)$ has universal power; it is capable of reaching any state, including the ones that are *not* reachable through actions whose relational interpretation is explicitly given in a model. Likewise, $[\sim\alpha]\phi$ means that ϕ holds in the *entire* state-space not reachable through α, and for instance $\langle\sim a\rangle\phi$[4] is satisfied in any model containing a state where ϕ holds, irrespective of whether the relation interpreting a is empty or not. It is clear then that this complement is not appropriate for our purposes. To be faithful to the intuitions for the notion of 'refraining', we need an action negation that defines the complement of the 'reachable' state-space as the effect of a negated action.

3.3. Relativized action negation

The general intuition for the relativized action negation, denoted $\wr^I$, is that it takes the relational complement with respect to all possible relations over the *reachable* state-space. An action $\alpha \cup \wr^I\alpha$ then only reaches all states that are reachable through any compound action. But the formal definition of this intuition is not straightforward. The definition and properties of the reachable state-space, actually depend on which action combinators are in the language. For instance, if we allow sequence, the reachable relation space is transitive. And if we allow iteration, the reachable relation space is transitive and reflexive. And if we allow converse, the reachable relation space is symmetric. So we cannot give, as for the complement with respect to the universal relation, a general definition of this action negation for all dynamic logics; each dynamic logic $DL(AC)$, where AC denotes a certain set of action combinators, comes with its own interpretation for the action negation. All in all we define 6 versions of the action negation: $\wr^K, \wr^B, \wr^{S4}, \wr^{K4}, \wr^{B4}, \wr^{S5}$. Each negation is suited for a dynamic logic encompassing some specific set of action combinators. The dynamic logics in the series are: $DL(\cup, \wr^K)$, $DL(^\leftarrow, \cup, \wr^B)$, $DL(;, \cup, \wr^{K4})$, $DL(^*, ;, \cup, \wr^{S4})$, $DL(^\leftarrow, ;, \cup, \wr^{B4})$, $DL(^*, ^\leftarrow, ;, \cup, \wr^{S5})$. Each subsequent logic in the series introduces a new action operator and redefines the relativized action negation operation accordingly. In general, if AC denotes a set of action combinators, $\wr^I$-logics are denoted by $DL(AC)$, where $\wr^I \in AC$, with I an annotation referring to properties of the relational space with respect to which the relational complement is taken. For the annotation I we adopt standard terminology from modal logic to refer to properties of the complement space (transitivity, reflexivity, etc.). The semantics of a $\wr^I$-logic $DL(AC)$ follows from the standard dynamic logic semantics of Section 2 by copying the definitions for connectives in AC and adding:

$$\text{for } DL(\cup, \wr^K): \qquad R(\wr^K\alpha) = \Big(\bigcup_{a\in\mathcal{A}} R(a)\Big) \setminus R(\alpha)$$

[4] The use of a instead of α is deliberate. An α stands for an arbitrary compound action, thus also possibly $\sim\alpha$. But then we get $\langle\sim\sim\alpha\rangle\phi$ which is equivalent to $\langle\alpha\rangle\phi$ for which the mentioned property does not hold.

for $DL(^{\leftarrow}, \cup, \wr^B)$: $$R(\wr^B \alpha) = \left(\bigcup_{a \in \mathcal{A}} R(a) \cup R(a^{\leftarrow})\right) \setminus R(\alpha)$$

for $DL(;, \cup, \wr^{K4})$: $$R(\wr^{K4} \alpha) = \left(\bigcup_{a \in \mathcal{A}} R(a)\right)^{+} \setminus R(\alpha)$$

for $DL(^*, ;, \cup, \wr^{S4})$: $$R(\wr^{S4} \alpha) = \left(\bigcup_{a \in \mathcal{A}} R(a)\right)^{*} \setminus R(\alpha)$$

for $DL(^{\leftarrow}, ;, \cup, \wr^{B4})$: $$R(\wr^{B4} \alpha) = \left(\bigcup_{a \in \mathcal{A}} \big(R(a) \cup R(a^{\leftarrow})\big)\right)^{+} \setminus R(\alpha)$$

for $DL(^*, ^{\leftarrow}, ;, \cup, \wr^{S5})$: $$R(\wr^{S5} \alpha) = \left(\bigcup_{a \in \mathcal{A}} \big(R(a) \cup R(a^{\leftarrow})\big)\right)^{*} \setminus R(\alpha)$$

We then introduce the relativized 'any'-action any^I and the 'subsumption action' combinator $\subseteq^I$ as syntactic extensions on $\cup$ and $\wr^I$:

$$\alpha \subseteq^I \beta \underset{\text{def}}{\equiv} \wr^I \alpha \cup \beta$$

$$any^I \underset{\text{def}}{\equiv} \alpha \cup \wr^I \alpha$$

The relativized versions of the any and the subsumption action are different from their non-relativized counterparts. In particular, the relativized any does not reach the complete state-space, but only the part that is reachable through (compound) action, as determined by the action language. In the same way, we may define other relativized actions and action combinators like $fail^I$ and $\cap^I$. But it is easy to prove (see [17]) that these have exactly the same interpretation as their non-relativized counterparts. Thus, for all interpretations of actions α and β on a model $\mathcal{M} = (S, R^{\mathcal{A}}, V^{\mathcal{P}})$ it holds that:

$$R(fail) = R(fail^I) \quad \text{with } fail^I \underset{\text{def}}{\equiv} \alpha \cap \wr^I \alpha$$

$$R(\alpha \cap \beta) = R(\alpha \cap^I \beta) \quad \text{with } \alpha \cap^I \beta \underset{\text{def}}{\equiv} \wr^I (\wr^I \alpha \cup \wr^I \beta)$$

So, in dynamic logics with a relativized action negation (in combination with a choice operation), we have standard intersection as a defined operation. This is of importance, since we use plain intersection to model concurrency. To get an impression of the kind of logic properties that are reflected by $\wr^I$-logics, we mention some general validities. Many validities are shared by all defined $\wr^I$-logics. A non-trivial example is:

$$\models \langle\alpha\rangle\phi \wedge [\wr^I \beta]\neg\phi \rightarrow \langle\alpha \cap \beta\rangle\phi \quad \text{(K-R)}$$

By way of illustrating the semantics of the relativized action negation, we give the proof of the soundness of this property. Consider an arbitrary model $\mathcal{M}$ and an arbitrary state s. In case that $\mathcal{M}, s \not\models \langle\alpha\rangle\varphi$ or $\mathcal{M}, s \not\models [\wr^I \beta]\neg\varphi$ it holds trivially that $\mathcal{M}, s \models \langle\alpha\rangle\varphi \wedge [\wr^I \beta]\neg\varphi \rightarrow \langle\alpha \cap \beta\rangle\varphi$. So, we assume that (1) $\mathcal{M}, s \models \langle\alpha\rangle\varphi$ and (2) $\mathcal{M}, s \models [\wr^I \beta]\neg\varphi$. From (1) it follows that $\alpha \neq fail$ and that there is a state t such that $(s, t) \in R(\alpha)$ and (1′) $\mathcal{M}, t \models \varphi$. Now either $(s, t) \in R(\beta)$ or $(s, t) \in R(\wr^I \beta)$ ($\wr^I \beta \cup \beta$ reaches any state reachable through

Table 1

DL variant	Trans-any	Symm-any	Refl-any	NegSeq-R
$\cup, \wr^{K}$	no	no	no	no
${}^{\leftarrow}, \cup, \wr^{B}$	no	yes	no	no
$;\,, \cup, \wr^{K4}$	yes	no	no	no
${}^{*}, ;\,, \cup, \wr^{S4}$	yes	no	yes	no
${}^{\leftarrow}, ;\,, \cup, \wr^{B4}$	yes	yes	no	yes
${}^{*}, {}^{\leftarrow}, ;\,, \cup, \wr^{S5}$	yes	yes	yes	yes

compound action). But from $(s,t) \in R(\wr^I \beta)$ and property 2 we would have to conclude that $\mathcal{M}, t \models \neg\varphi$, which contradicts $1'$. Then, from $(s,t) \in R(\beta)$ and $(s,t) \in R(\alpha)$ it follows that $(s,t) \in R(\alpha \cap \beta)$, which together with $1'$ gives $\mathcal{M}, s \models \langle \alpha \cap \beta \rangle \varphi$.

Other properties are valid for some $\wr^I$-logics and invalid for others. These properties therefore can be said to mark differences between the logics. Consider the following possible validities for $\wr^I$-logics:

$$\models \langle any^I \rangle \langle any^I \rangle \phi \to \langle any^I \rangle \phi \qquad \text{(Trans-any)}$$

$$\models \phi \to [any^I] \langle any^I \rangle \phi \qquad \text{(Symm-any)}$$

$$\models \phi \to \langle any^I \rangle \phi \qquad \text{(Refl-any)}$$

$$\models \langle \alpha \rangle [\wr^I \beta] \phi \to \big[\wr^I (\alpha; \beta)\big] \phi \qquad \text{(NegSeq-R)}$$

Now Table 1 lists which of these properties hold for which logic.

The proof of this proposition is certainly not hard, but also not straightforward. In particular the proof of the soundness of NegSeq-R for the strongest two logics in the table takes slightly more than a simple verification (see [17]). Another result from [17] concerns the relative strength of the logics. Under replacement of complements in formulas, the following inclusion relation between $\wr^I$-logics holds. In the picture, logics are represented by the type of complement they endorse, and arrows denote inclusion of validities.

$$\begin{tikzcd}
 & \wr^{K4} \arrow[r] \arrow[dr] & \wr^{S4} \arrow[dr] & \\
\wr^{K} \arrow[ur] \arrow[dr] & & & \wr^{S5} \\
 & \wr^{B} \arrow[r] & \wr^{B4} \arrow[ur] &
\end{tikzcd}$$

Relativized action complement dynamic logics are also expected to have better complexity properties then their universal complement counterpart logics. In particular, we conjecture that the logic $DL(;\,, \cup, \wr^{K4}, {}^{\leftarrow})$ (and thus $DL(;\,, \cup, \wr^{K4})$) is decidable, whereas $DL(;\,, \cup, \sim)$ (and thus $DL(;\,, \cup, \sim, {}^{\leftarrow})$) is undecidable. For the relativized complement dynamic logics, undecidability strikes for sure with the introduction of iteration in the language. This follows from the result that under syntactic interchange of the action combinators $\sim$ and $\wr^I$, the logics $DL(;\,, \cup, {}^{*}, {}^{\leftarrow}, \wr^{S4})$ and $DL(;\,, \cup, {}^{*}, {}^{\leftarrow}, \sim)$ are equivalent, that is, encompass the same set of validities. For a proof we again refer to [17].

Other results concern the expressiveness in terms of definability of classes of models and frames. Finally we want to mention that addition of the relativized complement to dynamic logic invalidates many well-known properties that are common to modal logics. For instance, the logics do not obey the tree model property, and do not preserve validity over bisimulations. Because of this latter property, according to van Benthem [25], we are not entitled to call the logics 'modal'.

4. Free-choice and inter-definability of deontic operators

In this section we address the problem of adopting a free-choice semantics and the problem of the inter-definability of deontic operators. But, before we discuss these issues, we first want to make clear how we interpret concurrency. We adopt an 'open action interpretation' of concurrency. Under this interpretation, execution of α may involve 'any concurrent action that includes α as a concurrent component'. The dynamic logic property that naturally fits our open concurrency interpretation is $[\alpha]\varphi \to [\alpha \cap \beta]\varphi$. This is easily seen. If $[\alpha]\varphi$ means that φ holds after any execution of α concurrently with some other action, then it holds in particular after $\alpha \cap \beta$. The open interpretation of actions is contrasted with the closed interpretation. Under a closed interpretation of concurrency, an action term α is interpreted as 'the action α in isolation, i.e., not concurrently with yet other actions'.

For ought-to-be and STIT-type logics, the free-choice principle can be described as the semantic position saying that formulas like $P(p \vee q) \wedge \neg P(p)$ are inconsistent. For these logics it is generally accepted that free-choice semantics are *not* intuitive. It is often argued (e.g., [27–29]) that the problems with free-choice permission are simply the result of misinterpretations of deontic logic formulas, accumulating in mis-translations between logic formulas and natural language expressions. We agree. However, for dynamic deontic logics, things are different. For dynamic deontic logic, the free-choice alternative can be described as the position that $P(a \cup b) \wedge \neg P(a)$ is inconsistent. We call the opposite position, namely that $P(a \cup b) \wedge \neg P(a)$ is consistent, the 'imposed choice semantics'. The original system of dynamic deontic logic by Meyer [14,26] was actually developed according to the imposed choice view. However, we have some difficulty accepting this view as a conceptual alternative for dynamic deontic logics. The imposed choice interpretation seems to suffer from an internal confusion: expressions $P(a \cup b)$ are interpreted to mean that the action $a \cup b$ is permitted, but it is not excluded that we have a violation when performing $a \cup b$. What justification is there for calling actions permitted, if they include the non-deterministic possibility, *not under control* of agents, of reaching states where a violation occurs? The imposed choice principle is sometimes defended with examples like the following: 'I permit you to drive my car' does not imply that 'I permit you to drive my car and drink (a concurrent action that by the open interpretation of concurrency, as explained above, counts as a way to perform the driving action)'. But the problem with such examples is that they do not attack the principle of free choice, but only reveal the incompleteness of normative assertions in normal discourse. In normal discourse, the actual meaning of assertions involves many implicit default assumptions. The agent enacting this norm meant 'I permit you to drive my car, unless at the same time you drink, use your telephone, violate traffic regulations, etc.'. With such an exhaustive set of exceptions added to it, the choice that is present in the car driving action is still free. But, in general, such exceptions are not spelled out explicitly: they are assumed as general background knowledge. Therefore, in Section 6, we define a reduction that respects the free-choice semantics.

The second problem we discuss in this section concerns the inter-definability of deontic operators. First of all we want to avoid $P(\alpha) = \neg F(\alpha)$, leaving room for actions that do not have a normative status. This enables us to define 'indifference' as $I(\alpha) \equiv_{\text{def}} \neg P(\alpha) \wedge \neg F(\alpha)$. Meyer's original system does not allow this. Also we want to avoid properties like $F(-\alpha) = O(\alpha)$. The relations between obligation and prohibition we are looking for cannot be expressed by such a simple identification. First of all, as we explained, we want it to be the case that the action $-\alpha$ is not the only action that is forbidden when α is obliged. Second, in many cases, for an obligation to perform an action it takes *more* than a prohibition to perform alternative action. Obligations that follow from prohibitions to the contrary can be said to be 'negatively' motivated. One of the aspects they seem to lack, and that distinguishes them from 'positive' obligations, is the principle of 'ought implies may'. Also in the other branches of ought-to-do deontic logics, the inter-definabilities have been disputed. Hintikka [20] avoids the inter-definabilities $F(\neg p) = O(p)$ and $\neg P(\neg p) = O(p)$ for reasons similar to ours. And von Wright emphasized [9] that since 'norm and action' [19], he has considered obligation and permission to be *not* inter-definable.

5. Some logic requirements

Based on the intuitions we explained in the previous sections we formulate a set of dynamic deontic logic requirements. We require that: no action can be (1) at the same time permitted and forbidden, (2) at the same time be obliged and not permitted, (3) obliged, while alternative actions (see Section 3) are not forbidden, (4) obliged, while an action that is different (again, see Section 3) is also obliged. In formulas:

(1) $\neg\big(P(\alpha) \wedge F(\alpha)\big)$

(2) $\neg\big(O(\alpha) \wedge \neg P(\alpha)\big)$

(3) $\neg\big(O(\alpha) \wedge \neg F(\beta)\big)$ if $R(\beta) \not\subseteq R(\alpha)$

(4) $\neg\big(O(\alpha) \wedge O(\beta)\big)$ if $R(\alpha) \neq R(\beta)$

The first two requirements together imply the requirement $\neg(O(\alpha) \wedge F(\alpha))$. Note that requirement 3 does not state that if α is obliged, the act of refraining from α is forbidden; it is much stronger than that. Not only the act of refraining itself is forbidden, any action that is an alternative to α (i.e., that includes the non-deterministic possibility of refraining from α) is too. Likewise, for requirement 4, if α is obliged, it should not only be implied that the act of refraining from α is not at the same time obliged; any action that is different in the dynamic logic sense cannot be at the same time obliged. This is quite a strong requirement. We discuss it in the next section, where we also show how to define a reduction that relaxes it.

The above requirements mainly concern the interactions between separate deontic modalities. Below we formulate for each modality individually how it interacts with (free) choice and (open action) concurrency. We require that: (5) permission to choose between

α and β is equivalent to permission to do α together with permission to perform β, (6) permission to perform α implies permission to perform α concurrently with β (due to the open action interpretation and free-choice), (7) prohibition to choose between α and β is equivalent to prohibition to do α or prohibition to perform β, (8) prohibition to perform α and β simultaneously implies prohibition to perform α and prohibition to perform β, (9) the obligation to perform α and β simultaneously is equivalent to the combination of the obligation to perform α and the obligation to perform β. In formulas:

(5) $P(\alpha \cup \beta) \leftrightarrow P(\alpha) \wedge P(\beta)$

(6) $P(\alpha \cap \beta) \leftarrow P(\alpha) \vee P(\beta)$

(7) $F(\alpha \cup \beta) \leftrightarrow F(\alpha) \vee F(\beta)$

(8) $F(\alpha \cap \beta) \rightarrow F(\alpha) \wedge F(\beta)$

(9) $O(\alpha \cap \beta) \leftrightarrow O(\alpha) \wedge O(\beta)$

At this point we again stress the importance of considering the actions in the above formulas as *open*. For instance, property 8 is *not* refuted by the example 'prohibition to push two buttons does not imply prohibition to push the first'. In an open reading of action concurrency, this example turns into 'prohibition to perform all possible ways to push two buttons implies prohibition to perform all possible ways to push the first'. This is valid, because in the open interpretation of concurrency, performing the concurrent two button push counts as 'a way' to perform the one button push.

6. A deontic-dynamic reduction

In this section we define a new deontic-dynamic reduction. The definition makes use of the relativized action negation $\wr^I \alpha$, and meets the logic requirements of the previous section. We avoid the inter-definabilities between permission, prohibition and obligation by introducing a 'violation constant' for each of them: V_F for the violation of a prohibition, V_O for the violation of an obligation, and V_P for lack of explicit permission. Now the deontic-dynamic reductions of the modal operators for permission, prohibition and obligation are defined by:

(a) $P(\alpha) \underset{\text{def}}{\equiv} [\alpha]\neg V_P$

(b) $F(\alpha) \underset{\text{def}}{\equiv} \langle\alpha\rangle V_F$

(c) $O(\alpha) \underset{\text{def}}{\equiv} P(\alpha) \wedge [\wr^I \alpha] V_O$

This deontic-dynamic reduction does not impose any relation between the deontic modalities. We may introduce these by relating the different types of violation. We may for instance require, as in Meyer [14], that there is no difference between the violation of an obligation, the violation of a prohibition, and the absence of accordance with a permission: $V_O \leftrightarrow V_F \leftrightarrow V_P$. But then we get back the strong inter-definabilities. We can be much more cautious. Here we require, that we cannot at the same time (1) be in accordance with an explicit permission and at the same time violate a prohibition, and (2) violate an obligation and not at the same time violate a prohibition. In formulas:

(d) $\neg(\neg V_P \wedge V_F)$

(e) $\neg(V_O \wedge \neg V_F)$

Note that under these constraints states may satisfy $\neg V_P \wedge \neg V_O \wedge \neg V_F$, indicating that there is room for indifference with respect to the normative status of actions.

Now we show that the above deontic dynamic reduction satisfies the requirements formulated in Section 5. Logic requirement 1 follows easily from the properties a, b and d. Requirement 2 follows directly from c. To show that requirement 3 holds, we first observe that the condition $\beta \not\subseteq \alpha$ can be formulated in $\wr^I$-logics as $\langle\beta \cap \wr^I\alpha\rangle\top$, which directly expresses: there is possible way to do β that is at the same time not a way to do α. Then requirement 3 becomes $\langle\beta \cap \wr^I\alpha\rangle\top \to \neg(O(\alpha) \wedge \neg F(\beta))$. Applying the reduction returns $\langle\beta \cap \wr^I\alpha\rangle\top \to \neg([\alpha]\neg V_P \wedge [\wr^I\alpha]V_O \wedge \neg\langle\beta\rangle V_F)$. It is fairly easy to verify that with the constraints (d) and (e) this is valid in any $\wr^I$-logic. Requirement 4 is verified in the same way. We express the condition $R(\alpha) \neq R(\beta)$ as $\langle\wr^I(\alpha \subseteq^I \beta) \cup \wr^I(\beta \subseteq^I \alpha)\rangle\top$. Thus, the requirement is $\langle\wr^I(\alpha \subseteq^I \beta) \cup \wr^I(\beta \subseteq^I \alpha)\rangle\top \to \neg(O(\alpha) \wedge O(\beta))$. This reduces to $\langle\wr^I(\alpha \subseteq^I \beta) \cup \wr^I(\beta \subseteq^I \alpha)\rangle\top \to \neg([\alpha]\neg V_P \wedge [\wr^I\alpha]V_O \wedge [\beta]\neg V_P \wedge [\wr^I\beta]V_O)$, which can be checked by using the property $\neg(V_O \wedge \neg V_P)$, that follows from (d) and (e). The association of the permission operator with a dynamic logic *box* operator (a) ensures that permission obeys the requirements for free-choice. The requirements for choice and concurrency all follow from the choice for either a box or a diamond operation for these modalities.

The given reduction thus satisfies the requirements of Section 5. We now come back to the point that some of these requirements seem quite strong. In particular requirements 4 and 9 for obligations. These requirements are in accordance with our view that goal-type obligations of complex actions cannot be reduced to obligations of sub-actions. In this sense, obligations are primitive. There is a relation between this view on obligations and the 'ought implies may' principle. It can be seen that the incorporation of this principle in the reduction definition, *induces* properties like 4 and 9. As an example we show how Ross's property [30] is avoided by the principle. Rephrased for the dynamic logic context, Ross's property says that the obligation to do some specific action implies the obligation to do any non-deterministic sub-action of that action. We take the instantiation: $O(\mathit{Listen})$ implies $O(\mathit{Listen} \cup \mathit{Leave})$. This is undesirable, because with the background information that $[\mathit{Listen} \cap \mathit{Leave}]\bot$ (it is not possible to listen and leave simultaneously), it follows that $O(\mathit{Listen})$ implies $F(\mathit{Leave})$. But, if $F(\mathit{Leave})$ and $O(\mathit{Listen} \cup \mathit{Leave})$ can hold at the same time, we would violate the 'ought implies may' principle $O(\alpha) \to P(\alpha)$. More generally, if we were to drop the 'ought implies may' principle from the reduction defined above, we would have the following alternative properties for requirements 4 and 9:

(9′) $O(\alpha \cup \beta) \leftarrow O(\alpha) \vee O(\beta)$

(9″) $O(\alpha \cap \beta) \to O(\alpha) \wedge O(\beta)$

(4′) $\neg\big(O(\alpha) \wedge O(\beta)\big)$ if $R(\alpha \cap \beta) = \emptyset$

This shows that the reduction is easily adapted in order to satisfy other logic requirements.

7. Conclusions

In this paper we proposed solutions to two problems with deontic dynamic logics. First we defined a notion of action negation that closely follows the intuition of 'refraining' from an action. We discussed how this notion of refraining enables us to formalize intuitively valid sentences as 'α's being obligatory implies that alternative actions are forbidden'. Second we defined a deontic-to-dynamic reduction that is faithful to the free-choice interpretation of norms, and that leaves room for actions whose normative status counts as 'indifferent'. The (series of) logics defined by the reduction and the dynamic logics with the new action negation were shown to obey a set of deontic logic requirements.

A clear asset of the present approach is that it gives intuitive definitions for all three central deontic notions, that is, permission, prohibition and obligation for actions exhibiting all relevant action combinators, that is, *choice*, *sequence*, *concurrent composition*, *converse*, and *iteration*.

Many issues had to be left unconsidered. We did not discuss how contrary to duty examples might be formalized in the logics. Hilpinen [13] argues that Castañeda's distinction between practitions and action propositions provides solutions to many of the contrary to duty benchmark examples, but not to the one about the 'gentle murderer'. We did not show that our approach is able to draw the right conclusions in this scenario. Also we did not consider deontic-to-dynamic reductions for the 'imposed' view on choice. And, we did not discuss the ought-implies-can principle that is relevant for practition-type deontic logics. Finally, we did not present complexity and completeness results for the series of dynamic logics we defined. These issues are postponed to a later occasion.

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