ON ANALYTIC FUNCTION GERMS OF TWO
COMPLEX VARIABLES

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1. INTRODUCTION

Let \( f(u, v) \), \( f(0, 0) = 0 \), be a complex analytic function of two complex variables defined near 0. There are two kinds of invariants that interest us. The first is the smallest integer \( r \) such that in the Taylor's expansion of \( f \), all terms of degree \( \geq r + 1 \) can be omitted without changing the local topological picture of \( f \). The second are the linking coefficients between the torus knots of the branches of \( f \). These knots are the intersection of the curve \( f(u, v) = 0 \) in \( \mathbb{C}^2 = \mathbb{R}^4 \) with a 3-sphere \( S^3 \) of sufficiently small radius centered at 0. The intersection consists of 1-dimensional circuits, or a system of knots (torus knots) linked together in \( S^3 \) ([6], §2; [8], p. 13).

One of our main purposes in this paper is, roughly speaking, to express the first invariant in terms of the second kind, this is our Theorem B below.

Generalizing a theorem for the real case ([3], Theorem 0), it was proved in [2] (Theorem 2, p. 878), that if

\[
|\text{Grad } f| \geq \epsilon \rho^a, \quad \text{for } (u, v) \text{ near } 0,
\]

(1.1)

where

\[
\rho = \sqrt{|u|^2 + |v|^2}, \quad \epsilon > 0, \quad \alpha > 0,
\]

then the complex jet \( j^{tr}(f) \), \( r = \langle \alpha \rangle \), is \( C^0 \)-sufficient. (Here \( \langle \alpha \rangle \) denotes the smallest integer \( > \alpha \).) That is, if \( g(u, v) \) is any analytic function whose Taylor's expansion near 0 agrees with that of \( f \) up to degree \( \leq r \), then there exists a local homeomorphism \( h: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \) such that \( f \circ h = g \).

In this paper, we shall first determine the smallest value of \( \alpha \) satisfying (1.1) from the Puiseux expansion of \( f \) (Theorem A in §2 and Theorem C in §4). This value is the so-called Łojasiewicz exponent of \( |\text{Grad } f| \), [4]. No use of the Puiseux expansions of \( f_u, f_v \) will be made, except in the proofs. This is already a considerable improvement of the results in [3], where, for real jets, \( \alpha \) is computed via the expansions of \( f_u \) and \( f_v \). In fact, Theorem A is the most difficult one in this paper. Next, in Theorem B, the smallest value of \( \alpha \) is interpreted geometrically as the way how certain branches (or places) of \( f \) are linked up.

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2. THE RESULTS

Without loss of generality, we can assume that the \( u \)-axis is not a tangent of \( f = 0 \) at 0. Then, by Puiseux's Theorem [7, p. 98], \( f \) can be factored into a product of fractional power series

\[
f(u, v) = \prod_{i=1}^{m} (u - \beta_i(v))
\]

(2.1)

\( O(\beta_i) \geq 1 \), where \( O(\beta_i) \) is the order of \( \beta_i \), thus \( m = O(f) \). We shall assume, throughout this paper, that no two \( \beta_i \)'s are identical.

For each root \( \beta_i \) in (2.1), let \( c_i = \max_{|\beta|} O(\beta_i - \beta) \). Now let \( g_{c_i}(\beta_i) \) denote \( \beta_i \) with its terms of degree \( c_i \) replaced by \( \xi v^c \), \( \xi \) a generic number (or an indeterminant), and all higher-order terms omitted. That is, we treat \( \xi \) either as an indeterminant, or as any complex number not belonging to a finite set \( \Sigma \subset \mathbb{C} \), where \( \Sigma \) is to be determined in the context.
We call $g_c(\beta)$ the generic perturbation of $\beta$ at degree $c_i$.

Example (2.1). $f(u, v) = u^3 + 3uv^2$, $k \geq 3$. We have $\beta_1 = 0$, $\beta_2, \beta_3 = \pm \sqrt{3}iv^{k/2}$; $O(f) = 3$; $c_i = k/2$. $g_c(\beta) = \xi^iiv^{k/2}$, $i = 1, 2, 3; O(f(g_c(\beta), v)) = O((\xi^i + 3\xi^iiv^{k/2}) = k/2$ Note that the last equality holds only for generic values of $\xi$; for $\xi = 0, \pm \sqrt{3}i$, the orders are $\infty$.

Theorem A. Let $l_i = O(f(g_c(\beta), v)), 1 \leq i \leq m$, and $\alpha = \text{Max} \{l_i - 1\}$. Then $\alpha$ is the smallest number which satisfies inequality (1.1) for some $\epsilon > 0$. In particular, $f^{(r)}(f), r = (\alpha)$, is $C^\omega$-sufficient.

Thus, in Example (2.1), $\alpha = (3k/2) - 1$.

Theorem A can be slightly improved. Let us call a root $\beta$, minimal if the following holds. Let $\beta_i$ be any root with $O(\beta_i - \beta_e) = \text{Max} O(\beta_i - \beta_i), 1 \leq i \leq m$, then there is no root $\beta_i$ satisfying $O(\beta_i - \beta_e) > O(\beta_i - \beta_e)$. (2.2)

Geometrically, the torus knot of the branch containing a minimal root $\beta$, does not contain (in the sense of [6, §2.3]) the torus knot of any other branch of $f$.

Theorem A'. The number $\alpha$ in Theorem A is also given by

$$\alpha - \text{Max} \{l_i - 1\}$$

where $i$ runs through all indices for which $\beta_i$ is minimal.

For a root, $\beta$, in (2.1), (omitting the subscript $i$ for simplicity) we rewrite it so as to display its characteristic pairs, as in [8, p. 7],

$$U = \sum a_1v^1 + b_1v^q/p_1 + \sum a_2v^{(q_1 + q_2)/p_1} + b_2v^{q_2/p_2} + \ldots + b_mv^{q_m/p_m} + \sum b_{\mathfrak{c}}v^{q_{\mathfrak{c}}/p_{\mathfrak{c}}}$$

(2.3)

where the symbols have the following significance: $p_i > 1$, $k_i = \lfloor (q_i - q_{i-1})/p_i \rfloor$ ($q_0 = 0$), $b_i \neq 0$, $q_i, n_i$ are relatively prime.

The branch, or place, $P$, containing $\beta$, consists of the following $\mathfrak{p} = p_1 \ldots p_\mathfrak{p}$ roots [7, p. 107, Theorem 4.1],

$$u = \sum a_1\varepsilon_{\mathfrak{p}}v^1 + b_1\varepsilon_{(h\mathfrak{p}/p_1)}v^{q_1/p_1} + \ldots + b_m\varepsilon_{(h\mathfrak{p}q_m/p_1 \ldots p_\mathfrak{p})} v^{q_m/p_1 \ldots p_\mathfrak{p}} + \ldots$$

with $0 \leq h \leq p - 1$,

(2.4)

where $\varepsilon$ is a $p$th primitive root of unity. Call $p$ the order of $P$: $O(P) = p$. Note that $\varepsilon_{hp} = 1$ so that $a_{ih} = a_{ih}\varepsilon_{hp}$ for $1 \leq i \leq k_i$; and when $h = 0$, (2.4) gives (2.3).

Let $\{P_1, \ldots, P_t\}$ be the branches of $f$. Consider a 3-dimensional sphere, $S^3$, in $\mathbb{C} = \mathbb{R}^4$, centered at 0, with sufficiently small radius. The intersections $S^3 \cap P_i$ where $P_i$ also denotes the zero set of $P_i$, are knots, $\mathcal{N}_i, 1 \leq i \leq t$, which are linked together in $S^3$. For each $\mathcal{N}_i$, we are interested in three kinds of geometrical invariants, the numbers $l_i$ in Theorem A' will then be expressed in terms of them. The first is the order $O(P_i)$; recall that this is also the intersection multiplicity of $P_i$ with the $u$-axis. The second is the linking coefficients, $L(\mathcal{N}_i, \mathcal{N}_j)$, for $j \neq i$. As was proved in [6, §4, p. 170], $L(\mathcal{N}_i, \mathcal{N}_j)$ equals the intersection multiplicity of $P_i$ and $P_j$. That is, let $P_i = \{\beta_1, \ldots, \beta_s\}$, $P_j = \{\beta_{s+1}, \ldots, \beta_{s+t}\}$, then

$$L(\mathcal{N}_i, \mathcal{N}_j) = \sum_{ij} O(\beta_i - \beta_{i+j}). \quad 1 \leq i \leq r, \quad 1 \leq j \leq s.$$  

(2.5)

The last invariant for $P_i$ is what we shall call the self-linking of $\mathcal{N}_i$. For a branch, $P$, whose roots are expressed as in (2.4), let $\mathcal{N}$ denote the torus knot of the $P$ consisting of all fractional power series

$$u = \cdots + (b_\mathfrak{p} + \delta)v^{\mathfrak{p}/\mathfrak{p}} + \cdots + \cdots$$

(2.6)
obtained by adding a small number \( \delta \neq 0 \) to the coefficient \( b_e \) in (2.4). Note that \( \tilde{P} \) is not necessarily a branch of \( f \). The self-linking of \( P \), or of the torus knot \( \mathcal{N} \), is defined to be

\[ \sigma = \mathcal{L}(\mathcal{N}, \tilde{\mathcal{N}}). \]

Geometrically, \( \mathcal{N} \) is the torus knot \( \Gamma_{q_1, p_1}^{q, p} \) which is a tubular knot of indices \( q_1, p_1 \) with axis \( \Gamma_{q_1, p_1}^{q, p} \), as explained in [8, p. 13]; then \( \tilde{\mathcal{N}} \) is \( \mathcal{N} \) rotated around its axis \( \Gamma_{q_1, p_1}^{q, p} \) by a small angle.

Let \( \sigma_i \) denote the self-linking of \( \mathcal{N}_i \).

For each \( i \), let \( \rho_i = \max \{ \mathcal{L}(\mathcal{N}_i, \mathcal{N}_j) \} \), where \( j \) runs through all indices for which \( \mathcal{N}_i \) is isotopic to \( \mathcal{N}_j \) [1, p. 8]. In case there is no such \( j \), we put \( \rho_i = 0 \).

**Theorem B.** For each minimal root \( \beta_n \), the number \( l \), in Theorem A is given by the formula:

\[ O(P_l) = \sum_{k} \mathcal{L}(\mathcal{N}_n, \mathcal{N}_k) + \max \{ \sigma_i, \rho_i \}, \tag{2.7} \]

where \( k \neq i \), \( 1 \leq k \leq t \), \( t \) is the total number of branches \( \{ P_1, \ldots, P_t \} \) of \( f \).

Let us call a knot \( \mathcal{N}_i \) uncoupled if it is not isotopic to another \( \mathcal{N}_j \); in this case, \( \rho_i = 0 \), \( \max \{ \sigma_i, \rho_i \} = \sigma_i \). We then have

**Corollary.** If the knot associated to a minimal root \( \beta \) is uncoupled, then

\[ O(P_l) = \sum_{k \neq i} \mathcal{L}(\mathcal{N}_n, \mathcal{N}_k) + \sigma_i. \tag{2.8} \]

In §4, the tree-model, \( M(f) \), of \( f \) is introduced; Theorem C asserts that \( l_n \) and hence \( \sigma \), can be computed directly from \( M(f) \).

The main ideas in this paper grow out of the observation that \( M(f_n) \) can be constructed from \( M(f) \), as explained in §4.

### §3. Newton Polygon

The notion of Newton Polygon is recalled briefly in this section. Some important lemmas, based on Walker’s constructive proof of Puiseaux’s Theorem [7, p. 98], are established; they will be used repeatedly in later sections.

Given \( f(x, y) = a_0(y) + a_1(y)x + \cdots + a_m(y)x^m \), where \( x, y \) are indeterminants, \( m \) any positive integer and \( a_i \) are fractional power series in \( y \). In a coordinate plane plot points \( P_i \) at \( (i, a_i) \), where \( 1 \leq i \leq m \), and \( a_i = O(\tilde{a}_i(y)) \). Call this the Newton Diagram of \( f \). Then join \( P_0 \) to \( P_m \) by a convex polygonal arc as in [7, p. 99]. This is the Newton Polygon of \( f \).

Let \( P_k, P_l, k < l \), be an edge, then there are exactly \( l - k \) roots \( x = \xi_i(y) \) of \( f(x, y) = 0 \), \( 1 \leq i \leq l - k \), with \( O(\xi_i) = (\alpha_i - \alpha_k)/(l - k) \). Moreover, the first (non-zero) coefficients of the \( \xi_i \)'s are the non-zero roots of the associated equation

\[ \varphi(x) = \sum a_k x^k = 0 \tag{3.1} \]

where \( a_k \) is the leading coefficient of \( \tilde{a}_k \), and the summation is over all \( h \) for which \( P_k \) lies on \( P_l P_l \).

The details can be found in [7]. It should be noticed, however, that we are solving \( x \) in terms of \( y \), while in [7], \( y \) is solved in terms of \( x \).

Let \( x = \beta(y), 1 \leq i \leq m \) denote the roots of \( f = 0 \), and \( x = \gamma_k(y), 1 \leq k \leq m - 1 \), those of \( f_l = 0 \).

**Lemma (3.2).** (i) \( P_i \) is a vertex of the Newton Polygon of \( f \) if, and only if \( \max \{ O(\beta_i) \} > \max \{ O(\gamma_k) \} \) for some \( i \).

\[ \max \{ O(\beta_i) \} = \alpha_0 - \alpha i, \max \{ O(\gamma_k) \} = (\alpha_j - \alpha_i)/(s - 1), \] when \( P_s, s \geq 2 \), is the vertex following \( P_i \).

(ii) \( P_i \) lies on the first edge \( P_l P_l, s \geq 2 \), if, and only if \( \max \{ O(\beta_i) \} = \max \{ O(\gamma_k) \} \). In this case, \( \max \{ O(\beta_i) \} = (\alpha_0 - \alpha_i)/s \).
(iii) $P_i$ lies above the first edge $P_0P_s$, $s \geq 2$, if, and only if $\max \{O(\beta_i)\} < \max \{O(\gamma_k)\}$.

In this case $\max \{O(\beta_i)\} = (\alpha_0 - \alpha_i)/s$.

(iv) $O(f_s(O, y)) = \alpha_1$.

In fact, a term $cx^{i-1}y^j, c \neq 0$, of $f(x, y)$ yields a non-zero term $icx^{i-1}y^j$ of $f_i$, if $i \neq 0$. Hence each $P_i = (i, \alpha_i)$ gives rise to a point $Q_{i-1} = (i-1, \alpha_i)$ of the Newton Diagram of $f_s = 0$, for $i \neq 0$. Hence the Lemma follows.

**Lemma (3.3).** For $\beta_i, \beta_j, i \neq j$, there is a $\gamma_k$ such that

$$O(\beta_i - \gamma_k) = O(\beta_j - \gamma_k) = O(\beta_i - \beta_j).$$

Conversely, given $\beta_i$ and $\gamma_k$, there is also a $\beta_j$ for which (3.4) holds. Moreover, for given $\beta_i$ and $d > 0$, the number of $\beta_j$'s with $O(\beta_i - \beta_j) = d$ is the same as the number of $\gamma_k$'s with $O(\beta_i - \gamma_k) = d$.

By this lemma, one can obtain the tree-model (to be defined in §4) of $f_s = 0$ directly out of that of $f = 0$. See §4.

For a proof of Lemma 3.3, let $\beta_{ij}$ denote $\beta_j$ with its terms of degree $O(\gamma_k - \beta_i)$ replaced by $\xi_{ij}^{\alpha_0 - \gamma_k}$, $\xi$ a generic number, and all terms of higher degrees omitted. That is, $\beta_{ij}$ is the generic perturbation of $\beta_j$ at degree $O(\gamma_k - \beta_i)$. Under the change of variables

$$\bar{x} = x - \beta_{ij}(y), \quad \bar{y} = y,$$

$f(x, y)$ becomes $\bar{f}(\bar{x}, \bar{y}) = f(\bar{x} + \beta_{ij}(\bar{y}), \bar{y})$, and

$$\bar{f}(\bar{x}, \bar{y}) = \prod_{k=1}^{m} (\bar{x} - \beta_{ij}(\bar{y})).$$

where

$$\beta_{ij} = \beta_i(\bar{y}) - \beta_{ij}(\bar{y}).$$

Since $\bar{f} = f_s$, we have $f_s = \prod_{k=1}^{m} (x - \gamma_k)$ where $\gamma_k = \gamma_k(\bar{y}) - \beta_{ij}(\bar{y})$.

Now, $\xi$ being generic, $O(\beta_{ij}) = O(\beta_{ij}) \supseteq O(\beta_i)$ for all $l$. Hence $\beta_{ij}, \beta_i$ are given rise by the first edge $P_0P_s$ of the Newton Polygon of $f_i$ and we have $(\alpha_0 - \alpha_i)/s = O(\beta_i)$. Moreover, all the $s$ roots of the associated equation (see (3.1))

$$\varphi(x) = \sum a_n x^n \in \bar{P}_0 \bar{P}_s,$$

are non-zero and they are the leading coefficients of the $\beta_{ij}$ with $O(\beta_{ij}) = (\alpha_0 - \alpha_i)/s$.

On the other hand, we can choose the generic number $\xi$ so that all roots of $\frac{d \varphi}{dx} = 0$ are also non-zero. Note that $\frac{d \varphi}{dx} = 0$ is the associated equation of the first edge $\tilde{Q}_0 \tilde{Q}_{i-1}, \tilde{Q}_i = (i, \alpha_i)$, of the Newton Polygon of $f_s = 0$. There are as many roots of $\varphi = 0$ which are different from the leading coefficient of $\beta_i$, say $b_i$, as there are roots of $\frac{d \varphi}{dx} = 0$ which are different from $b_i$; each root of the later gives rise to a root $\gamma_k$ of $f_s = 0$ with $O(\gamma_k - \beta_i) = O(\beta_i)$.

Now, let $\beta_i, \gamma_k$ be given. Consider the change of variables

$$\tilde{x} = x - \mu_{\alpha_0}(y), \quad \tilde{y} = y$$

where $\mu_{\alpha_0}$ is the generic perturbation of $\beta_i$ at degree $O(\beta_i - \gamma_k)$. A similar argument shows that there is a $\beta_i$ satisfying (3.4).

**Corollary (3.5).** For any given $i$, $1 \leq i \leq m$,

$$\sum_{j \neq i} O(\beta_i - \beta_j) = \sum_{k=1}^{m-1} O(\beta_i - \gamma_k).$$

**Lemma (3.6).** Suppose $O(\gamma_k) > O(\beta_i) = \max \{O(\beta_i)\}$. Then for $|c|$ small,
Moreover, \(|f_{\partial}(cy^{a_0}, y)| \geq \varepsilon |y|^{a_0-1}, \varepsilon > 0.

Let \(P_{\alpha}P_{\beta}\) be the first edge of the Newton polygon of \(f\), and
\[
\varphi(x) - \sum a_n x^n, \quad P_n \in P_{\alpha}P_{\beta},
\]
the associated equation.

Observe that all roots of \(\varphi\) are non-zero, \(O(\beta_i) = (\alpha_o - \alpha_i)/s\), and the leading coefficient of \(\beta_i\) is a root of \(\varphi = 0\).

Now consider the form
\[
\varphi^*(x, y) = \sum a_n x^n y^{a_n}, \quad \alpha_n = \alpha_0 - h[O(\beta_i)],
\]
which consists of precisely those (non-zero) terms of \(f\) lying on \(P_{\alpha}P_{\beta}\). Observe that \(\varphi^*(t^{a_0}, y)\) is a homogeneous form of degree \(a_0\), and hence, by Euler's Theorem,
\[
O(\beta_i) x\varphi^* + y\varphi^* = \alpha_0 \varphi^*
\]
(In [5], \(\varphi^*\) is called a weighted homogeneous form.)

Since \(O(\gamma_k) > O(\beta_i)\), \(x\) is a factor of \(d\varphi/dx = 0\), which is the associated equation of an edge of the Newton Polygon of \(f\). But \(x\) is not a factor of \(\varphi = 0\). Hence, by (3.13) \(x\) is not a factor of \(\varphi^*\); in particular, for \(|c|\) small, \(x - cy^{a_0}\) does not divide \(\varphi^*\), and so
\[
|\varphi^*(cy^{a_0}, y)| \geq \varepsilon |y|^{a_0-1}.
\]
Now, the Newton Diagram of \(f\) has points at \((i, a_i - 1)\); its first edge consists of precisely those terms of \(\varphi^*\). Thus the lemma follows.

4. THE TREE-MODEL

We have assumed \(O(\beta_i - \beta_j) < \infty, i \neq j\). Call \(O(\beta_i - \beta_j)\) the order of contact of \(\beta_i, \beta_j\).

The tree-model, \(M(f)\), of \(f(x, y) = \prod_{i=1}^m (x - \beta_i)\), is constructed as follows. First, draw a vertical line segment, called the main trunk, marking the number \(m = O(f)\) by its side. Then draw a bar, \(B_i\), on top of the main trunk, marking the number \(b_i\), by \(B_i\), where \(b_i\) is the smallest number such that there are at least two roots \(\beta_i, \beta_j\) with \(O(\beta_i - \beta_j) = b_i\). Call \(b_i\) the height of \(B_i\). Let \(c_i\) denote the maximal number of roots such that any two have order of contact \(b_i\). All roots are now divided into \(c_i\) groups; in each group, any two members have order of contact \(> b_i\). Now draw \(c_i\) vertical line segments on \(B_i\), called trunks; each represents one of the above groups and then mark the number of members in each group by the side of the trunk; call them the multiplicities of the trunks. Then repeat what we did to the main trunks to each of these trunks on \(B_i\), getting more bars and then more trunks, . . . , etc. A trunk of multiplicity \(i\) is called a twig, no more bar will be drawn on top of it. Thus our construction ends at the stage where all trunks are twigs.

The multiplicity of a twig will not be marked out.

Example (4.1). \(f(x, y) = x^2 - y^3\). \(M(f)\):
\[
\begin{array}{c}
\text{3/2} \\
\text{2}
\end{array}
\]

Example (4.2). \(f(x, y) = (x^2 - y^3 + y^5) = (x - y^{2\alpha} + \cdots) (x + y^{2\alpha} + \cdots)\). The model is the same as in above example.

Note that terms in any \(\beta_i\) of degree \(> \text{Max} \{O(\beta_i - \beta_j)\}, j \neq i\), can be omitted without affecting the model.

Example (4.3). \(f(x, y) = x - y^2 + y^3 + \cdots)(x - y^2 + 2y^3 + \cdots)(x - y^2 + 3y^3 + \cdots)(x - y^2 + 4y^3 + \cdots)(x + y^2 + \cdots)(x + 2y^2 + \cdots)
\]
\[
M(f):
\begin{array}{c}
\text{3} \\
\text{4} \\
\text{6}
\end{array}
\]
Note that the multiplicity of each trunk is the total number of twigs supported by the trunk; so it is not really necessary to mark out the multiplicity of any trunk in the model.

Tracing from the main trunk up to a twig amounts to identifying a root \( \beta \). The number \( l \) for \( \beta \) in Theorem A can be computed as follows. While tracing up, we have the main trunk, of multiplicity \( \tau_1 = m \), then a bar, say of height \( b_1 \), then a trunk, say of multiplicity \( \tau_2 \), then a bar, \ldots, and finally, a bar, say of height \( b_k \), followed by a twig (thus identifying \( \beta \)). We have, by definition, the following

**Theorem C.** The numbers \( l \) in Theorem A are given by

\[
l_i = b_i \tau_i + b_{i-1}(\tau_{i-1} - \tau_i) + \cdots + b_1(\tau_1 - \tau_i)
\]

\[
= \sum_{j=1}^{i} h_j \tau_j - \sum_{j=1}^{i-1} h_j \tau_j.
\]

Of course, twigs on a same bar yield the same number \( l \).

In Example (4.3), the four highest twigs yield \( l_i = 3 \cdot 4 + 2(6 - 4) = 15 \) which is larger than the number yielded by the other two lower twigs. Hence \( a = 15 \).

A root \( \beta \) is minimal if, and only if on the highest bar, which supports the twig representing \( \beta \) there are twigs only (i.e. no trunk of multiplicity \( \geq 2 \)).

**Contact between** \( M(f) \) **and** \( M(f_\lambda) \). **Let** \( M(f) \) **be known. We shall point out in the following how to get important information about** \( M(f_\lambda) \), **without computing the roots of** \( f_\lambda \). **Although it is not possible to construct the whole picture of** \( M(f_\lambda) \) **from** \( M(f) \) **(see Example (4.4)), we can determine, from** \( M(f) \) **alone, the orders of contact between the roots of** \( f \) **and that of** \( f_\lambda \).

We first construct a new tree, denoted by \( M^*(f_\lambda) \), from \( M(f) \). Let \( B \) be a bar in \( M(f) \), and let there be \( k \) trunks on \( B \), having multiplicities \( w_i \geq 1 \), \( 1 \leq i \leq k \), (trunks of multiplicity \( w_i = 1 \) are twigs.) To construct \( M^*(f_\lambda) \), replace each trunk of multiplicity \( w_i \) on \( B \) by a trunk of multiplicity \( w_i - 1 \); note that, in particular, all twigs on \( B \) are removed; and then add \((k-1)\) new twigs on \( B \). These will be trunks and twigs of \( M^*(f_\lambda) \). The height of \( B \), as a bar of \( M^*(f_\lambda) \), is the same as that of \( M(f) \). Special attention should be paid to a bar which supports only two twigs (and no trunk), such a bar disappears in \( M^*(f_\lambda) \).

It is worthwhile to draw \( M(f), M^*(f_\lambda) \) together, with all twigs of \( M^*(f_\lambda) \) indicated by dotted segment, not coinciding with any twig or trunk of \( M(f) \).

Here are some illustrative examples:
Note that the order of contact between a twig of \(M(f)\) and one of \(M^*(f_2)\) can be read out directly from the model.

An important observation is this: By Lemma (3.3), the model \(M(f_2)\) is obtained by collecting certain twigs of \(M^*(f_2)\), which lie on a same bar, to form a trunk, and then draw more bars and trunks etc. From \(M(f)\) we are unable to tell which twigs of \(M(f)\) should be collected together to form a trunk of \(M(f_2)\). But as far as the order of contact between a twig of \(M(f)\) and one of \(M^*(f_2)\) is concerned, this does not matter. To sum up, twigs of \(M^*(f_2)\) and \(M^*(f_2)\) are in one-one correspondence, the order of contact between a twig of \(M(f)\) and one of \(M^*(f_2)\) equals that of the same twig of \(M(f)\) and the corresponding one of \(M^*(f_2)\).

Example (4.4). \(f(x,y) = x^3 - y^3 + xy^4\); \(g(x,y) = x^3 - y^3 - x^2y^2\).

\[
\begin{array}{c|c|c|c|c}
M(f) - M(g) & M^*(f) = M^*(g) & M(f_2) & M(g_2) \\
\end{array}
\]

\[\text{§5. COMPLEX HORN-NEIGHBORHOOD}\]

Let \(\xi(v) = \sum a_i v^{n_i}\) be a fractional power series. Since fractional powers of a complex variable \(v\) is not well-defined, an inequality like \(|u - \sum a_i v^{n_i}| \leq w|v|^d\), which is used in defining horn-neighborhood in the real case [3], is meaningless.

We put it right as follows. Let \(\Gamma: \mathbb{C}^2 \to \mathbb{C}^2\) be the covering map defined by \(u = u, v = t^e\), where \((u, v)\) and \((u, v)\) denote the coordinate systems in the domain and range respectively. Let \(\xi^{-1}\) denote the analytic curve \(u = \sum a_i t^{n_i}\) in the \((u, t)\)-space, and for \(d > 0, w > 0\), consider the complex neighborhood of \(\xi^{-1}\)

\[
H_d(\xi^{-1}; w) = \left\{(u, t) \mid |u - \sum a_i t^{n_i}| \leq w|t|^d\right\},
\]

which is well-defined.

Note that when \(d\) is an integer, \((u, t) \in H_d\) if, and only if

\[
u - \sum a_i t^{n_i} = \omega t^d,
\]

where \(\omega \in \mathbb{C}, |\omega| \leq w\).

Now, in the \((u, v)\)-space, define the horn-neighborhood of \(\xi\) of degree \(d\), width \(w\), as

\[
H_d(\xi; w) = \Gamma(H_{ad}(\xi^{-1}; w))
\]

There is a minor difference between complex and real horn-neighborhoods. Let \(e\) denote an \(n\)th primitive root of unity, and \(\xi_e(v) = \sum a_i e^{n_i} v^{n_i}\). The \(\xi^{-1} = \xi^{-1}\) and hence \(H_d(\xi; w) = H_d(\xi; w)\). Therefore, the roots of \(f(u, v) = 0\) belonging to a same place have identical horn-neighborhood. Thus, for example, \(u^2 - v^3 = 0\) has two roots (but only one place) \(u = \pm v^{3/2}\); their horn-neighborhoods are identical.

The relation between roots and places is referred to in [7, p. 107, Theorem (4.1)].

\[\text{§6. PROOF OF THEOREM A}\]

The roots of \(f = 0\) are \(\beta_i\). Let \(\gamma_k, 1 \leq k \leq m - 1\), denote the roots of \(f_\gamma = 0\). Let \(n\) denote the l.c.m. of the denominators of the fractional powers appearing in all \(\beta_i\) and \(\gamma_k\).

For each \(\gamma_k\), choose \(i\) so that \(O(\beta_i - \gamma_k) = \max \{O(\beta_i - \gamma_k)\}\) and then choose \(j\) so that (3.4) holds.

Let \(d_k = O(\beta_i - \gamma_k)\), and consider the complex horn-neighborhood \(H_{d_k}(\gamma_k; w)\), where \(w > 0\) is a fixed sufficiently small constant throughout this paper.

Choose a parametric representation of \(\gamma_k\),

\[
u = \sum b_i t^{n_i}, \quad v = t^e
\]

where \(n \leq n_1 < n_2 < \cdots\), (they may have a common factor). And we shall examine
306 TZEE-CHAR KUO AND YUNG-CHEN LU

\[ \text{Grad}_{u \in \mathcal{O}} f(u, t^*) \text{ in } H_{\mathcal{O}}(\gamma_k^{-1}; w). \]

In the \((u, t)\)-space, consider the coordinate transformation

\[ \tilde{u} = u - \sum b_i t^i, \quad \tilde{t} = t \]

which is analytic, and under which the curve \(\gamma_k^{-1}\) becomes the \(t\)-axis, while \(H_{\mathcal{O}}(\gamma_k^{-1}; w)\) becomes \(H_{\mathcal{O}} = \{(u, t); |\tilde{u}| \equiv w|\tilde{t}|^M\}. \) Let the vertices of the Newton Diagram of \(f(u, t)\) be denoted by \(\hat{P}_i = (\tilde{u}_i, \tilde{t}_i)\).

Since \(f_0 = 0\) along \(u = 0\), \(\hat{P}_i = (1, \infty)\) and \(\hat{P}_i\) does not appear in the diagram. Let \(\hat{P}_0, s \geq 2\), be the second vertex of the first edge \(\hat{P}_0\hat{P}_s\). Then, by Lemma (3.2),

\[ (\tilde{a_0} - \tilde{a}_s)/s = \text{Max}_{1 \leq i, j \leq m} \{O(\hat{b}_i, \hat{b}_j)\} = nd_k \]

where \(\tilde{b}_i\) is the image of \(b_i(t^*)\) under (6.1). Now, the first vertex of \(f_i\) has coordinates \((0, \tilde{a}_0 - 1)\); and, by Lemma (3.6),

\[ f_i(\tilde{u}, \tilde{t}) \geq \tilde{e}_i |\tilde{t}|^{n_i - 1}, \quad (\tilde{u}, \tilde{t}) \in \hat{H}_{\mathcal{O}}. \]

It remains to compute \(\tilde{a}_0\). Considering \(v^{1/n}\) as an indeterminant, and \(u^* = u - \sum b_i v^{n_i} \), \(f\) becomes a function in \(u^*\) and \(v^{1/n}\), whose Newton Diagram has points \(P_i^* = (i, a_i^*), \) where \(a_i^* = (1/n) \tilde{a}_i. \) Moreover, \(O(f(\tilde{b}_i, \tilde{b}_j), v)) = a_i^* = (1/n) \tilde{a}_0. \) Hence \(l_i = a_i^* \leq \alpha + 1,\) and by (6.3),

\[ |f_i| \geq (1/n) |(1/|t^*|^{n_i - 1})|f_i| \geq \text{Max}_{1 \leq i, j \leq m} \{O(\hat{b}_i, \hat{b}_j)\} = \text{Max}_{1 \leq i, j \leq m} \{O(\hat{b}_i, \hat{b}_j)\} = nd_k \]

and (1.1) follows.

To complete the proof of Theorem A, it remains to establish (1.1) outside of \(U \cup \bigcup_{k=1}^{m} H_{\mathcal{O}}(\gamma_k ; w). \)

Morally, using the Curve Selection Lemma, (which is not available in the complex case,) it suffices to establish (1.1) along any arc \(\lambda: u = \sum \alpha_i t^i, \quad v = t. \) In case \(O(\lambda - \gamma_k) \leq \text{Max}_{1 \leq i, j \leq m} \{O(\hat{b}_i, \hat{b}_j)\} \) for some \(k, \Gamma(\lambda) \) lies in \(U \cup \bigcup_{k=1}^{m} H_{\mathcal{O}}(\gamma_k; w),\) and then we have (1.1). In case \(O(\lambda - \gamma_k) \geq \text{Max}_{1 \leq i, j \leq m} \{O(\hat{b}_i, \hat{b}_j)\} \) for some \(i,\) we perform a change of variables, \(\tilde{u} = u - \beta_i, \tilde{v} = v,\) and then apply Lemma (3.2) to establish (1.1) along the \(v\)-axis.

The rigorous proof proceeds as follows. As we do not have the Curve Selection Lemma at our disposal, we shall consider a finite family of horn-neighborhoods of \(\beta_i, \{H_{\mathcal{O}}(\beta_i; w)\}, \) where \(1 \leq i \leq m, e_i = O(\beta_i - \beta_i). \) Let

\[ \hat{H}_{\mathcal{O}_i} = H_{\mathcal{O}_i} - \bigcup_{k=1}^{m} H_{\mathcal{O}}(\gamma_k; w) - \bigcup_{r,s} H_{\mathcal{O}}(\beta_r; w) \]

where \(r,s\) run through all indices such that \(H_{\mathcal{O}_i}\) is a proper subset of \(H_{\mathcal{O}}.\)

It then suffices to establish (1.1) in each \(\hat{H}_{\mathcal{O}_i}. \) Given \(\hat{H}_{\mathcal{O}_i}\) and any \(\gamma_k,\) either \(O(\beta_i - \gamma_k) \leq e_i,\) or \(O(\beta_i - \gamma_k) > e_i. \) In the former case, \(H_{\mathcal{O}_i} \cap H_{\mathcal{O}}(\beta_i - \gamma_k) = \{0\}\) since \(w\) is sufficiently small. Then, for \((u, v) \in H_{\mathcal{O}_i},\)

\[ |u - \gamma_k(v)| \geq w|v|^{O(\beta_i - \gamma_k)}, \quad (u, v) \in \hat{H}_{\mathcal{O}_i}. \]

that is,

\[ |u - \gamma_k(t^*)| \geq w|t|^{O(\beta_i - \gamma_k)}. \]

In the latter case, choose \(l \neq i\) such that

\[ O(\beta_i - \gamma_k) = \text{Max}_{1 \leq i, j \leq m} \{O(\beta_i - \gamma_j)\}. \]

By Lemma (3.3), there exists \(l\) such that \(O(\beta_i - \gamma_k) = \text{Max}_{1 \leq i, j \leq m} \{O(\beta_i - \gamma_j)\}. \)

In case \(O(\beta_i - \gamma_k) = O(\beta_i - \gamma_k),\) we have \(d_i = O(\beta_i - \gamma_k)\) and so

\[ |u - \gamma_k(v)| \geq w|v|^{O(\beta_i - \gamma_k)}, \quad (u, v) \in \hat{H}_{\mathcal{O}_i}. \]

In case \(O(\beta_i - \gamma_k) > O(\beta_i - \gamma_k),\) we have \(O(\beta_i - \gamma_k) = O(\beta_i - \beta_i)\) and \(H_{\mathcal{O}}(\gamma_k; w) = H_{\mathcal{O}}(\beta_i - \gamma_k; w) \subset H_{\mathcal{O}}(\beta_i; w).\)

Note that the last set \(H_{\mathcal{O}_i}\) is a proper subset of \(H_{\mathcal{O}}. \) Hence, for \((u, v) \in \hat{H}_{\mathcal{O}_i},\) we still have

\[ |u - \gamma_k(v)| \geq w|v|^{O(\beta_i - \gamma_k)} \]
Combining (6.4), (6.6), and (6.7), we find that for \((u, v) \in \hat{H}_{nv}, |f_\nu(u, v)| \geq \varepsilon |v|^p\), where
\[ p = \sum_{i=1}^{m-1} O(\beta_i - \gamma_i). \]

Now,
\[ p = \sum_{i \neq i} O(\beta_i - \beta_i) = \sum_{i \neq i} O(g_\nu(\beta_i) - \beta_i) = \sum_{i \neq i} O(g_\nu(\beta_i) - \beta_i) - c_i = l - c; \leq l_1 - 1, \]
where the first equality follows from Corollary (3.5); the 2nd and 3rd follow by definition.

Thus (1.1) holds in each \(\hat{H}_{nv}\) and hence in \(\bigcup_{ij} H_{nv}\).

In the complement of \(\bigcup_{ij} H_{nv} \bigcup (\bigcup_{ij} H_{\alpha}(\gamma_i; w))\), we have \(|f_\nu(u, v)| \geq \varepsilon r^n\), where \(\rho = \sqrt{(|u|^2 + |v|^2)}\), \(q = \text{Min } \{l_i\}\).

The proof of Theorem A is now complete.

\section*{7. PROOF OF THEOREM A'}

Let a root \(\beta_i\) be not minimal. Choose \(\beta_n, \beta_s\) such that
\[ O(\beta_i - \beta_i) = \max_{j \neq i} \{O(\beta_i - \beta_j)\} = c, \]
and
\[ O(\beta_n - \beta_s) > O(\beta_i - \beta_i). \]

It suffices to show that \(l_n > l_1\). The roots of \(f\) can be divided into two classes. The first consists of all \(\beta_i\) satisfying
\[ O(\beta_i - \beta_i) = O(\beta_n - \beta_s) = O(\beta_i - \beta_i), \]
such as shown in Fig. 1.

The second consists of all \(\beta_i\) for which
\[ O(\beta_i - \beta_i) < O(\beta_n - \beta_s) \]
such as shown in Fig. 2.

Now, for \(\beta_i\) in the first class,
\[ O(g_\nu(\beta_i) - \beta_i) = O(g_\nu(\beta_n) - \beta_s) \]
and for \(\beta_i\) in the second class,
\[ O(g_\nu(\beta_i) - \beta_i) < O(g_\nu(\beta_n) - \beta_s). \]
Hence \(l_n > l_1\), proving Theorem A'.
PROOF OF THEOREM B

For a branch $P$, whose roots are expressed in (2.4), let us consider the sequence

$$\mathcal{H}(h): (1, \ldots, 1; e^{h\rho q_{1}/p_1}, \ldots, e^{h\rho q_{k}/p_{k_1}}, \ldots, e^{h\rho q_{k}/p_{k_2}}, \ldots)$$

(8.1)

The $p$ roots of $P$ are in one-one correspondence with $\mathcal{H}(h)$ for $0 \leq h \leq p - 1$. Note that for any $s$, $e^{h\rho q_{1}/p_1} \ldots p_i$, $0 \leq h \leq p - 1$, are $(p_1 \ldots p_i)$th roots of unity (with repetitions). Moreover,

$$e^{h\rho q_{1}/p_1} = e^{h\rho q_{1}/p_1} \ldots p_i = e^{h\rho q_{1}/p_1} \ldots p_i$$

hold simultaneously if, and only if

$$h_1 = h_2 \mod p_1 \ldots p_s$$

(8.2)

and in this case, we have

$$e^{h \rho q_{1}/p_1} \ldots p_i - e^{h \rho q_{1}/p_1} \ldots p_i, \quad \text{for } 1 \leq j \leq s, \quad 0 \leq i \leq k.$$ (8.3)

Therefore $\mathcal{H}(h)$, $0 \leq h \leq p - 1$, are in one-one correspondence with the sequences

$$(e^{h\rho q_{1}/p_1}, e^{h\rho q_{1}/p_1} \ldots p_i, \ldots, e^{h\rho q_{1}/p_1} \ldots p_s), \quad 0 \leq h \leq p - 1,$$

which will be denoted by the same symbol $\mathcal{H}(h)$.

The above argument yields in particular the tree-model $M(P)$ of the branch $P$. It has the following simple form: The main trunk has multiplicity $p$, the bar on top of it has height $q_1/p_1$; there are then $p_1$ trunks on top of this bar, each having multiplicity $p^{(1)}$, where $p = p^{(1)} p_1$; on each of these $p_1$ trunks, there is a bar of height $q_1/p_1 p_2$; then on each of these $p_1$ bars, there are $p_2$ trunks, all of multiplicity $p^{(2)}$, where $p_2 = p^{(2)} p_2$; finally, the highest bars are those having height $q_1/p_1 \ldots p_s$, on top of each there are $p_s$ twigs. The construction of $M(P)$ is then finished.

A typical case is shown in Fig. 3 (where $g = 2$).

Let $\bar{P}$ be another branch consisting of roots $\bar{\beta}_1, \ldots, \bar{\beta}_{p_1}$, with characteristic pairs \{(0, $\bar{q}_i$), \ldots, ($\bar{q}_s$, $\bar{q}_d$). Put $C(P, \bar{P}) = \max_{i,j} \{O(\beta, \bar{\beta}_i)\}$, where $\beta_i \in P$, $\bar{\beta}_i \in \bar{P}$, and then choose largest $\nu$, satisfying

$$C(P, \bar{P}) > q_1/p_1 \ldots p_\nu.$$ (8.4)

It follows that $\bar{p}_i = p_i$, $\bar{q}_i = q_i$ for $1 \leq j \leq \nu$, and that the tree-model $M(\bar{P})$ of $\bar{P}$ is the same as $M(P)$ up to the bars of heights $\leq q_j/p_1 \ldots p_\nu$, except that the trunks of $M(\bar{P})$ have multiplicities $\bar{p}_i$, $\bar{p}^{(1)}$, $\bar{p}^{(2)}$, etc. Let us consider the product

$$P\bar{P} = \prod (x - \beta_i) \cdot \prod (x - \bar{\beta}_i).$$

For the same reason, its tree-model $M(P\bar{P})$ is the same as $M(P)$ (and $M(\bar{P})$) up to bars of heights $\leq q_j/p_1 \ldots p_\nu$, while its trunks have multiplicities $p + \bar{p}_i$, $p^{(1)} + \bar{p}^{(1)}$, $p^{(2)} + \bar{p}^{(2)}$, etc. On the other hand, on each bar of height $q_j/p_1 \ldots p_\nu$, there are $p_\nu$ trunks, among them $p_\nu$ have multiplicity $p^{(\nu)}$, and $\bar{p}_\nu$ have multiplicity $\bar{p}^{(\nu)}$, they represent trunks of $P$ and $\bar{P}$ respectively. It is on these bars where $M(P)$ and $M(\bar{P})$ begin to separate.

Let $d_i = \sum_{j=1}^{\nu} O(\beta_i - \bar{\beta}_i), 1 \leq i \leq \nu$. Then, by looking at $M(P\bar{P})$ constructed above, we have
where $\mathcal{N}, \tilde{\mathcal{N}}$ are the torus knots of $P, \tilde{P}$ respectively.

We are now ready to prove Theorem B.

Let $\beta_i$ be a minimal root. By permuting the indices if necessary, we may assume $\beta_i \in P_i$. Then

$$l_i = O(f_i(\beta_i, \nu)) = \sum_{i=1}^{d_i} O(g_{\beta_i}(\xi_i) - \beta_i)$$

$$= \sum_a O(g_{\beta_a}(\beta_a) - \beta_a) + \sum_b O(g_{\beta_b}(\beta_b) - \beta_b)$$

where $a$ (respectively $b$) runs through all indices for which $\beta_a \in P_i$ (respectively $\beta_b \in P_i$). Note that

$$\sum_a O(g_{\beta_a}(\beta_a) - \beta_a) = \sum_b O(\beta_b - \beta_b) = (1/O(P_i))$$

$$\times \sum_{i=1}^{d_i} \mathcal{L}(\mathcal{N}_i, \tilde{\mathcal{N}}_i). \quad (8.4)$$

To complete the proof, it remains to show that

$$O(P_i) \sum_{i=1}^{d_i} O(g_{\beta_i}(\beta_i) - \beta_a) = \operatorname{Max}_{i \neq j \leq m} \{ \sigma_i, \rho_i \}. \quad (8.5)$$

Choose $e$ so that

$$O(\beta_i - \beta_a) = c_i = \operatorname{Max}_{i \neq j \leq m} \{ O(\beta_i - \beta_a) \}, \quad 1 \leq j \leq m.$$ 

There are two possibilities:

$$\beta_a \in P_i \text{ and } \beta_i \in P_i. \quad (8.6)$$

First, suppose $\beta_a \in P_i$. Then

$$c_i = O(\beta_i - \beta_a) = q_a/p_1 \ldots p_{e}$$

and so

$$\sum_a O(g_{\beta_a}(\beta_a) - \beta_a) = (1/O(P_i)) \mathcal{L}(\mathcal{N}_i, \tilde{\mathcal{N}}_i) = (1/O(P_i)) \sigma_i. \quad (8.8)$$

Let $\tilde{P}$ be any branch of $f$, whose torus knot, $\tilde{\mathcal{N}}$, is isotopic to $\mathcal{N}$. Since the characteristic pairs are topological invariants [8, p. 13], $P, \tilde{P}$ have same characteristic pairs:

$$\tilde{p}_1 = p_1, \tilde{q}_1 = q_1, \ldots, \tilde{p}_e = p_e, \tilde{q}_e = q_e : (\tilde{q} = q),$$

and in particular

$$O(P_i) = O(\tilde{P}) = p = \tilde{p}. \quad (8.9)$$

Now, by the above consideration on the tree-models, we find

$$(1/O(P_i)) \mathcal{L}(\mathcal{N}_i, \tilde{\mathcal{N}}) = \sum_{\beta_a \in P_i} O(\beta_i - \beta_a) = \sum_{\beta_a \in P_i} O(g_{\beta_a}(\beta_a) - \beta_a)$$

$$\leq \sum_{\beta_a \in P_i} O(g_{\beta_a}(\beta_a) - \beta_a) = (1/O(P_i)) \sigma_i. \quad (8.10)$$

where the inequality in the middle can be seen from the tree-model $M(P, \tilde{P})$, taking into account of (8.9). Hence $\sigma_i \geq \rho_i$, and (8.5) follows.

Now suppose $\beta_i \in P_i$. Then $O(\beta_i - \beta_a) \geq q_a/p_1 \ldots p_{e}$. By (2.2), $q_a/p_1 \ldots p_{e}$ is also the height of the highest bars of $M(\tilde{P})$ where $\tilde{P}$ is the branch containing $\beta_a$. Hence the characteristic pairs...
of $P$, and $\tilde{P}$ coincide (in particular $O(P) = O(\tilde{P})$), and their torus knots $N$, $\tilde{N}$ are isotopic. Next we have

$$\sum_{p \in P} O(\beta_i - \tilde{\beta}_i) = \sum_{i} O(g_i(\beta_i) - \tilde{\beta}_i) \geq \sum_{p \in P} O(g_i(\beta_i) - \beta_p).$$

(8.11)

Therefore, $L(N, \tilde{N}) \geq \sigma, \rho \geq \sigma$. Now, since $\beta_i$ is minimal,

$$O(\beta_i, \beta_i) = O(\beta_i - \beta_i)$$

for any $\beta_i \in P_i$. Hence

$$O(P) \sum_{i} O(g_i(\beta_i) - \beta_i) = L(N, \tilde{N}).$$

On the other hand, by the choice of $\beta_i$, we find $L(N, \tilde{N}) = \rho$. Hence (8.5) follows.

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