Functorial Representation Theorems for $\mathbb{MV}_\Delta$ Algebras with Additional Operators

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We prove functorial representation theorems for $\mathbb{MV}_\Delta$ algebras, and for varieties obtained from $\mathbb{MV}_\Delta$ algebras by the adding of additional operators corresponding to natural operations in the real interval $[0, 1]$, namely $\mathbb{PMV}_\Delta$ algebras, obtained by the adding of product, and $\mathbb{LII}$ algebras, obtained by the adding of product and of its residuum. Our first result is that the category of $\mathbb{MV}_\Delta$ algebras is equivalent to that of lattice ordered abelian groups with strong unit and with some kind of characteristic function. Our second result is that the category of $\mathbb{PMV}_\Delta$ algebras is equivalent to a category of commutative f-rings with strong unit, again with a suitable characteristic function, whose members (modulo a forgetful functor) are isomorphic to subdirect products of linearly ordered domains of integrity. Our third result (in our opinion, the most interesting) is that the category of $\mathbb{LII}$ algebras is equivalent to a category whose members are regular commutative f-rings with unit, equipped with an ideal with suitable properties.

Key Words: fuzzy logic; fuzzy algebras; functorial equivalence.

1. INTRODUCTION

The three most important fuzzy logics (Łukasiewicz logic, Gödel logic, and product logic) arise from the three basic t-norms on the real interval $[0, 1]$ and from the corresponding residua (Łukasiewicz implication, Gödel implication, and product implication), which will be reviewed in Section 2. Łukasiewicz logic is probably the most important of these logics. However, we believe that the study of the relationships between the above-mentioned connectives contributes to a better understanding of fuzzy logic. This line of research has been pursued by many authors (cf. [H98, TT92]). In [EG98], [Mo98], and [EGM98], the authors introduce a formal system, called $\mathbb{LII}$, and the corresponding variety, called the variety of $\mathbb{LII}$ algebras.
in which Łukasiewicz logic, Gödel logic, and product logic are faithfully interpretable. In [Mo98] and [EGM98] the authors investigate an extension of ŁΠ obtained by the adding of a constant $\frac{1}{2}$ for one half, thus obtaining a logic $\check{\text{ŁΠ}}_{\frac{1}{2}}$ in which rational Pavelka logic and rational product logic are also interpretable. Thus $\check{\text{ŁΠ}}_{\frac{1}{2}}$ seems to be the strongest natural fuzzy logic.

A very general project of research would be the investigation of all systems lying in the interval whose extrema are Hájek basic logic $\text{BL}$, the logic of all continuous t-norms and their residua, hence the weakest natural fuzzy logic, and $\text{ŁΠ}_{\frac{1}{2}}$, the strongest natural system of propositional fuzzy logic. As a starting point, we plan to deal with the easiest (but still far from trivial) part of the project, namely the investigation of the most important fuzzy logics in the final segment of the above interval, whose extrema are Łukasiewicz logic plus Baaz operator $\Delta$ (cf. [Ba96]) and $\text{ŁΠ}_{\frac{1}{2}}$.

The operator $\Delta$ results from a combination of Łukasiewicz negation and Product negation inducings a negation $\neg$ letting $\neg x = x \to 0$. In the real interval $[0, 1]$, $\Delta(x)$ is defined by $\Delta(x) = 1$ if $x = 1$, and $\Delta(x) = 0$ otherwise. We believe that such logics are easier than the other ones because classical logic can be interpreted in any such logic. This increases the expressive power of the logic considerably. A relevant example, considered in [Mo98], is constituted by Łukasiewicz logic with product, but without product implication. At the moment, no axiomatization is known for this logic. In fact, axiomatizing this logic amounts to axiomatizing the subvariety of the variety of commutative $f$-rings with unit generated by linearly ordered integral domains, and it follows from [K95] that such a logic is not finitely axiomatizable. The problem with this logic is that we cannot express by means of equations the fact that there are no zero divisors. In the presence of $\Delta$, it is possible to overcome this difficulty, as shown, e.g., in [EGM98] (cf. also Section 2 of the present paper).

Since Gödel Logic is already interpretable in Łukasiewicz logic plus $\Delta$, there are four relevant logics between Łukasiewicz logic plus $\Delta$ and $\text{ŁΠ}_{\frac{1}{2}}$: Łukasiewicz logic plus $\Delta$: Łukasiewicz logic plus $\Delta$ plus product conjunction (but without product implication); $\text{ŁΠ}_{\frac{1}{2}}$, i.e., Łukasiewicz logic plus product logic (in which both $\Delta$ and the connectives of Gödel logic are definable); and $\check{\text{ŁΠ}}_{\frac{1}{2}}$.

In this paper, we give functorial representation theorems for the varieties corresponding to these logics. In algebraic logic, a representation theorem is usually a theorem that establishes an equivalence between a category of algebras arising from logic, and another category, possibly arising from other fields of mathematics. For example, Stone’s Theorem shows an equivalence between Boolean algebras, which arise from classical logic, and Stone spaces, which have a topological interest. Mundici’s Theorem [Mu86] establishes an equivalence between the category of $MV$ algebras, the algebras corresponding to Łukasiewicz logic, and the category
of lattice-ordered abelian groups with strong unit, a category which is relevant in many fields of mathematics.

In [Mo98], an analogous result was given for ŁΠ\(^1_2\) algebras, i.e., the algebras corresponding to the logic ŁΠ\(^1_2\). There, it is shown that the category of ŁΠ\(^1_2\) algebras is equivalent to that of the so-called f-semifields, namely, regular commutative f-rings with unit and with an explicit seminverse operation. Somewhat surprisingly, these structures constitute a variety, whose elements are isomorphic to subdirect products of linearly ordered fields.

In the present paper we prove similar representation theorems for the MV\(_\Delta\) algebras, for the PMV\(_\Delta\) algebras, and for the ŁΠ algebras. The category corresponding to MV\(_\Delta\) algebras is the category of the so-called \(\Delta\)-l-groups with strong unit, i.e., of lattice-ordered abelian groups with strong unit and with an additional unary operator \(\delta\) which plays the role of \(\Delta\). This result is certainly not surprising and, as observed by the referee, can be easily derived from [Mu86] and from [To89].

A similar result holds for PMV\(_\Delta\) algebras and the so-called \(\delta\)-f-rings with strong unit, i.e., commutative f-rings with strong unit with an additional unary operator \(\delta\), which once again plays the role of \(\Delta\).

The third result concerns a representation theorem for ŁΠ algebras. This result is conceptually more difficult and, in our opinion, more surprising. Indeed, even though ŁΠ algebras can be thought of as substructures of ŁΠ\(^1_2\) algebras (modulo a forgetful functor), they can be also obtained from ŁΠ\(^1_2\) algebras by a more complicated construction, involving ideals. This construction leads to our final result that the category of ŁΠ algebras is equivalent to that of f-semifields equipped by an ideal with suitable properties. Thus, even though ŁΠ algebras are easier to describe than ŁΠ\(^1_2\) algebras, the corresponding algebraic category is more complicated than the one corresponding to ŁΠ\(^1_2\) algebras.

2. PRELIMINARIES

Almost all the results of this section are proved in [Mo98]. However, for some of them we sketch a simpler proof, based on the theory of discriminator varieties [BS81].

We consider the real interval [0, 1] equipped with the operations \(\oplus\), \(\neg\), \(\cdot\), and \(\div\), where \(x \oplus y = \min(x + y, 0)\); \(\neg x = 1 - x\); \(\cdot\) is ordinary product, and \(\div\) is defined by

\[
x \div y = \begin{cases} 
1 & x \leq y \\
\frac{y}{x} & \text{otherwise}.
\end{cases}
\]
DEFINITION 2.1. We define

\[
x \rightarrow y = (\neg x \oplus y) \quad x \circ y = \neg (x \rightarrow \neg y)
\]

\[
x \oplus y = \neg (x \rightarrow y) \quad |x - y| = (x \circ y) \oplus (y \circ x)
\]

\[
x \lor y = x \oplus (y \circ x) \quad x \land y = x \oplus (x \circ y)
\]

\[
\neg_p x = x \rightarrow_p 0 \quad \Delta(x) = \neg_p \neg x \quad \nabla(x) = \neg_p \neg_p(x)
\]

\[
x/y = y \rightarrow_p x.
\]

We have

\[
x \rightarrow y = \min\{1 - x + y, 1\} \quad x \circ y = \max\{x + y - 1, 0\}
\]

\[
x \oplus y = \max\{x - y, 0\} \quad |x - y| = \max\{x - y, y - x\}
\]

\[
x \lor y = \max\{x, y\} \quad x \land y = \min\{x, y\}
\]

\[
\neg_p x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases} \quad \Delta(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\nabla(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases} \quad x/y = \begin{cases} x & \text{if } x \geq y \\ y & \text{otherwise.} \end{cases}
\]

In [H98], the algebras of the variety generated by \( \mathcal{J}_0 = \langle [0, 1], \oplus, \neg, \Delta, 0, 1 \rangle \) are called \( \text{MV}_\Delta \) \( \text{algebras} \). There, the author proves that such variety is axiomatized by

(i) The axioms of \( \text{MV} \) algebras (cf. [COM95]).

(ii) \( \Delta(1) = 1 \).

(iii) \( \Delta(x) \land \Delta(x \rightarrow y) \leq \Delta(y) \).

(iv) \( \Delta(x) \leq x \).

(v) \( \Delta(x) \lor \neg \Delta(x) = 1 \).

(vi) \( \Delta(\Delta(x)) = \Delta(x) \).

The algebras of the variety generated by \( \mathcal{J}_1 = \langle [0, 1], \oplus, \cdot, \neg, \Delta, 0, 1 \rangle \) are called \( \text{PMV}_\Delta \) \( \text{algebras} \). In [Mo98] and in [EGM98], the algebras of the variety generated by the structure \( \mathcal{J} = \langle [0, 1], \oplus, \neg, \cdot, \rightarrow_p, 0, 1 \rangle \) are called \( \text{L}_{\Pi} \) \( \text{algebras} \), and the algebras of the variety generated by \( \mathcal{J}_2 = \langle [0, 1], \oplus, \neg, \cdot, \rightarrow_p, \frac{1}{2}, 0, 1 \rangle \) are called \( \text{L}_{\Pi}^{\frac{1}{2}} \) \( \text{algebras} \).
PROPOSITION 2.2 [EGM98]. The variety of \( \mathcal{LII} \) algebras is axiomatized by the following identities:

(i) Those of \( MV^1_\Delta \) algebras (cf. [H98]).

(ii) Identities of commutative monoids for \( \cdot \) and 1.

(iii) \( x \cdot (y \ominus z) = x \cdot y \ominus x \cdot z \).

(iv) \( \Delta(x \rightarrow (y \rightarrow_p z)) = \Delta((x \cdot y) \rightarrow z) \).

(v) \( \Delta(- (x \cdot y)) = \Delta(- x) \lor \Delta(- y) \).

(vi) \( (\Delta(x \leftrightarrow y) \land \Delta(u \leftrightarrow v)) \leq ((x \cdot u) \leftrightarrow (y \cdot v)) \).

(vii) \( (\Delta(x \leftrightarrow y) \land \Delta(u \leftrightarrow v)) \leq ((x \rightarrow_p u) \leftrightarrow (y \rightarrow_p v)) \).

Moreover, the variety of \( \mathcal{LII}^1_\Delta \) algebras is axiomatized by (i) \cdots (vii) plus

(viii) \( \frac{1}{\frac{1}{2}} = -\frac{1}{2} \).

Note that axioms (vi) and (vii) imply that any \( \mathcal{LII} \) algebra has the same congruences as its underlying \( MV^1_\Delta \) algebra. Axiom (iv) says that \( \rightarrow_p \) is the residuum of \( \cdot \), and axiom (v) says that every subdirectly irreducible algebra has no zero divisors (with respect to \( \cdot \)). Finally, axiom (iii) implies that whenever \( x \odot y = 0 \), \( (x \odot y) \cdot z = (x \cdot z) \oplus (y \cdot z) \). By the same means used in [EGM98] and in [Mo98], it is easy to prove

PROPOSITION 2.3. \( PMV^1_\Delta \) algebras are axiomatized by axioms (i), (ii), (iii), (v), and (vi) of Proposition 2.2.

In the sequel, with reference to \( MV^1_\Delta \), \( PMV^1_\Delta \), \( \mathcal{LII} \), and \( \mathcal{LII}^1_\Delta \) algebras, respectively, we will tacitly use the fact that any identity which is valid in \( \mathcal{R}_0 \) (\( \mathcal{R}^i \), \( \mathcal{R} \), and \( \mathcal{R}^1_\mathcal{L} \), respectively) is valid in every algebra of the corresponding variety.

In [H98], the author shows that every \( MV^1_\Delta \) algebra is isomorphic to a subdirect product of linearly ordered \( MV^1_\Delta \) algebras, and that in any linearly ordered \( MV^1_\Delta \) algebra one has \( \Delta(x) = 1 \) if \( x = 1 \), and \( \Delta(x) = 0 \) otherwise. Moreover, a \( MV^1_\Delta \) algebra is subdirectly irreducible iff it is linearly ordered. Since every \( PMV^1_\Delta \) algebra (\( \mathcal{LII} \), \( \mathcal{LII}^1_\Delta \) algebra, respectively) has the same congruences as the underlying \( MV^1_\Delta \) algebra [Mo98], the same results hold for \( PMV^1_\Delta \) algebras, for \( \mathcal{LII} \) algebras, and for \( \mathcal{LII}^1_\Delta \) algebras as well. Now all linearly ordered \( MV^1_\Delta \) (\( PMV^1_\Delta \), \( \mathcal{LII} \), \( \mathcal{LII}^1_\Delta \), respectively) algebras have the same discriminator function, namely \( d(x, y, z) = (\Delta(x \leftrightarrow y) \land z) \lor (\neg \Delta(x \leftrightarrow y) \land x) \). It follows that \( MV^1_\Delta \) algebras, \( PMV^1_\Delta \) algebras, \( \mathcal{LII} \) algebras, and \( \mathcal{LII}^1_\Delta \) algebras are discriminator varieties [BS81]. Hence these varieties share all properties of discriminator varieties. In particular, they can be represented as Boolean products [BS81, p. 165], and open formulas can be written by means of equations [MK75, p. 188].
PROPOSITION 2.4. (i) Every MV\(_{\Delta}\) (PMV\(_{\Delta}\), ŁII, ŁII\(_{\frac{1}{2}}\)) algebra is isomorphic with a Boolean product of simple (hence subdirectly irreducible, hence linearly ordered) MV\(_{\Delta}\) (PMV\(_{\Delta}\), ŁII, ŁII\(_{\frac{1}{2}}\)) algebras.

(ii) For every open first-order formula \(\varphi(x_1, \ldots, x_n)\) of the language of MV\(_{\Delta}\) (PMV\(_{\Delta}\), ŁII, ŁII\(_{\frac{1}{2}}\)) algebras there is an equation \(e_{\varphi}(x_1, \ldots, x_n)\) such that for every linearly ordered MV\(_{\Delta}\) (PMV\(_{\Delta}\), ŁII, ŁII\(_{\frac{1}{2}}\)) algebra \(\mathcal{A}\) and for all \(a_1, \ldots, a_n \in \mathcal{A}\) one has \(\mathcal{A} = \varphi(a_1, \ldots, a_n)\) iff \(\mathcal{A} \models e_{\varphi}(a_1, \ldots, a_n)\).

As observed by the referee, property (i) might also be derived from [To89]. First one observes that every MV\(_{\Delta}\) algebra is pseudocomplemented (cf. [BD74] for the definition of pseudocomplemented lattice): \(\neg_p x\) is the pseudocomplement of \(x\). Now Torrens [To89] shows that a MV algebra is pseudocomplemented if and only if it is representable as a Boolean product of linearly ordered MV algebras, and property (i) follows. Note that \(\nabla\) is a closure operator, and that \(\nabla(x)\) is the smallest complemented element above \(x\). Moreover, \(\Delta(x)\) is the greatest complemented element below \(x\).

We want to strengthen Proposition 2.4. First of all, it is well known [COM95, To89] that the congruences of a MV algebra are uniquely determined by the congruence classes of 0 (ideals) or, equivalently, by the congruence classes of 1 (filters). We can extend this property to MV\(_{\Delta}\), PMV\(_{\Delta}\), ŁII, and ŁII\(_{\frac{1}{2}}\) algebras.

PROPOSITION 2.5. (i) MV\(_{\Delta}\) (PMV\(_{\Delta}\), ŁII, ŁII\(_{\frac{1}{2}}\)) algebras are ideal determined [GU84], i.e., their congruences are uniquely determined by the corresponding ideal (congruence class of 0).

(ii) Ideals of MV\(_{\Delta}\) algebras are precisely the lattice ideals which are closed under \(\oplus\) and under \(\nabla\).

Proof. In [Mo98], (i) and (ii) are shown for ŁII and for ŁII\(_{\frac{1}{2}}\) algebras, but the same argument holds for MV\(_{\Delta}\) algebras and for PMV\(_{\Delta}\) algebras as well (the only difference is that we do not have to check compatibility with product implication \(\rightarrow_p\)). Note that the congruence \(\theta_I\) corresponding to an ideal \(I\) is defined by \(x \theta_I y \iff |x - y| \in I\) (or, equivalently, iff \(\nabla|x - y| \in I\)).

It follows from [BS81, pp. 158–165] that the factors occurring in the Boolean representation of an algebra \(\mathcal{A}\) in a discriminator variety are those of the form \(\mathcal{A}/\theta : \theta\) a maximal congruence of \(\mathcal{A}\). By Proposition 2.5, we can replace maximal congruences by maximal ideals. Hence, every MV\(_{\Delta}\) (PMV\(_{\Delta}\), ŁII, ŁII\(_{\frac{1}{2}}\), respectively) algebra \(\mathcal{A}\) can be represented as a Boolean product of the family \(\{a_M : M \in M(\mathcal{A})\}\). Thus every \(a \in \mathcal{A}\) can be identified with the function \(a\) on \(M(\mathcal{A})\) defined, for all \(M \in M(\mathcal{A})\), by \(a(M) = a_M\), where \(a_M\) denotes the congruence class of \(a\) modulo the congruence determined by \(M\).
Notation 1. For \( a \in \mathcal{A} \), we define \( a^\circ = \{ M \in M(\mathcal{A}) : a \not\in M \} \).

We continue our comments about Proposition 2.4. The topology \( \mathcal{T}(\mathcal{A}) \) on \( M(\mathcal{A}) \) occurring in the Boolean representation of \( \mathcal{A} \) is generated by the set \( T(\mathcal{A}) = \{ E(a, b), D(a, b) : a, b \in \mathcal{A} \} \), where \( E(a, b) = \{ M \in M(\mathcal{A}) : a \theta_M b \} \), and \( D(a, b) \) is the complement of \( E(a, b) \) relative to \( M(\mathcal{A}) \). Now \( T(\mathcal{A}) = \{ a^\circ : a \in \mathcal{A} \} \). Indeed, for \( a \in \mathcal{A} \), \( a^\circ = D(a, 0) \); moreover, for all \( a, b \in \mathcal{A} \), \( E(a, b) = \Delta(a \leftrightarrow b)^\circ \), and \( D(a, b) = (\neg \Delta(a \leftrightarrow b))^\circ \). It follows that \( \mathcal{T}(\mathcal{A}) \) is the topology generated by \( T(\mathcal{A}) \). We want to connect \( \mathcal{T}(\mathcal{A}) \) with the topology of the dual space of the algebra of idempotents of \( \mathcal{A} \).

An element \( a \) of an \( MV_\Delta \) algebra (possibly with additional operations) is said to be idempotent iff \( a \oplus a = a \). Using Boolean representations, we obtain that the idempotent elements of an \( MV_\Delta \) (\( PMV_\Delta \), \( LII, LII_1 \), respectively) algebra \( \mathcal{A} \) are precisely those elements \( a \) such that for every \( M \in M(\mathcal{A}) \), \( a_M \in \{ 0_M, 1_M \} \). These elements can be also characterized as the members of \( \text{range}(\Delta) \), as the members of \( \text{range}(\nabla) \), or as the elements \( a \) such that \( a \ominus a = a \).

Notation 2. Let \( \mathcal{A} \) be a \( MV_\Delta \) (\( PMV_\Delta \), \( LII, LII_1 \), respectively) algebra. The idempotent elements of \( \mathcal{A} \) constitute a Boolean algebra [COM95]. In the sequel, such algebra is denoted by \( B_d \). The set of maximal ideals of \( B_d \) is denoted by \( M(\mathcal{B}_d) \).

For \( M \in M(\mathcal{A}) \) and for \( a \in \mathcal{A} \), one has \( a \in M \) iff \( \nabla(a) \in M \cap B_d \). Moreover, \( M \cap B_d \) is a maximal ideal of \( B_d \). Conversely, given a maximal ideal \( J \) of \( B_d \), the set \( \{ a \in \mathcal{A} : \nabla(a) \in J \} \) is a maximal ideal of \( \mathcal{A} \) [Mo98]. It follows:

**Proposition 2.6** [Mo98]. The map \( \Phi \) from \( M(\mathcal{A}) \) into \( M(\mathcal{B}_d) \) defined, for \( M \in M(\mathcal{A}) \), by \( \Phi(M) = M \cap B_d \) is a bijection from \( M(\mathcal{A}) \) onto \( M(\mathcal{B}_d) \) whose inverse \( \Psi \) is defined, for every \( J \in M(\mathcal{B}_d) \), by \( \Psi(J) = \{ a \in \mathcal{A} : \nabla(a) \in J \} \).

By Proposition 2.6, we can represent any \( MV_\Delta \) (\( PMV_\Delta \), \( LII, LII_1 \), respectively) algebra \( \mathcal{A} \) as a Boolean product, using maximal ideals of \( B_d \) instead of maximal ideals of \( \mathcal{A} \).

**Notation 3.** For all \( a \in \mathcal{A} \) and for all \( M \in M(\mathcal{B}_d) \) we write \( \mathcal{A}_M \) and \( a_M \) instead of \( \mathcal{A}_{\Psi(M)} \) and \( a_{\Psi(M)} \), respectively, where \( \Psi \) is defined as in Proposition 2.6. Moreover, if \( \mathcal{E} \) is another \( MV_\Delta \) (\( PMV_\Delta \), \( LII, LII_1 \), respectively) algebra such that \( \mathcal{B}_d = \mathcal{B}_e \), then we regard the elements of both \( \mathcal{A} \) and \( \mathcal{E} \) as maps with the same domain \( M(\mathcal{B}_d) = M(\mathcal{B}_e) \).
In particular, for \( a \in \mathcal{A} \), we can identify \( a^\circ \) with the set \( \{ M \in M(\mathfrak{A}) : \forall(a) \notin M \} \). It follows that modulo homeomorphism, the topology \( \mathcal{T}(\mathcal{A}) \) is the Stone topology of the dual space \( \mathfrak{R}_\mathcal{A}^* \) of \( \mathcal{A} \). Summing up, we have

**Proposition 2.7.** (i) Any \( MV_{\Delta} \) \( (PMV_{\Delta}, L.II, L.II_{1/2}, \) respectively) algebra \( \mathcal{A} \) can be represented as a suitable set of functions as on \( M(\mathfrak{R}_\mathcal{A}) \) which associate to every \( M \in M(\mathfrak{R}_\mathcal{A}) \) the element \( a_M \in \mathcal{A} \), with operations defined componentwise.

(ii) The elements of \( \mathfrak{R}_\mathcal{A}^* \) are precisely those elements of \( \mathcal{A} \) which are represented as functions \( a \) such that for all \( M \in M(\mathfrak{R}_\mathcal{A}) \), \( a_M \in \{0,1\} \).

(iii) The topology \( \mathcal{T}(\mathcal{A}) \) occurring in the Boolean representation of \( \mathcal{A} \) is homeomorphic to the topology of the dual space \( \mathfrak{R}_\mathcal{A}^* \) of \( \mathfrak{R}_\mathcal{A} \).

**Proposition 2.8.** Let \( \mathcal{A} \) be an \( MV_{\Delta} \) \( (PMV_{\Delta}, L, L.II_{1/2}, \) respectively) algebra. Let for every ideal \( J \) of \( \mathcal{A} \), \( J^* = \{ M \in M(\mathcal{A}) : J \subseteq M \} \). Then \( J^* \) is a closed set in \( \mathcal{T}(\mathcal{A}) \).

**Proof.** For all \( M \in M(\mathcal{A}) \) one has \( M \in J^* \) iff for all \( a \in J \), \( a \in M \) iff for all \( a \in J \), \( \forall(a) \subseteq M \) iff for all \( a \in J \), \( \Delta(\neg a) \subseteq M \) iff for all \( a \in J \), \( M \subseteq (\Delta(\neg a))^\circ \). Thus \( J^* = \bigcap_{a \in J} (\Delta(\neg a))^\circ \), and \( J^* \) is closed. □

We can also improve claim (ii) of Proposition 2.4. Given an open formula \( \varphi(x_1,\ldots,x_n) \), let \( e_\varphi(x_1,\ldots,x_n) \) be as in Proposition 2.4 (ii). Clearly we can write \( e_\varphi \) as \( s(\varphi) = 1 \) for a suitable term \( s(\varphi) \). Letting \( t(\varphi) = \Delta(s(\varphi)) \), we obtain

**Proposition 2.9.** For every open formula \( \varphi(x_1,\ldots,x_n) \) of the language of \( MV_{\Delta} \) \( (PMV_{\Delta}, L, L.II_{1/2}, \) respectively) algebras there is a term \( t(\varphi)(x_1,\ldots,x_n) \) of the corresponding language such that for every \( MV_{\Delta} \) \( (PMV_{\Delta}, L.II, L.II_{1/2}, \) respectively) algebra \( \mathcal{A} \) and for all \( a^1,\ldots,a^n \in \mathcal{A} \) the following conditions hold:

- \( t(\varphi)(a^1,\ldots,a^n) \in \mathfrak{R}_\mathcal{A}^* \).
- For all \( M \in M(\mathfrak{R}_\mathcal{A}) \), \( t(\varphi)(a^1,\ldots,a^n)_M = 1 \) if \( \mathcal{A}_M \models \varphi(a^1_M,\ldots,a^n_M) \), and \( t(\varphi)(a^1,\ldots,a^n)_M = 0 \) otherwise.

\( L.II_{1/2} \) algebras have another important property (cf. [Mo98]). To introduce it, we recall the following definition.

**Definition 2.10.** An \( f \)-semifield is a structure which is isomorphic to a subdirect product of a family of linearly ordered fields equipped with an additional operation \( ^{-1} \) defined as follows: if \( x \neq 0 \), then \( x^{-1} \) is the multiplicative inverse of \( x \); otherwise, \( x^{-1} = 0 \).
In [Mo98] it is shown that f-semifields constitute a variety. Moreover, the following is proved:

**Proposition 2.11.** (i) Let \( \mathcal{F} \) be a f-semifield, and let \( \mathcal{A}(\mathcal{F}) \) be the structure whose domain is the interval \([0,1]\) of \( \mathcal{F} \), and whose operations are defined by \( x \oplus y = (x + y) \wedge 1 \); \( \neg x = 1 - x \); \( \cdot \) is the restriction of product of \( \mathcal{F} \) to \([0,1]\); \( x \rightarrow_p y = 1 \wedge (y \cdot x^{-1} + (1 - x \cdot x^{-1})) \); \( \frac{1}{2} = (1 + 1)^{-1} \). Then \( \mathcal{A}(\mathcal{F}) \) is a \( \mathcal{L}\Pi_1^1 \) algebra.

(ii) For every \( \mathcal{L}\Pi_1^1 \) algebra there is a (unique up to isomorphism) f-semifield \( \mathcal{F} \) such that \( \mathcal{A} = \mathcal{A}(\mathcal{F}) \).

(iii) The functor \( \Pi \) associating to every f-semifield \( \mathcal{F} \) the \( \mathcal{L}\Pi_1^1 \) algebra \( \mathcal{A}(\mathcal{F}) \) defined in (i) and to every morphism \( h \) from a f-semifield \( \mathcal{F} \) to a f-semifield \( \mathcal{G} \) its restriction to \( \mathcal{A}(\mathcal{F}) \) is an equivalence between the category of f-semifields and that of \( \mathcal{L}\Pi_1^1 \) algebras.

**Corollary 2.12.** Let \( \mathcal{I}_0 \) be the subalgebra of \( \mathcal{I}_2 \) whose domain is \( \mathbb{Q} \cap [0,1] \), and let \( \mathcal{A} \) be any nontrivial \( \mathcal{L}\Pi_1^1 \) algebra. Then there is a unique monomorphism from \( \mathcal{I}_0 \) into \( \mathcal{A} \).

**Proof.** It is readily seen that for every nontrivial f-semifield \( \mathcal{F} \) there is a unique monomorphism from \( \mathcal{I}_0 \) into \( \mathcal{F} \). The claim follows from Proposition 2.11.

**Notation 4.** By abuse of notation, with reference to any fixed \( \mathcal{L}\Pi_1^1 \) algebra \( \mathcal{A} \), we identify every rational \( q \in \mathbb{Q} \cap [0,1] \) with its image under the unique monomorphism from \( \mathcal{I}_0 \) into \( \mathcal{A} \).

**Proposition 2.13 ([Mo98]).**

- Modulo an extension by definitions, every linearly ordered \( \mathcal{L}\Pi_1 \) algebra with more than two elements is a \( \mathcal{L}\Pi_1^1 \) algebra.
- If \( \mathcal{A} \) is a \( \mathcal{L}\Pi_1 \) algebra, then, for all \( M \in \mathcal{M}(\mathcal{A}) \), \( \mathcal{A}_M \) is either a \( \mathcal{L}\Pi_1^1 \) algebra (modulo an extension by definitions) or the two-element Boolean algebra (In this case, we write \( \mathcal{A}_M = 2 \)).
- If \( \mathcal{A} \) is a nontrivial \( \mathcal{L}\Pi_1^1 \) algebra, then for all \( M \in \mathcal{M}(\mathcal{A}) \), \( \mathcal{A}_M \neq 2 \).

Finally, Boolean representations are preserved under the functor \( \Pi \) and under its inverse. Therefore

**Proposition 2.14 (cf. [Mo98]).** Every f-semifield \( \mathcal{F} \) can be represented as a Boolean product of the family \( \{ \mathcal{F}_M : M \in \mathcal{M}(\Pi(\mathcal{F})) \} \).
3. REPRESENTATION THEOREMS FOR $MV_{\Delta}$ AND FOR PMV ALGEBRAS

We start from the following definitions.

**Definition 3.1 [BKW77].** A lattice-ordered abelian group is a system
\[ \mathcal{G} = \langle G, +, -, \lor, \land, 0 \rangle \]
satisfying the following conditions:
\begin{itemize}
  \item [(i)] \( \langle G, +, -, 0, \rangle \) is an abelian group.
  \item [(ii)] \( \langle G, \lor, \land \rangle \) is a lattice.
  \item [(iii)] If \( \leq \) denotes the partial order on \( G \) induced by \( \lor \) and \( \land \),
    then, for all \( a, b, x \in G \), if \( a \leq b \), then \( a + x \leq b + x \).
\end{itemize}

**Notation 5.** In the sequel, if \( \mathcal{G} \) is a lattice-ordered abelian group
(possibly with additional operations), if \( x \in \mathcal{G} \) and \( n \in \mathbb{N} \) we define
\( nx = 0 \) if \( n = 0 \); \( nx = x + \cdots + x \) if \( n > 0 \). We define further:
\( (n^{-1})x = n \text{ times} \). In this way we have defined \( xz \) for all \( z \in \mathbb{Z} \) and for all \( x \in \mathcal{G} \). We also write \( x^+ \) for \( x \lor 0 \), \( x^- \) for \( (x) \lor 0 \), and \( |x| \) for \( x \lor (-x) \).

**Definition 3.2.** A lattice-ordered group with strong unit is a system
\[ \mathcal{G} = \langle G, +, -, \lor, \land, 0, u \rangle \]
such that \( \langle G, +, -, \lor, \land, 0 \rangle \) is a lattice-ordered abelian group, and for all \( a \in \mathcal{G} \) there is a positive natural
number \( n \) such that \( a \leq nu \).

**Notation 6.** If \( \mathcal{G} \) is a (nontrivial) lattice-ordered abelian group
with strong unit, \( u \), there is a unique embedding \( \psi \) of \( \mathbb{Z} \) (thought of as a lattice-ordered abelian group) into \( \mathcal{G} \)
such that \( \psi(1) = u \). In the sequel for every \( z \in \mathbb{Z} \) we identify \( \psi(z) \) with \( z \).

To give a functorial representation theorem for \( MV_{\Delta} \) algebras, we need
the following variant of the concept of lattice-ordered abelian group.

**Definition 3.3.** A \( \delta \) lattice-ordered abelian group (for short: a \( \delta \)-l-group)
is a lattice \( \mathcal{G} = \langle G, +, -, \lor, \land, 0, 1 \rangle \) where \( \langle G, +, -, \lor, \land, 0 \rangle \) is
a lattice-ordered abelian group, \( 1 \in G \), and \( \delta \) is a unary function on \( G \)
such that the following equations hold:
\begin{itemize}
  \item [(a)] \( \delta(1) = 1 \).
  \item [(b)] \( \delta(x) \land \delta(y^+ + (1 - |x|)^+) \leq \delta(y) \).
  \item [(c)] \( \delta(x) \leq |x| \land 1 \).
  \item [(d)] \( \delta(x) \lor (1 - \delta(x)) = 1 \).
  \item [(e)] \( \delta(\delta(x)) = \delta(x) \).
  \item [(f)] \( 0 \leq \delta(x) \).
  \item [(g)] \( \delta(x) = \delta(|x| \land 1) \).
\end{itemize}
We say that a \( \delta \)-l-group \( \mathcal{G} = \langle G, +, -, \lor, \land, \delta, 0, 1 \rangle \) is a \( \delta \)-l-group with strong unit if \( 1 \) is a strong unit of the underlying lattice-ordered abelian group.

Axioms (a), (b), (c), (d), (e), and (f) of \( \delta \)-l-groups imply that the range of \( \delta \) is included in \([0, 1]\), and its restriction to \([0, 1]\) satisfies the axioms of Baa\text{'}s \( \delta \) when \( \to \) is defined by \( x \to y = 1 \land (y^+ + (1 - x)^+) \). (Note that with this definition if \( x, y \in [0, 1] \), then \( x \to y = 1 \land (y + 1 - x) \).) Moreover, axiom (g) ensures that \( \delta \) is uniquely determined by its restriction to \([0, 1]\). Note that there are nontrivial \( \delta \)-l-groups where \( 0 = 1 \) (in which case \( \delta \) is constantly equal to 0). This possibility is excluded in nontrivial \( \delta \)-l-groups with strong unit: the strong unit of a nontrivial \( \delta \)-l-group must be nonzero, and axiom (a) guarantees that \( \delta \) cannot be constantly equal to 0. Another easy property of \( \delta \)-l-groups is the following:

**Lemma 3.4.** In every linearly ordered \( \delta \)-l-group, \( \delta \) is the characteristic function of the complement of the interval \((-1, 1)\).

Let \( \Gamma \) denote Mundici\’s functor \([\text{Mu86}]\), and let \( \Gamma^{-1} \) be a functor from \( MV \) algebras into lattice-ordered abelian groups with strong unit which inverts \( \Gamma \) (cf. \([\text{Mu86}]\)).

**Lemma 3.5.** Let \( \mathcal{G} = \langle G, +, -, \lor, \land, \delta, 0, 1 \rangle \) be a \( \delta \)-l-group with strong unit, and let \( \mathcal{G}' \) be the underlying lattice-ordered abelian group with strong unit. Let \( \mathcal{A} \) be a \( MV_\Delta \) algebra, and let \( \mathcal{A}' \) be the underlying \( MV \) algebra. Finally, let \( h \) be a homomorphism from \( \mathcal{G} \) into a \( \delta \)-l-group with strong unit \( \mathcal{G} \), and let \( k \) be a homomorphism from \( \mathcal{A} \) into a \( MV_\Delta \) algebra \( \mathcal{B} \). The following conditions hold:

(a) The restriction \( \Delta \) of \( \delta \) to \( \Gamma(\mathcal{G}') \) makes \( \Gamma(\mathcal{G}') \) a \( MV_\Delta \) algebra.

(b) If \( \Delta' \) makes \( \Gamma(\mathcal{G}') \) a \( MV_\Delta \) algebra, then \( \Delta' \) is the restriction of \( \delta \) to \( \Gamma(\mathcal{G}') \).

(c) The operator \( \delta \) on \( \Gamma^{-1}(\mathcal{A}') \) defined by \( \delta(x) = \Delta(|x| \land 1) \) makes \( \Gamma^{-1}(\mathcal{A}') \) a \( \delta \)-l-group with strong unit.

(d) If \( \delta' \) makes \( \Gamma^{-1}(\mathcal{A}') \) a \( \delta \)-l-group with strong unit, then for all \( x \in \Gamma^{-1}(\mathcal{A}') \), one has \( \delta'(x) = \Delta(|x| \land 1) \).

(e) \( \Gamma(h) \) is a homomorphism of \( MV \) algebras which is compatible with the operator \( \Delta \) defined as in (a).

(f) \( \Gamma^{-1}(k) \) is a homomorphism of lattice-ordered abelian groups with strong unit which preserves the operator \( \delta \) defined as in (b).

**Proof.** By \([\text{Mu86}]\), \( \Gamma(\mathcal{G}') \) is a \( MV \) algebra. Moreover, identities (c) and (d) in Definition 3.3 imply that the range of \( \delta \) is included in \([0, 1]\), and axioms (a), (b), (c), (d), and (e) guarantee that the restriction \( \Delta \) of \( \delta \) to
[0, 1] satisfies axioms (ii), (iii), (iv), (v), and (vi) of $MV_\Delta$ algebras, respectively. Finally, identity (g) implies that $\delta$ is uniquely determined by its restriction to [0, 1]. Thus an operation $\delta$ on $\mathcal{G}'$ makes it a $\delta$-l-group with strong unit iff its restriction $\Delta$ to [0, 1] makes $\Gamma(\mathcal{G}')$ a $MV_\Delta$ algebra, and $\delta(x) = \Delta(|x| \land 1)$. This proves (a), (b), (c), and (d).

We prove (e). That $\Gamma(h)$ is a homomorphism of $MV$ algebras follows from [Mu86]. Since $h$ commutes with $\delta$, its restriction to [0, 1] commutes with the restriction of $\delta$ to [0, 1], i.e., with $\Delta$.

We prove (f). That $\Gamma^{-1}(k)$ is a homomorphism of lattice-ordered abelian groups with strong unit follows from [Mu86]. Moreover, since $k$ coincides with $\Gamma^{-1}(k)$ on $\mathcal{A}'$, and $\delta$ coincides with $\Delta$ in $\mathcal{A}'$, for all $x \in \Gamma^{-1}(\mathcal{A}')$, one has

$$\Gamma^{-1}(k)(\delta(x)) = \Gamma^{-1}(k)(\Delta(|x| \land 1)) = k(\Delta(|x| \land 1)) = \Delta(k(|x| \land 1))$$

$$= \delta(k(|x| \land 1)) = \delta(\Gamma^{-1}(k)(|x| \land 1))$$

$$= \delta(\Gamma^{-1}(k)(x) \land 1) = \delta(\Gamma^{-1}(k)(x)).$$

**Definition 3.6.** We define a functor $\Gamma_\Delta$ from the category of $\delta$-l-groups with strong unit into the category of $MV_\Delta$ algebras as follows:

- For every $\delta$-l-group $\mathcal{G}$ with strong unit, $\Gamma_\Delta(\mathcal{G})$ is defined as follows: let $\mathcal{G}'$ denote the underlying lattice-ordered abelian group with strong unit. Then $\Gamma_\Delta(\mathcal{G})$ is $\Gamma(\mathcal{G}')$ equipped with the restriction of $\delta$ to $\Gamma(\mathcal{G}')$.

- For every homomorphism $h$ from a $\delta$-l-group $\mathcal{G}$ with strong unit into a $\delta$-l-group $\mathcal{H}$, again with strong unit, we define $\Gamma_\Delta(h) = \Gamma(h)$ (by Lemma 3.5, (e), $\Gamma(h)$ commutes with $\Delta$; therefore it is a homomorphism of $MV_\Delta$ algebras).

We further define a functor $\Gamma_\Delta^{-1}$ from the category of $MV_\Delta$ algebras to $\delta$-l-groups with strong unit as follows:

- For every $MV_\Delta$ algebra $\mathcal{A}$, let $\mathcal{A}'$ be the underlying $MV$ algebra. Then $\Gamma_\Delta^{-1}(\mathcal{A})$ is $\Gamma^{-1}(\mathcal{A}')$ equipped with the operator $\delta$ defined by $\delta(x) = \Delta(|x| \land 1)$.

- For every morphism $k$ of $MV_\Delta$ algebras, $\Gamma_\Delta^{-1}(k) = \Delta^{-1}(k)$ (by Lemma 3.5(f), $\Gamma^{-1}(k)$ preserves $\delta$; therefore it is a morphism of $\delta$-l-groups with strong unit).

By Lemma 3.5 (a) and (e), $\Gamma_\Delta$ is a functor from the category of $\delta$-l-groups with strong unit into the category of $MV_\Delta$ algebras. Moreover, by Lemma 3.5 (c) and (f), $\Gamma_\Delta^{-1}$ is a functor from the category of $MV_\Delta$.
algebras into the category of $\delta$-l-groups with strong unit. Finally, by Lemma 3.5 (b), for every $MV_\Delta$ algebra $\mathcal{A}$, $\Gamma_\Delta(\mathcal{A})$ is isomorphic to $\mathcal{A}$, and by Lemma 3.5 (d), for every $\delta$-l-group $\mathcal{F}$ with strong unit, $\Gamma_\Delta^{-1}(\mathcal{F})$ is isomorphic to $\mathcal{F}$. Now $\Gamma_\Delta(\Gamma_\Delta^{-1}(\mathcal{F}))$ coincides with $\mathcal{F}$ (with $\Gamma^{-1}$, respectively) on morphisms of $\delta$-l-groups with strong unit (of $MV_\Delta$ algebras, respectively). Since $\Gamma^{-1}$ inverts $\Gamma$, we have proved

**Theorem 3.7.** The functor $\Gamma_\Delta$ is an equivalence of categories.

Now subdirect representations are preserved under equivalences of categories. Thus we can represent every $\delta$-l-group $\mathcal{G}$ with strong unit as a subdirect product of the family $\{\mathcal{F}_M : M \in M(\mathcal{A}(\mathcal{F}))) \}$, where $\mathcal{F}_M = $ $\Gamma_\Delta^{-1}(\mathcal{F}))$. Every element $a \in A$ can be regarded as a function on $M(\mathcal{A}(\mathcal{F}))$. As usual, the value of such a function on $M \in M(\mathcal{A}(\mathcal{F}))$ is denoted by $a_M$. The elements of $\mathcal{A}(\mathcal{F}))$ correspond to $\{0, 1\}$-valued functions. These elements can also be characterized as the elements of $\text{range} \delta$, or as the characteristic elements in the sense of [G86]. Finally, $\Gamma_\Delta$ and $\Gamma_\Delta^{-1}$ preserve the property of being linearly ordered, therefore

**Corollary 3.8.** Every $\delta$-l-group with strong unit can be represented as a subdirect product of linearly ordered $\delta$-l-groups with strong unit.

Of course we can give $M(\mathcal{A}(\mathcal{F}))$ the Stone topology of the dual space of $\mathcal{A}(\mathcal{F}))$. For $a, b \in \mathcal{A}$, the sets

$$E(a, b) = \{ M \in M(\mathcal{A}(\mathcal{F})) : a_M = b_M \} \quad \text{and}$$

$$D(a, b) = \{ M \in M(\mathcal{A}(\mathcal{F})) : a_M \neq b_M \}$$

are clopen, because

$$E(a, b) = \{ M \in M(\mathcal{A}(\mathcal{F})) : \delta(1 - (1 \land |a - b|)) \notin M \}$$

$$= (\delta(1 - (1 \land |a - b|)))^\circ, \quad \text{and}$$

$$D(a, b) = \{ M \in M(\mathcal{A}(\mathcal{F})) : 1 - \delta(1 - (1 \land |a - b|)) \notin M \}$$

$$= (1 - \delta(1 - (1 \land |a - b|)))^\circ.$$

**Theorem 3.9.** Let $\mathcal{F}$ be a $\delta$-l-group with strong unit. Then for every element $g \in \mathcal{F}$ there are $z_1, \ldots, z_k \in \mathbb{Z}$, $b_1, \ldots, b_k \in \mathcal{A}(\mathcal{F}))$ and $\alpha_1, \ldots, \alpha_k \in \Gamma_\Delta(\mathcal{F})$ such that the following conditions hold:

(i) For $i, j = 1, \ldots, k$, if $i \neq j$, then $b_i \land b_j = 0$.

(ii) For $i = 1, \ldots, k$ and for $M \in M(\mathcal{A}(\mathcal{F}))$, $(\alpha_i)_M \leq (b_i)_M$, and $(\alpha_i)_M < 1$.

(iii) $g = \sum_{i=1}^{k}(z_ib_i + \alpha_i).$
Proof. Since 1 is a strong unit of \( \mathcal{R} \), there is \( n \in \mathbb{N} \) such that \( |g| < n \). Let \( I_\varepsilon = \{-n, -n + 1, \ldots, 0, 1, \ldots, n - 1\} \). Let for \( i \in I_\varepsilon \),

\[
C_i = \{ M \in \mathcal{M}(\mathcal{R}(\Gamma_M(\mathcal{R}))) : i \leq g_M < (i + 1) \}.
\]

Clearly, the \( C_i \)'s form a partition of \( \mathcal{M}(\mathcal{R}(\Gamma_M(\mathcal{R}))) \). Moreover, each \( C_i \) is clopen, as \( C_i = E(i, g \wedge i) \cap D(i + 1, g \wedge (i + 1)) \). Thus for \( i \in I_\varepsilon \) there is \( b_i \in \mathcal{R}(\Gamma_M(\mathcal{R})) \) such that \( C_i = b_i^\circ \). Since the \( C_i \)'s are mutually disjoint, if \( i \neq j \), then \( b_i \wedge b_j = 0 \). This proves (i).

Now let for \( i \in I_\varepsilon \), \( \alpha_i = 0 \lor (b_i \wedge (g - z_j)) \). Clearly \( \alpha_i \leq b_i \). Moreover, for \( M \in \mathcal{M}(\mathcal{R}(\Gamma_M(\mathcal{R}))) \), if \( M \notin b_i^\circ \), then \( (\alpha_i)_M = 0 \), and if \( M \in b_i^\circ \) then \( (\alpha_i)_M = (g - z_j)_M < 1 \). This proves (ii).

Finally, for \( M \in \mathcal{M}(\mathcal{R}(\Gamma_M(\mathcal{R}))) \), there is a unique \( i \in I_\varepsilon \) such that \( M \in b_i^\circ \). For such \( i \), \( i \leq g_M < (i + 1) \), and \( (\alpha_i)_M = (g - i)_M \). Hence \( g_M = i + (\alpha_i)_M \). Moreover, for \( j \neq i \), \( (b_j)_M = (\alpha_i)_M = 0 \). Thus \( (\sum_{i \in I_\varepsilon} z_i b_i + \alpha_i)_M = z_i + (\alpha_i)_M = g_M \). Since \( M \) is arbitrary, (iii) is proved.

To obtain an equivalence of categories for PMV₄ algebras, we introduce a variety whose algebras are similar to \( \delta \)-l-groups, but have in addition the structure of ring. We start from some preliminary definitions.

Definition 3.10 [BKW77]. A lattice-ordered ring is a system \( \mathcal{R} = \langle R, +, -, \cdot, \vee, \wedge, 0 \rangle \) such that \( \langle R, +, -, \cdot, 0 \rangle \) is a ring, \( \langle R, +, -, \vee, \wedge, 0 \rangle \) is a lattice-ordered abelian group, and, for all \( a, b, x \in R \), if \( a \leq b \) and \( x \geq 0 \), then \( a \cdot x \leq b \cdot x \).

An \( f \)-ring is a lattice-ordered ring which is isomorphic to a subdirect product of a family of totally ordered rings.

In [BKW77, Theorem 9.1.2] it is shown that \( f \)-rings constitute an equational class. Such a class is axiomatized by the equations of lattice-ordered rings plus

\[
x^+ \wedge (x^- \cdot y^+) = x^+ \wedge (y^+ \cdot x^-) = 0.
\]

Definition 3.11. A \( \delta \)-f-ring is a system \( \mathcal{R} = \langle R, +, -, \vee, \wedge, \cdot, \delta, 0, 1 \rangle \) such that \( \langle R, +, -, \vee, \wedge, 0, 1 \rangle \) is a \( \delta \)-l-group, \( \langle R, +, -, \vee, \wedge, \cdot, 0, 1 \rangle \) is a commutative \( f \)-ring with unit 1, and for all \( x, y \in \mathcal{R} \) one has

\[
\delta(1 - (1 \wedge |x \cdot y|)) = \delta(1 - (1 \wedge |x|)) \lor \delta(1 - (1 \wedge |y|)). \tag{3.11.1}
\]

A \( \delta \)-f-ring with strong unit is a \( \delta \)-f-ring whose underlying \( \delta \)-l-group is a \( \delta \)-l-group with strong unit.

Lemma 3.12. Every linearly ordered \( \delta \)-f-ring is a domain of integrity.

Proof. Let \( \mathcal{R} \) be a linearly ordered \( \delta \)-f-ring, and let \( x, y \in \mathcal{R} \). If \( x \cdot y = 0 \), then \( \delta(1 - (1 \wedge |x \cdot y|)) = 1 \). By the identity (3.11.1) in Definition
ordered, either $\delta(1 - (1 \land |x|)) \lor \delta(1 - (1 \land |y|)) = 1$. Since $\mathcal{R}$ is linearly ordered, either $\delta(1 - (1 \land |x|)) = 1$ or $\delta(1 - (1 \land |y|)) = 1$. By Lemma 3.4, it follows that either $x = 0$ or $y = 0$.

**Definition 3.13.** We define a functor $\Pi_3^\delta$ from the category of $\delta$-f-rings with strong unit into the category of $PMV_\Delta$ algebras as follows:

- For every $\delta$-f-ring $\mathcal{R}$ with strong unit, let $\mathcal{G}$ be the underlying $\delta$-l-group with strong unit. Then $\Pi_3^\delta(\mathcal{R})$ is $\Gamma_3^\delta(\mathcal{G})$, equipped with the restriction of product to the interval $[0, 1]$.

- For every homomorphism $h$ of $\delta$-f-rings with strong unit, we define $\Pi_3^\delta(h) = \Gamma_3^\delta(h)$.

It is easy to check that $\Pi_3^\delta$ is a functor as claimed in Definition 3.13. We will prove that $\Pi_3^\delta$ is an equivalence of categories.

**Lemma 3.14.** Let $\mathcal{A}$ be a $PMV_\Delta$ algebra, let $\mathcal{A}'$ be the underlying $MV_\Delta$ algebra, and let $\mathcal{G}' = \Gamma_3^{-1}(\mathcal{A}')$. Then there is a unique product $\star$ on $\mathcal{G}'$ such that $\star$ makes $\mathcal{G}'$ a $\delta$-f-ring with strong unit, and $\Gamma_3(\mathcal{G}')$ equipped with the restriction of $\star$ to $[0, 1]$ is isomorphic to $\mathcal{A}$.

**Proof.** If $\mathcal{A}$ is a linearly ordered $PMV_\Delta$ algebra and $\mathcal{A}'$ is the underlying $MV_\Delta$ algebra, then $\Gamma_3^{-1}(\mathcal{A}')$ consists of all elements of the form $(z, \alpha)$ with $z \in \mathbb{Z}$ and $\alpha \in \mathcal{A}' - \{1\}$, with lattice operations induced by the lexicographic order, where $\delta$ is the characteristic function of the complement of the interval $(-1, 1)$, and the sum $\hat{+}$ is defined by

$$(z, \alpha) \hat{+}(z', \alpha') = \begin{cases} (z + z', \alpha \oplus \alpha') & \text{if } \alpha \oplus \alpha' < 1 \\ (z + z' + 1, \alpha \odot \alpha') & \text{otherwise.} \end{cases}$$

If we identify an integer $z$ with $(z, 0)$ and an element $\alpha \in [0, 1)$ with $(0, \alpha)$, then $(z, \alpha) = z \hat{+} \alpha$. In [Mo98] it is shown that (modulo the above identifications) if we define

$$(z \hat{+} \alpha) \hat{\cdot}(z' \hat{+} \alpha') = zz' \hat{+} z\alpha' \hat{+} z'\alpha \hat{+} \alpha \cdot \alpha', \quad (3.14.1)$$

where $\cdot$ is the product in $\mathcal{A}$, then $\hat{\cdot}$ is commutative, associative, and compatible with the order, and the distributive property holds. Moreover, $1 = (1, 0)$ is the multiplicative unit, and there are no zero divisors with respect to $\hat{\cdot}$. Thus $\hat{\cdot}$ makes $\Gamma_3^{-1}(\mathcal{A}')$ a $\delta$-f-ring with strong unit.

Now let $\mathcal{G}$ be an arbitrary $PMV_\Delta$ algebra, let $\mathcal{G}'$ be the underlying $MV_\Delta$ algebra, and let $\mathcal{A} = \Gamma_3^{-1}(\mathcal{G}')$. Modulo isomorphism, we can assume $\mathcal{G}' = \Gamma_3(\mathcal{G})$. We want to define a product $\star$ in $\mathcal{G}$. Let $a, b \in \mathcal{G}$. By Theorem 3.9, there are $b_1, \ldots, b_n, c_1, \ldots, c_m \in \mathcal{A}(\mathcal{G}')$, $z_1, \ldots, z_n$, $u_1, \ldots, u_m \in \mathbb{Z}$, and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathcal{G}'$ such that the following
conditions hold:

(i) For \( i, h = 1, \ldots, n \) with \( i \neq h \), for \( j, k = 1, \ldots, m \) with \( j \neq k \), and for \( M \in M(\mathcal{B}(\mathcal{C})) \), one has \( b_i \land b_h = c_j \land c_k = 0 \), \( \alpha_i \leq b_i \), \( \beta_j \leq c_j \), \( (\alpha_i)_M < 1 \), and \( (\beta_j)_M < 1 \).

(ii) \( a = \sum_{i=1}^{n} (z_i b_i + \alpha_i) \), and \( b = \sum_{j=1}^{m} (y_j c_j + \beta_j) \).

Now define

\[
a \star b = \sum_{i=1}^{m} \sum_{j=1}^{n} \left( (z_i y_j) (b_i \cdot c_j) + y_j (\alpha_i \cdot c_j) + z_i (\beta_j \cdot b_i) + \alpha_i \cdot \beta_j \right),
\]

where \( \cdot \) is the product operation in \( \mathcal{C} \), and \( + \) is the sum in \( \mathcal{C} \).

It is clear that the restriction of \( \star \) to \( \mathcal{C} \) coincides with \( \cdot \). We want to prove that \( \star \) makes \( \mathcal{F} \) a \( \delta \)-f-ring with strong unit.

For \( M \in M(\mathcal{B}(\mathcal{C})) \) and for every operation \( \circ \) of \( \mathcal{C} \) or of \( \mathcal{C} \), we denote by \( \circ_M \) the corresponding operation on \( \mathcal{C}_M \) (\( \mathcal{C}_M \), respectively). One has \( (a \circ b)_M = a_M \circ_M b_M \). Now for every \( M \in M(\mathcal{B}(\mathcal{C})) \), either \( a_M = 0 \), or \( b_M = 0 \) (in which case, \( (a \star b)_M = 0 \)), or else there are exactly one \( i \) and one \( j \) such that \( (b_i)_M = 0 \), \( (c_j)_M \neq 0 \), and either \( z_i \neq 0 \) (respectively: \( y_j \neq 0 \)), or \( (\alpha_i)_M = 0 \) (respectively: \( (\beta_j)_M = 0 \)). In this case, \( a_M = z_i y_j (\alpha_i)_M \cdot (\beta_j)_M \) and \( (a \star b)_M = z_i y_j (\alpha_i)_M + (\beta_j)_M \).

It follows that if we define \( a_M \star_M b_M = (a \star b)_M \), then \( \star_M \) is well defined and satisfies (3.14.1). Thus \( \mathcal{C}_M \) equipped with \( \star_M \) is a \( \delta \)-f-ring with strong unit. By the arbitrariness of \( M \), \( \star \) makes \( \mathcal{F} \) a \( \delta \)-f-ring with strong unit.

From the distributive law it follows that any operation \( \bullet \) which extends \( \cdot \) and makes \( \mathcal{F} \) a \( \delta \)-f-ring with strong unit must satisfy Eq. (3.14.2) and therefore coincides with \( \star \).

**Definition 3.15.** We define \((P_{\Gamma_3})^{-1}\) on the category of \( PMV_{\Delta} \) algebras as follows:

- For every \( PMV_{\Delta} \) algebra \( \mathcal{C} \), \((P_{\Gamma_3})^{-1}(\mathcal{C})\) is \( \Gamma_{-1}(\mathcal{C}) \) equipped with the product \( \star \) defined according to Lemma 3.14.

- For every homomorphism \( h \) of \( PMV_{\Delta} \) algebras, \((P_{\Gamma_3})^{-1}(h) = (\Gamma_3^{-1})^{-1}(h)\).

**Lemma 3.16.** For every homomorphism \( h \) from a \( PMV_{\Delta} \) algebra \( \mathcal{A} \) into a \( PMV_{\Delta} \) algebra \( \mathcal{C} \), \((P_{\Gamma_3})^{-1}(h) \) is a homomorphism of \( \delta \)-f-rings with strong unit from \((P_{\Gamma_3})^{-1}(\mathcal{A})\) into \((P_{\Gamma_3})^{-1}(\mathcal{C})\).
Proof. Let \( a = \sum_{i=1}^{n}(z_i b_i + \alpha_i) \) and \( b = \sum_{j=1}^{m}(y_j c_j + \beta_j) \) be arbitrary elements of \( M \), where \( b_i, c_j, z_i, y_j, \alpha_i, \beta_j \) are as in Lemma 3.14. By (3.14.2),

\[
P \Gamma^{-1}_\alpha(h)(a) \star P \Gamma^{-1}_\alpha(h)(b)
= \sum_{i=1}^{n} \sum_{j=1}^{m} \left( (z_i y_j) (h(b_i) \cdot h(c_j)) + y_j (h(\alpha_i) \cdot h(c_j)) + z_i h(\beta_j) \cdot h(b_i) + h(\alpha_i) \cdot h(\beta_j) \right)
= P \Gamma^{-1}_\alpha(h)(a \star b).
\]

By Lemma 3.16, \((P \Gamma_\alpha)^{-1}\) is a functor from the category of \( PMV_\alpha \) algebras into the category of \( \delta \)-f-rings with strong unit. Moreover, by Lemma 3.14, \((P \Gamma_\alpha)^{-1}\) inverts \( P \Gamma_\alpha \). It follows:

**Theorem 3.17.** The functor \( P \Gamma_\alpha \) defined in Definition 3.13 is an equivalence between the categories of \( \delta \)-f-rings with strong unit and the category of \( PMV_\alpha \) algebras.

### 4. Representation Theorems for \( LII \) Algebras

Proposition 2.11 shown in [Mo98] gives us a representation theorem for \( LII_\frac{1}{2} \) algebras by means of a categorical equivalence with the \( f \)-semifields. In this section, we prove an analogous result for \( LII \) algebras. One possible starting point is the following: every \( LII \) algebra is isomorphic to a subdirect product of algebras which are either \( LII_\frac{1}{2} \) algebras or Boolean \( \frac{1}{2} \) algebras. Both Boolean algebras and \( LII_\frac{1}{2} \) algebras have nice representation theorems; therefore we might try to combine them to get a representation theorem for \( LII \) algebras. However, we meet a difficulty: there are \( LII \) algebras which are not isomorphic to a direct product of a \( LII_\frac{1}{2} \) algebra and a Boolean algebra, as shown by the following example:

**Example 1.** Let \( \mathcal{F}_{fin} \) be the subset of \( (Q \cap [0, 1])^\mathbb{N} \) consisting of those maps \( f \) from \( \mathbb{N} \) into \( Q \cap [0, 1] \) such that \( f_i \in \{0, 1\} \) for almost all \( i \). Clearly, \( \mathcal{F}_{fin} \) is the domain of a \( LII \) subalgebra \( \mathcal{F}_{fin} \) of \( \mathcal{F}_Q \) (cf. Corollary 2.12 for the definition of \( \mathcal{F}_Q \)).

It is easy to check that \( \mathcal{F}_{fin} \) is not isomorphic to the direct product of a \( LII_\frac{1}{2} \) algebra and a Boolean algebra.

The example above suggests two different attempts to modify the representation of \( LII_\frac{1}{2} \) algebras in Proposition 2.11 to obtain a representation of \( LII \) algebras:
(a) Replace f-semifields by more complicated structures, e.g., subdirect products of ordered fields and of Boolean algebras (thought of as Boolean rings).

(b) Leave the structure of f-semifield unchanged, but change the way of obtaining a Ł II algebra from an f-semifield.

As a matter of fact, we succeeded with attempt (b). To explain the intuitive idea of our construction, we start from the example shown above. The Ł II algebra $\mathcal{F}_{\text{fin}}$ is obtained as follows: start from the f-semifield $\mathbb{Q}^N$, and consider the ideal $J = \{ x \in \mathbb{Q}^N : x_i = 0 \text{ for almost all } i \}$. Then, the domain of $\mathcal{F}_{\text{fin}}$ is the set of all $f \in \mathbb{Q}^N$ such that $f \wedge (1 - f) \in J$, and its operations are defined as in Proposition 2.11. Alternatively, $\mathcal{F}_{\text{fin}}$ may be obtained by a similar construction, but replacing $\mathbb{Q}^N$ with its subalgebra $\mathbb{Q}^N_{\text{fin}}$ consisting of all $f \in \mathbb{Q}^N$ having finite range. This second choice is in a sense minimal (no redundant elements are introduced) and is characterized by the following properties:

(P1) The domain of $\mathcal{F}_{\text{fin}}$ is the set of all $f \in \mathcal{F}_{\text{fin}}$ such that $0 \leq f \leq 1$ and $f \wedge (1 - f) \in J$, and the operations on $\mathcal{F}_{\text{fin}}$ are defined from the operations of $\mathcal{F}_{\text{fin}}$ as in Proposition 2.11.

(P2) For every maximal ideal $M \supseteq J$, $\mathcal{F}_{M}$ is isomorphic to $\mathbb{Q}$. Note that (P2) is no longer true if we replace $\mathcal{F}_{\text{fin}}$ with $\mathbb{Q}^N$, because in this case, if $M$ is a maximal ideal extending $J$, $\mathbb{Q}^N_M$ has infinitesimal elements.

The example above suggests that we proceed as follows.

**Definition 4.1.** An ideal $J$ of an f-semifield $\mathcal{F}$ is said to be a Q-ideal if for every maximal ideal $M \supseteq J$, $\mathcal{F}_M$ is isomorphic to $\mathbb{Q}$.

• An ideal $J$ of a Ł II algebra $\mathcal{A}$ is said to be an $I_0$-ideal iff for every maximal ideal $M \supseteq J$, $\mathcal{A}_M$ is isomorphic to $\mathcal{A}_Q$.

• A Q-f-semifield is a pair $\langle \mathcal{F}, J \rangle$, where $\mathcal{F}$ is a f-semifield, and $J$ is a Q-ideal of $\mathcal{F}$.

• A Q-Ł II algebra is a pair $\langle \mathcal{A}, J \rangle$, where $\mathcal{A}$ is a Ł II algebra, and $J$ is an $I_0$ ideal of $\mathcal{A}$.

• A morphism from a Q-f-semifield $\langle \mathcal{F}, I \rangle$ into a Q-f-semifield $\langle \mathcal{G}, J \rangle$ is a homomorphism $h$ from $\mathcal{F}$ into $\mathcal{G}$ such that $h(x) \in J$ for all $x \in I$.

• A morphism from a Q-Ł II algebra $\langle \mathcal{A}, I \rangle$ into a Q-Ł II algebra $\langle \mathcal{B}, J \rangle$ is a homomorphism $h$ from $\mathcal{A}$ into $\mathcal{B}$ such that $h(x) \in J$ for all $x \in I$.

**Definition 4.2.** Let $\Pi$ be the functor introduced in Proposition 2.11. Define, for every Q-f-semifield $\langle \mathcal{F}, I \rangle$, $\Pi_0(\langle \mathcal{F}, I \rangle) = \langle \mathcal{A}, I \rangle$, where $\mathcal{A} = \Pi(\mathcal{F})$ and $J = I \cap \mathcal{A}$. Moreover, for every morphism $h$ from a Q-f-semifield $\langle \mathcal{F}, I \rangle$ into $\langle \mathcal{G}, H \rangle$, let $\Pi_0(h)$ be the restriction of $h$ to $\Pi(\mathcal{F})$. 


The following lemma is an easy consequence of Proposition 2.11

**Lemma 4.3.** The map \( \Pi_Q \) introduced in Definition 4.2 is an equivalence between the categories of \( Q \)-f-semifields and the category of \( Q \)-\( \Pi_1 \) algebras.

**Lemma 4.4.**

(a) Let \( \langle \mathcal{F}, J \rangle \) be a \( Q \)-f-semifield, and let \( \langle \langle \mathcal{F}, J \rangle \rangle_0 \) be the algebra whose domain is the set \( \{ f \in \mathcal{F} : 0 \leq f \leq 1 \text{ and } f \wedge (1 - f) \in J \} \), and whose operations are defined from the operations of \( \mathcal{F} \) into \( \langle \langle \mathcal{F}, J \rangle \rangle_0 \) and \( \langle \langle \mathcal{F}, J \rangle \rangle_0 \),

(b) If \( \langle \mathcal{A}, I \rangle \) is a \( Q \)-\( \Pi_1 \) algebra, then the set \( \{ f \in \mathcal{A} : f \wedge (1 - f) \in I \} \) is the domain of a \( \Pi_1 \)-subalgebra \( \langle \langle \mathcal{A}, I \rangle \rangle^- \) of the \( \Pi_1 \)-algebra underlying \( \mathcal{A} \).

(c) If \( h \) is a morphism from a \( Q \)-f-semifield \( \langle \mathcal{F}, I \rangle \) into a \( Q \)-f-semifield \( \langle \mathcal{G}, J \rangle \), the restriction of \( h \) to \( \langle \langle \mathcal{F}, I \rangle \rangle_0 \) is a morphism from \( \langle \langle \mathcal{F}, I \rangle \rangle_0 \) into \( \langle \langle \mathcal{G}, J \rangle \rangle_0 \).

(d) If \( h \) is a morphism from a \( Q \)-\( \Pi_1 \) algebra \( \langle \mathcal{A}, I \rangle \) into a \( Q \)-\( \Pi_1 \) algebra \( \langle \mathcal{B}, H \rangle \), the restriction of \( h \) to \( \langle \langle \mathcal{A}, I \rangle \rangle^- \) is a morphism from \( \langle \langle \mathcal{A}, I \rangle \rangle^- \) into \( \langle \langle \mathcal{B}, H \rangle \rangle^- \).

**Proof.** By Lemma 4.3, it suffices to prove (b) and (d). With regard to (b), let \( \langle \mathcal{A}, I \rangle \) be a \( Q \)-\( \Pi_1 \) algebra, let \( \theta_I \) be the congruence determined by \( I \), let \( \mathcal{A}_I \) denote \( \mathcal{A} \) modulo \( \theta_I \), and let, for \( a \in \mathcal{A} \), denote \( a_I \) the equivalence class of \( a \) modulo \( \theta_I \). Then the domain of \( \langle \langle \mathcal{A}, I \rangle \rangle^- \) is the set of all \( a \in \mathcal{A} \) such that \( a_I \in \mathcal{A}_I \). Since \( \mathcal{A}_I \) is closed under all operations of \( \Pi_1 \) algebras, so is the domain of \( \langle \langle \mathcal{A}, I \rangle \rangle^- \), and the claim follows.

With regard to (d), it suffices to prove that if \( h \) is a morphism from a \( Q \)-\( \Pi_1 \) algebra \( \langle \mathcal{A}, I \rangle \) into a \( Q \)-\( \Pi_1 \) algebra \( \langle \mathcal{B}, J \rangle \), then the restriction of \( h \) to \( \langle \langle \mathcal{A}, I \rangle \rangle^- \) maps \( \langle \langle \mathcal{A}, I \rangle \rangle^- \) into \( \langle \langle \mathcal{B}, J \rangle \rangle^- \). In other words, it suffices to prove that if \( a \wedge \neg a \in I \), then \( h(a) \wedge \neg h(a) \in J \). But this is obvious, since \( h \) maps \( I \) into \( J \).

**Definition 4.5.**

(a) We define, for every \( Q \)-\( \Pi_1 \) algebra \( \langle \mathcal{A}, I \rangle \), \( \Pi^- \langle \langle \mathcal{A}, I \rangle \rangle = \langle \langle \mathcal{A}, I \rangle \rangle^- \).

(b) We define, for every morphism \( h \) from a \( Q \)-\( \Pi_1 \) algebra \( \langle \mathcal{A}, I \rangle \) into a \( Q \)-\( \Pi_1 \) algebra \( \langle \mathcal{B}, J \rangle \), \( \Pi^- (h) \) to be the restriction of \( h \) to \( \langle \langle \mathcal{A}, I \rangle \rangle^- \).

(c) We define, for every \( Q \)-f semifield \( \langle \mathcal{F}, I \rangle \), \( \Pi_0 (\langle \mathcal{F}, I \rangle) = \langle \langle \mathcal{F}, I \rangle \rangle_0 \).

(d) We define, for every morphism \( h \) from a \( Q \)-f-semifield \( \langle \mathcal{F}, I \rangle \) into a \( Q \)-f-semifield \( \langle \mathcal{F}, J \rangle \), \( \Pi_0 (h) \) to be the restriction of \( h \) to \( \langle \langle \mathcal{F}, I \rangle \rangle_0 \).

We are ready to state the main result of this section.
Theorem 4.6. The map $\Pi_0$ is an equivalence between the category of $Q$-f-semifields and that of $LI\Pi_0$ algebras.

Since $\Pi_0$ is the composition $\Pi^- \circ \Pi_O$, by Lemma 4.3 it suffices to prove

Theorem 4.7. The map $\Pi^-$ is an equivalence between the category of $Q$-$LI\Pi_0$ algebras and that of $LI\Pi$ algebras.

Proof. The proof of Theorem 4.7 is given in three steps.

At Step 1 we associate to every $LI\Pi$ algebra $A$ a $Q$-$LI\Pi_0^1$ algebra $\tilde{A} = \langle A', J \rangle$, and we prove that $\tilde{A}$ is isomorphic to $A$. We also prove that up to isomorphism $\tilde{A}$ is a subalgebra of $A'$, and that $A'$ is generated by $A$.

At Step 2 we prove that if $\langle A', J \rangle$ is a $Q$-$LI\Pi_0^1$ algebra such that $(\langle A', J \rangle)^{-1}$ is isomorphic to $\tilde{A}$, then $\langle A', J \rangle$ is isomorphic to the structure $A$ constructed in Step 1.

At Step 3 we prove that every homomorphism from a $LI\Pi$ algebra $A$ into a $LI\Pi$ algebra $\tilde{A}$ has a unique extension to a morphism from $\tilde{A}$ into $\tilde{A}$.

We begin with Step 1. Let $A$ be any $LI\Pi$ algebra. By Proposition 2.7, we can represent $A$ as a suitable set of functions which map every $M \in M(A)$ into an element of $A^M$, with operations defined componentwise.

Lemma 4.8. Let $A$ be any nontrivial $LI\Pi$ algebra, let $b \in A^B$ be such that for all $M \in b^o$, $A^M \not= 2$, and let $q \in Q \cap [0, 1]$. Then there is a $q^b \in A$ such that for all $M \in b^o$, $q^b_M = q$.

Proof. It suffices to prove the claim for all $q$ of the form $\frac{1}{n}, n \geq 2$: if $q = \frac{m}{n}$, and $(\langle A', J \rangle)^{-1}$ is isomorphic to $A$, then $(\langle A', J \rangle)^{-1}$ is isomorphic to the structure $A$ constructed in Step 1.

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Notation 7. The structure $\Pi_{M \in M(\mathcal{A})} \mathcal{A}_M$ is denoted by $\mathcal{A}^*$. Let $b^1, \ldots, b^n \in \mathcal{A}$ such that for $i \neq j$ and $b^i \wedge b^j = 0$, let $q^i, \ldots, q^n \in \mathbb{Q} \cap [0, 1]$, and let $a \in \mathcal{A}$. We denote $\langle (b^1, q^1), \ldots, (b^n, q^n), a \rangle$ the element of $\mathcal{A}^*$ defined, for $M \in M(\mathcal{A})$, by

$$\left( \langle (b^1, q^1), \ldots, (b^n, q^n), a \rangle \right)_M = \begin{cases} q^1 & \text{if } M \in b^1, \\ \cdots & \cdots \\ q^n & \text{if } M \in b^n, \\ a_M & \text{otherwise}. \end{cases}$$

The set of all elements of the form $\langle (b^1, q^1), \ldots, (b^n, q^n), a \rangle$ as specified above is denoted by $A^+$. 

Lemma 4.9. The set $A^+$ defined in Notation 7 is the domain of a $\mathcal{L}\Pi_2^n$ subalgebra of $\mathcal{A}^*$.

Proof. We start from the following observation. Let $a \in \mathcal{A}$ and let $\sigma(a) = \nabla(a) \wedge \nabla(\neg a)$. Note that $\sigma(a) \in \mathcal{B}_\mathcal{A}$. Moreover, if $M \in M(\mathcal{A})$, then $\sigma(a)_M = 0$ if $a_M \in \{0, 1\}$, and $\sigma(a)_M = 1$ otherwise. It follows that if $M \in (\sigma(a))^\mathcal{B}$, then $\mathcal{A}_M \neq 2$; therefore by Lemma 4.8 for every $q \in \mathbb{Q} \cap [0, 1]$ there is $q^{\sigma(a)} \in \mathcal{A}$ such that for all $M \in (\sigma(a))^\mathcal{B}$, $q^{\sigma(a)} = q$. Now let $\times$ be a binary operation of $\mathcal{L}\Pi_2^n$ algebras, and let $\alpha = \langle (b^1, q^1), \ldots, (b^n, q^n), a \rangle$, $\beta = \langle (d^1, r^1), \ldots, (d^n, q^n), c \rangle$ be elements of $A^+$. Let

$$e^{i,j} = b^i \wedge d^j, \quad f^i = \left( \bigwedge_{i=1}^n \neg b^i \right) \wedge d^j \wedge \sigma(a),$$

$$g^i = \left( \bigwedge_{i=1}^m \neg d^i \right) \wedge d^j \wedge \Delta(a), \quad h^i = \left( \bigwedge_{i=1}^n \neg b^i \right) \wedge d^j \wedge \Delta(\neg a),$$

$$k^i = \left( \bigwedge_{i=1}^m \neg d^i \right) \wedge b^i \wedge \sigma(c), \quad l^i = \left( \bigwedge_{i=1}^m \neg d^i \right) \wedge b^i \wedge \Delta(c),$$

$$n^i = \left( \bigwedge_{i=1}^m \neg d^i \right) \wedge b^i \wedge \Delta(\neg c), \quad p = \left( \bigwedge_{i=1}^n \neg b^i \right) \wedge \left( \bigwedge_{i=1}^m \neg d^i \right).$$

$E = \{ e^{i,j} : i = 1, \ldots, n; j = 1, \ldots, m \}, \quad F = \{ f^i : j = 1, \ldots, m \},$

$G = \{ g^j : j = 1, \ldots, m \}, \quad H = \{ h^i : i = 1, \ldots, n \},$

$K = \{ k^i : j = 1, \ldots, n \}, \quad L = \{ l^i : i = 1, \ldots, n \},$

$N = \{ n^i : i = 1, \ldots, n \}, \quad X = E \cup F \cup G \cup H \cup K \cup L \cup M \cup \{ p \}.$
The following facts are easy to prove:

(i) \( X \) is a subset of \( \mathcal{B}_x \), and if \( s, t \in X \) and \( s \neq t \), then \( s \land t = 0 \).

(ii) If \( s \in F \cup K \), and \( M \in s^0 \), then \( \mathcal{A}_M \neq 2 \); therefore by Lemma 4.8, for every \( q \in Q \cap [0, 1] \) there is \( q' \in \mathcal{A} \) such that for all \( M \in s^0 \), \( q'_M = q \).

(iii) If \( g \in G \), then for all \( M \in g^0 a_M = 1 \).

(iv) If \( h \in H \), then for all \( M \in h^0, c_M = 1 \).

(v) If \( l \in L \), then for all \( M \in l^0, a_M = 0 \).

(vi) If \( n \in N \), then for all \( M \in n^0, c_M = 0 \).

It follows that for all \( M \in M(\mathcal{B}_x) \) we have

\[
(a \times \beta)_M = \begin{cases} 
q'_i \times q'_j & \text{if } M \in (e^{i,j})^0 \\
1 \times q'_j & \text{if } M \in (g^j)^0 \\
0 \times q'_j & \text{if } M \in (h^j)^0 \\
q'_i \times 1 & \text{if } M \in (l^i)^0 \\
q'_i \times 0 & \text{if } M \in (m^i)^0 \\
(q'_j)^{i'} \times a & \text{if } M \in (f^{i'})^0 \\
c \times (q'_j)^{k'} & \text{if } M \in (k^i)^0 \\
a \times c & \text{otherwise.} 
\end{cases} \tag{2.9.1}
\]

Now for \( i = 1, \ldots, n \) and for \( j = 1, \ldots, m \), \( q'_i \times q'_j, 1 \times q'_j, q_i \times 1, 0 \times q'_i \), and \( q'_i \times 1 \) are in \( Q \cap [0, 1] \). Moreover, \( e^{i,j}, g^j, h^j, l^i, m^i \), are in \( \mathcal{B}_x \).

Finally, the last three clauses in (2.9.1) can be replaced by

\[
\left( (q'_j)^{i'} \times a \right) \cdot f^{i'} \lor \left( (c \times (q'_j)^{k'}) \cdot k^i \lor ((a \times c)) \cdot p \right) \text{ otherwise.}
\]

Since \( ((q'_j)^{i'} \times a) \cdot f^{i'} \lor ((c \times (q'_j)^{k'}) \cdot k^i) \lor ((a \times c)) \cdot p \in \mathcal{A} \), we conclude that \( \mathcal{A} \times \beta \in A^* \). Thus \( A^* \) is closed under all binary operations of \( \mathcal{L} \Pi \) algebras. A similar argument shows that \( A^+ \) is closed under \( \neg \). Finally, \( \frac{1}{2} \in A^+ \). So \( A^+ \) is the domain of a subalgebra of \( \mathcal{A}^* \), as desired.

**Notation 8.** The subalgebra of \( \mathcal{A}^* \) which Lemma 4.9 refers to is denoted \( \mathcal{A}^+ \).

**Lemma 4.10.** (i) \( \mathcal{A}^+ \) is the \( \mathcal{L} \Pi \frac{1}{2} \) subalgebra of \( \mathcal{A}^* \) generated by \( \mathcal{A} \).

(ii) \( \mathcal{B}_{x'} = \mathcal{B}_{x'} \).

**Proof.** (i) It is readily seen that for all \( \alpha = \langle (b^i, q^i), \ldots, (b^k, q^k), a \rangle \) one has

\[
\alpha = \left( \bigvee_{i=1}^{k} (b^i \cdot q^i) \right) \lor \left( \left( \bigwedge_{i=1}^{k} \neg b^i \right) \cdot a \right).
\]
Now for $i = 1, \ldots, k$, $b^i \in \mathcal{A}$. Moreover, $a \in \mathcal{A}$, and for $i = 1, \ldots, k$, $q^i$ belongs to any $L\Pi_1^\sigma$ algebra. The claim follows.

(ii) Let $\alpha = \langle (b^1, q^1), \ldots, (b^k, q^k), a \rangle$ be an idempotent element of $\mathcal{A}^+$. We can assume without loss of generality $b^i \neq 0$ for $i = 1, \ldots, k$, and that either $\bigwedge_{i=1}^k \neg b^i \neq 0$ or $a = 0$. Now if $\alpha$ is idempotent, then for all $M \in \mathcal{B}_\mathcal{A}$ we must have $\alpha_M \in \{0, 1\}$. Thus, taking $M \in (b^i)^\circ$, we obtain $q^i \in \{0, 1\}$. Moreover, for all $M \in (\bigwedge_{i=1}^k \neg b^i)^\circ$ we have $\alpha_M \in \{0, 1\}$. Thus for all $M \in M(\mathcal{B}_\mathcal{A})$ we have $((\bigwedge_{i=1}^k \neg b^i) \land a)_M \in \{0, 1\}$, and $(\bigwedge_{i=1}^k \neg b^i) \land a \in B_{\mathcal{A}}$. Since $\alpha = (\bigvee_{i\neq j} b^i) \lor ((\bigwedge_{i=1}^k \neg b^i) \land a)$, we conclude: $\alpha \in \mathcal{B}_\mathcal{A}$, as desired. \[\square\]

**Definition 4.11.** Let $\mathcal{A}$ be any $L\Pi$ algebra, and let $\mathcal{A}^+$ be defined as in Notation 8. We define $J(\mathcal{A}) = \{ a \in \mathcal{A}^+ : \forall M \in M(\mathcal{B}_\mathcal{A})(\mathcal{A}_M = 2 \Rightarrow a_M = 0) \}$; $X(\mathcal{A}) = \{ M \in M(\mathcal{B}_\mathcal{A}) : \mathcal{A}_M = 2 \}.$

**Lemma 4.12.** (i) For all $a \in \mathcal{A}$, $a \land \neg a \in J(\mathcal{A})$.

(ii) $X(\mathcal{A}) = (J(\mathcal{A}))^\circ$.

(iii) If $M \in (J(\mathcal{A}))^\circ$, then $\mathcal{A}_M^+ = \mathcal{J}_Q$.

**Proof.** (i) If $a \in \mathcal{A}$, $M \in M(\mathcal{B}_\mathcal{A})$, and $\mathcal{A}_M = 2$, then $a_M \in \{0, 1\}$. Therefore, $(a \land \neg a)_M = a_M \land \neg a_M = 0$, and $a \land \neg a \in J$.

(ii) If $M \in X(\mathcal{A})$, then $\mathcal{A}_M = 2$. Thus by the definition of $J(\mathcal{A})$, for all $a \in J(\mathcal{A})$ one has $a_M = 0$, i.e., $a \in M$. In other words, if $M \in X(\mathcal{A})$, then $M \supseteq J(\mathcal{A})$, i.e., $M \in (J(\mathcal{A}))^\circ$.

Vice versa, suppose $M \in (J(\mathcal{A}))^\circ$. By (i), if $a \in \mathcal{A}$, then $a \land \neg a \in J(\mathcal{A})$. Therefore $a \land \neg a \in M$, i.e., $a_M \land \neg a_M = 0$. Since this conclusion is independent of $a \in \mathcal{A}$, we conclude that if $M \in (J(\mathcal{A}))^\circ$ then $\mathcal{A}_M = 2$. Hence $J(\mathcal{A})^\circ \subseteq X(\mathcal{A})$.

(iii) It is sufficient to prove that for all $a \in \mathcal{A}^+$ and for all $M \in (J(\mathcal{A}))^\circ$, $\alpha_M \in Q$. Let $\alpha = \langle (b^1, q^1), \ldots, (b^k, q^k), a \rangle$, and let $M \in (J(\mathcal{A}))^\circ$. If for some $i \leq k$, $M \in (b^i)^\circ$, then $\alpha_M = q^i$ and the claim follows. Otherwise, $\alpha_M = a_M$. By (ii), $M \in X(\mathcal{A})$; therefore $\mathcal{A}_M = 2$. Since $a \in \mathcal{A}$, $a_M \in \{0, 1\}$; therefore $\alpha_M = a_M \in Q$. \[\square\]

**Lemma 4.13.** $\mathcal{A} = \{ a \in \mathcal{A}^+ : a \land \neg a \in J(\mathcal{A}) \}$.

**Proof.** One inclusion is Lemma 4.12 (i). With regard to the other inclusion, suppose $a \in \mathcal{A}^+$, and $a \land \neg a \in J(\mathcal{A})$. Let $\alpha = \langle (b^1, q^1), \ldots, (b^k, q^k), a \rangle$. We can assume without loss of generality $q^i \notin \{0, 1\}$. Indeed, if $q^i = 0$ we can write $\alpha$ as $\alpha = \langle (b^1, q^1), \ldots, (b^{i-1}, q^{i-1}), (b^{i+1}, q^{i+1}), \ldots, (b^k, q^k), a \land b^i \rangle$. If $q^i = 1$, we can write $\alpha$ as $\alpha = \langle (b^1, q^1), \ldots, (b^{i-1}, q^{i-1}), (b^{i+1}, q^{i+1}), \ldots, (b^k, q^k), a \lor b^i \rangle$. Now if $M \in (b^i)^\circ$, $\alpha_M = q^i \notin \{0, 1\}$. Hence, $(a \land \neg a)_M \neq 0$. Since $a \land \neg a \in J(\mathcal{A})$, $M \notin (J(\mathcal{A}))^\circ$. Hence $M \notin X(\mathcal{A})$. By Lemma 4.8, for $i = 1, \ldots, k$
there is $(q^i)_M \in \mathcal{A}$ such that for all $M \in (b^i)^o$, $((q^i)_M)_M = q^i$. Thus $\alpha = (\forall_{i=1}^k (b^i \cdot (q^i)_M)) \lor ((\land_{k=1}^i \neg b^i \cdot a))$. Hence $\alpha \in \mathcal{A}$. 

Summing up, we have associated to every $\mathfrak{L}_{\mathfrak{II}}^1$ algebra $\mathcal{A}$ a $\mathfrak{Q}$-$\mathfrak{L}_{\mathfrak{II}}^1$ algebra $\langle \mathcal{A}^+, J(\mathcal{A}) \rangle$ such that $(\mathcal{A}^+, J(\mathcal{A}))$ is isomorphic to $\mathcal{A}$, and we have also shown that up to isomorphism $\mathcal{A}$ is a $\mathfrak{L}_{\mathfrak{II}}$ subalgebra of $\mathcal{A}^+$, and that $\mathcal{A}^+$ is a $\mathfrak{L}_{\mathfrak{II}}^1$ algebra generated by $\mathcal{A}$. Thus we have concluded Step 1.

We continue the proof of Theorem 4.7 by performing Step 2. We have already seen (Lemma 4.4) that if $\mathfrak{E} = \langle \mathcal{E}, J \rangle$ is a $\mathfrak{Q}$-$\mathfrak{L}_{\mathfrak{II}}^1$ algebra, and $\mathcal{A} = (\mathfrak{E}^+, J(\mathcal{E}))$, then $\mathcal{A}$ is a $\mathfrak{L}_{\mathfrak{II}}$ subalgebra of $\mathcal{E}$. To conclude Step 2 we must prove that $(\langle \mathcal{A}^+, J(\mathcal{A}) \rangle)$ is isomorphic to $\mathfrak{E}$. This is proved by means of the following lemmas.

**Lemma 4.14.** If $\hat{\mathfrak{E}}$, $\mathcal{E}$, and $\mathcal{A}$ are as above, then $\mathcal{B}_\mathcal{A} = \mathcal{B}_{\hat{\mathfrak{E}}}$.

**Proof.** One inclusion is trivial; with regard to the other one, if $b \in \mathcal{B}_\mathcal{A}$, then $b \land \neg b = 0$; therefore $b \land \neg b \in J$, and $b \in \mathcal{A}$, as $\mathcal{A} = (\hat{\mathfrak{E}})^+$, hence $b$ is an idempotent element of $\mathcal{A}$, and $b \in \mathcal{B}_\mathcal{A}$.

**Lemma 4.15.** If $\hat{\mathfrak{E}}$, $\mathcal{E}$, and $\mathcal{A}$ are as above, and $\mathcal{A}^+$ is defined from $\mathcal{A}$ as in Notation 8, then $\mathcal{E}$ is isomorphic to $\mathcal{A}^+$.

**Proof.** Modulo isomorphism, we can assume that both $\mathcal{A}$ and $\mathcal{J}_o$ are subalgebras of both $\mathcal{E}$ and $\mathcal{A}^+$. Now let $c$ be any element of $\mathcal{E}$. Since for all $M \in J^*$ $\mathfrak{E}_M$ is isomorphic to $\mathcal{J}_o$, for all $M \in J^*$ we can choose $q^c_M \in \mathbb{Q} \cap [0, 1]$ such that $\mathfrak{E}_M = q^c_M$. Thus letting $\delta^M = \mathfrak{E}_c \leftrightarrow q^c_M$, for all $M \in J^*$, we have $(\delta^M)_M = 1$, therefore $M \in (\delta^M)^0$. Thus $\mathcal{D} = (\delta^M)^0 : M \in J^*$ is an open cover of $J^*$. Since $J^*$ is compact, $\mathcal{D}$ has a finite subcover, $\{\delta^1 \cdots \delta^n\}$ say. Note that for $i = 1, \ldots, n$, $\delta^i \in \mathcal{B}_\mathcal{A}$, hence $\delta^i \in \mathcal{B}_\mathcal{E}$. Now for $i = 1, \ldots, n$ there is $q^i \in \mathbb{Q} \cap [0, 1]$ such that $\delta^i = \mathfrak{E}_c \leftrightarrow q^i$. Note that for $M \in (\delta^i)^0$, $c_M = q^i$. Moreover, for all $M \in J^*$ there is $i \in \{1, \ldots, n\}$ such that $\delta^i = 1$; therefore $c_M = q^i$. Now let $a = c \lor \bigvee_{i=1}^n \delta^i$. If $M \in J^*$, then $M \in \bigcup_{i=1}^n (\delta^i)^0$; therefore $(\bigvee_{i=1}^n \delta^i) = 1$, and $a = M$, i.e., $a \in M$. It follows that $a \in M$ for every maximal ideal $M \supseteq J$. Thus $a \in J$. A fortiori, $a \land \neg a \in J$, and $a \in \mathcal{A}$. At this point, for all $M \in \mathcal{B}_\mathcal{A}$ we have

$$c_M = \begin{cases} q^i & \text{if } M \in (\delta^i)^0 \\ \cdots & \cdots \\ q^n & \text{if } M \in (\delta^n)^0 \\ a_M & \text{otherwise.} \end{cases}$$

Since $a \in \mathcal{A}$ and for $i = 1, \ldots, n$, $\delta^i \in \mathcal{B}_\mathcal{A}$ and $q^i \in \mathbb{Q} \cap [0, 1]$, $c \in \mathcal{A}^+$. It follows that, modulo idomorphism, $\mathfrak{E}$ is a subalgebra of $\mathcal{A}^+$. Vice versa,
for all \( a \in \mathcal{A} \), for all \( b^1, \ldots, b^n \in \mathcal{B}_\mathcal{A} \) such that \( b^i \land b^j = 0 \) whenever \( i \neq j \), and for all \( q^1, \ldots, q^n \in \mathcal{Q} \cap [0, 1] \), the element \( \alpha = \langle (b^1, q^1), \ldots, (b^n, q^n), a \rangle \) must belong to any \( \mathcal{L}_\Pi^1 \) algebra having both \( \mathcal{A} \) and \( \mathcal{J} \) as \( \mathcal{L}_\Pi \) subalgebras, because

\[
\alpha = \left( \bigvee_{i=1}^{n} (b^i \cdot q^i) \right) \lor \left( \bigwedge_{i=1}^{n} \neg b^i \right) \land a.
\]

Thus modulo isomorphism \( \mathcal{C} = \mathcal{A}^+ \).

**Lemma 4.16.** If \( \mathcal{C} \), \( \mathcal{A} \), and \( J \) are as in Lemma 4.15, then \( J = J(\mathcal{A}) \).

**Proof.** We have to prove that for all \( c \in \mathcal{C} \), \( c \in J \) iff for all \( M \in M(\mathcal{B}_\mathcal{A}) \), if \( \mathcal{A}_M = 2 \), then \( c_M = 0 \).

\[ \Rightarrow \text{ if } c \in J, \text{ then } \frac{1}{2} \cdot c \in J. \]

Now let \( M \in M(\mathcal{B}_\mathcal{A}) \) be such that \( \mathcal{A}_M = 2 \). Then, either \( \left( \frac{1}{2} \cdot c \right)_M = 0 \), or \( \left( \frac{1}{2} \cdot c \right)_M = 1 \). The second possibility is excluded, since \( \frac{1}{2} \cdot c \leq 1 \). Thus \( \left( \frac{1}{2} \cdot c \right)_M = 0 \). Moreover, if \( M \in \mathcal{N}(c)_\mathcal{A} \), then \( \mathcal{N}(c)_M = 1 \); therefore \( c_M \neq 0 \). Thus \( \mathcal{A}_M \neq 2 \) (otherwise \( c_M = 0 \) by our assumption about \( c \)). Thus \( \mathcal{N}(c)^\mathcal{A} \cap X(\mathcal{A}) = \emptyset \). By Lemma 4.8, there is \( h \in \mathcal{A} \) such that for all \( M \in \mathcal{N}(c)_\mathcal{A} \), \( h_M = \frac{1}{2} \). Now \( h \cdot \mathcal{N}(c) \in \mathcal{A} \), and \( h \cdot \mathcal{N}(c) \leq \neg (h \cdot \mathcal{N}(c)) \). Since the domain of \( \mathcal{A} \) is the set of all \( e \in \mathcal{C} \) such that \( e \land \neg e \in J \), \( h \cdot \mathcal{N}(c) = h \cdot \mathcal{N}(c) \land \neg (h \cdot \mathcal{N}(c)) \in J \), as desired.

Summing up, we have proved that if \( \mathcal{C} = \langle \mathcal{C}, J \rangle \) is a \( \mathcal{Q} \)-\( \mathcal{L}_\Pi^1 \) algebra and \( \mathcal{A} = \mathcal{C}^+ \), then modulo isomorphism \( \mathcal{C}^+ = \mathcal{C} \), and \( J(\mathcal{A}) = J \). This completes Step 2.

We complete the proof of Theorem 4.7, performing Step 3.

**Lemma 4.17.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( \mathcal{L}_\Pi \) algebras, and let \( \mathcal{C} = \mathcal{A}^+ \), and \( \mathcal{E} = \mathcal{B}^+ \) (up to isomorphism we can assume that \( \mathcal{A} \) and \( \mathcal{B} \) are \( \mathcal{L}_\Pi \) subalgebras of \( \mathcal{C} \) and \( \mathcal{E} \), respectively). Then every homomorphism \( h \) from \( \mathcal{A} \) into \( \mathcal{E} \) has a unique extension to a homomorphism of \( \mathcal{L}_\Pi^1 \) algebras from \( \mathcal{C} \) into \( \mathcal{E} \).

**Proof.** Uniqueness is immediate; if \( \alpha = \langle (b^1, q^1) \cdots (b^k, q^k), a \rangle \in \mathcal{C} \), and \( h \) is a homomorphism from \( \mathcal{A} \) into \( \mathcal{B} \), then every homomorphism \( \hat{h} \) from \( \mathcal{C} \) into \( \mathcal{E} \) which extends \( h \) must satisfy \( \hat{h}(\alpha) = h(\alpha) \lor \bigwedge_{i=1}^{k} h(b^i) \cdot q^i \).

We prove the existence of such extension. We can assume without loss of generality that \( h \) is onto \( \mathcal{B} \), and that \( \mathcal{B} \) is the quotient of \( \mathcal{A} \) modulo some ideal \( J \) of \( \mathcal{A} \). By Propositions 2.4 and 2.7, we can represent \( \mathcal{A} \) and \( \mathcal{C} \) as Boolean products of the families \( \{ \mathcal{A}_M : M \in \mathcal{B}_\mathcal{A} \} \), and \( \{ \mathcal{C}_M : M \in \mathcal{B}_\mathcal{C} \} \). Moreover, we can think of \( h \) as the projection on the set \( J^* = \{ M \in \)}
$M(\mathcal{D}_\alpha) : M \supseteq J$. Now let $\hat{h}$ be the projection of $\mathcal{E}$ on $J^*$. Clearly, $\hat{h}$ is a homomorphism with domain $\mathcal{E}$. Moreover, for every $\alpha = \langle (b^1, q^1), \ldots, (b^k, q^k), a \rangle \in \mathcal{E}$ (thought of as a function on $M(\mathcal{D}_\alpha)$), $\hat{h}(\alpha) \in \mathcal{E}$, because for all $M \in J^*$, one has

$$
(\hat{h}(\alpha))_M = \begin{cases} 
q^1 & \text{if } M \in (b^1)^o \\
\vdots & \vdots \\
q^k & \text{if } M \in (b^k)^o \\
\alpha_M & \text{otherwise.}
\end{cases}
$$

Thus $\hat{h}(\alpha) = h(a) \lor \lor_{i=1}^k h(b^i) \cdot q^i \in \mathcal{E}$. Hence, $\hat{h}$ is a homomorphism from $\mathcal{E}$ into $\mathcal{E}$ which extends $h$, and Lemma 4.17 is proved.

By Lemma 4.17, Step 3 is complete, and Theorem 4.7 is proved.

**Example 2.** (a) Let $\mathcal{B}$ be a Boolean algebra, and let $\mathcal{F} = \langle X, T \rangle$ be its Stone space. Let $J = \{0\}$, and let $\mathcal{A}$ be the set of continuous maps from $X$ into $Q \cap [0,1]$, with operations of $\mathcal{L}\mathbb{II}_1^1$ algebras defined pointwise. Then $\mathcal{A}$ is a $\mathcal{L}\mathbb{II}_1^1$ algebra, $\mathcal{B} = \Pi^-(\langle \mathcal{A}, J \rangle)$, and $\langle \mathcal{A}, J \rangle = \mathcal{B}^+$. Note that by a compactness argument we can see that all $f \in \mathcal{A}$ have finite range.

(b) An $\mathcal{L}\mathbb{II}_1^1$ algebra which is the reduct of a $\mathcal{L}\mathbb{II}_1^1$ algebra $\mathcal{A}$ is represented as $\Pi^-(\langle \mathcal{A}, I \rangle)$, where $I$ is the improper ideal constituted by the whole $\mathcal{A}$. Note that the condition that $\mathcal{A}_M = \mathcal{I}_Q$ for each $M \in J^*$ is trivially satisfied, as $J^* = \emptyset$.

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**References**


REPRESENTATION THEOREMS FOR MV\_\_


