A non-Abelian black ring

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ABSTRACT

We construct a supersymmetric black ring solution of SU(2) $\mathcal{N} = 1$, $d = 5$ Super-Einstein–Yang–Mills (SEYM) theory by adding a *distorted* BPST instanton to an Abelian black ring solution of the same theory. The change cannot be observed from spatial infinity: neither the mass, nor the angular momenta or the values of the scalars at infinity differ from those of the Abelian ring. The entropy is, however, sensitive to the presence of the non-Abelian instanton, and it is smaller than that of the Abelian ring, in analogy to what happens in the supersymmetric colored black holes recently constructed in the same theory and in $\mathcal{N} = 2$, $d = 4$ SEYM. By taking the limit in which the two angular momenta become equal we derive a non-Abelian generalization of the BMPV rotating black-hole solution.

0. Introduction

The discovery of black rings by Emparan and Reall in Ref. [1] showed how two important properties of 4-dimensional asymptotically-flat black holes, uniqueness/no-hair and spherical topology of the event horizon (which, for the 5-dimensional black ring, is $S^2 \times S^1$), could be violated in higher dimensions.[1] For a range of values of the conserved charges (mass, angular momenta) that may characterize an uncharged black ring, a different black-ring and a black-hole solutions are also possible. For charged black rings (the first of which was constructed in Ref. [5]) the non-uniqueness becomes infinite; for the same conserved electric charges one can construct black rings with regular horizons with magnetic dipole momenta taking continuous values in some interval [6]. Despite being innocuous to the conserved charges, these dipole momenta do contribute to the BH entropy. The construction of supersymmetric black-ring solutions in minimal [7] or matter-coupled $\mathcal{N} = 1$, $d = 5$ supergravity [8–12] using the general classification of supersymmetric solutions of these theories started in Ref. [13] opened up the possibility of constructing very general families of black-ring solutions with various kinds of electric charges and moduli in which these issues could be studied.

The violation of the no-hair conjecture by non-Abelian fields in 4-dimensions is also a well-known but less stressed fact, perhaps because the first solutions in which this was observed [14–16], black-hole generalizations of the "Bartnik–McKinnon particle" [17] with asymptotically vanishing gauge charges, were purely numerical, which makes more difficult their study and understanding.[2] The first black-holes with non-Abelian hair (not related to the embedding of an Abelian field into a non-Abelian one through a singular gauge transformation) given in an analytical form were found using supersymmetry techniques in the context of $\mathcal{N} = 2$, $d = 4$ Super-Einstein–Yang–Mills (SEYM) theory[3] in Refs. [20] and [21] using the general classification of the timelike supersymmetric solutions of these theories made in Ref. [22]. The black-hole solutions constructed in Ref. [21] include the field of an SU(2) *colored* monopole found by Protogenov in [23] which also has asymptotically vanishing gauge charge. The monopole charge does contribute to the entropy, though. These black holes, which can be seen as the result of adding the colored monopole to a standard black hole with Abelian charges, modifying the entropy but none of the asymptotic charges, were called *colored black holes* and they seem to be ubiquitous [24].

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[1] See, for instance, the reviews [2–4] and references therein.

[2] For a review on hairy and non-Abelian black-hole solutions see Ref. [18] or the more recent Ref. [19].

[3] This theory is the simplest $\mathcal{N} = 2$ supersymmetric generalization of the Einstein–Yang–Mills theory. This supersymmetrization requires the addition of scalar fields to the pure Einstein–Yang–Mills theory in order to complete $\mathcal{N} = 2$, $d = 4$ vector supermultiplets and, often, the addition of full vector supermultiplets to fulfill the requirements of Special Geometry. There may be more than one way of performing this supersymmetrization. Thus, there are more than one $\mathcal{N} = 2$, $d = 4$ SEYM theory with gauge group SU(2), for instance. These theories are also known as non-Abelian gauged $\mathcal{N} = 2$, $d = 4$ supergravity coupled to vector supermultiplets.

The results of Ref. [22] have been used more recently to construct new single-center and two-center non-Abelian solutions of $\mathcal{N}=2$, $d=4$ SEYM models that can be obtained by dimensional reduction of $\mathcal{N}=1$, $d=5$ SEYM models[4] in Ref. [25].

One of the main goals of that exercise was to open the possibility for the construction of the first non-Abelian black-hole solutions in $d=5$ by oxidation to $d=5$ of those solutions, because the direct construction using the general classification of timelike supersymmetric solutions of Refs. [26,27] turns out to be too complicated. This can only be done for certain models of the lower dimensional theory. The oxidation itself turned out to be a non-trivial exercise if one wanted to construct solutions without spatial translation isometries (which would be black strings instead of black holes), but, as was shown in Ref. [28], one can use non-trivial cycles to perform the reduction and still preserve supersymmetry, basically using Kronheimer's mechanism [29]. Both kinds of black solutions (strings and holes) were recently constructed in Ref. [30].

The $d=5$ non-Abelian black holes constructed there are, again, colored black holes, with asymptotically vanishing gauge fields. They can be understood as the result of adding a BPST instanton to a black hole with Abelian charges, leaving the mass and electric charges unmodified. Just as in the 4-dimensional case, the non-Abelian field does contribute to the entropy. The BPST instanton field turns out to be related by dimensional redox to the colored monopole at the heart of the 4-dimensional colored black holes.

It is natural to try to see if black-rings also admit the addition of non-Abelian instanton fields and the effect this addition may have on the mass and entropy. In this paper we are going to construct and study a regular supersymmetric black-ring solution of $\mathcal{N}=1$, $d=5$ SEYM with a *distorted* BPST instanton. We start by reviewing in Section 1 the recipe that we are going to use to construct timelike supersymmetric solutions, which was obtained in Ref. [30]. In Section 2 we will carry out the construction of the solution after which we will study its regularity and we will compute its essential properties. In Section 3 we will study the limit in which the black ring becomes a non-Abelian rotating black hole. Our conclusions are in Section 4.

1. The recipe to construct solutions

$\mathcal{N}=1$, $d=5$ Super-Einstein–Yang–Mills (SEYM) theories describe a supergravity multiplet (constituted by the graviton $e^a{}_\mu$, the gravitino ψ^i_μ and the graviphoton $A^0{}_\mu$) coupled to n_v vector multiplets labeled by $x=1,\cdots,n_v$ (each containing a real vector field $A^x{}_\mu$, a real scalar ϕ^x and a gaugino $\lambda^{i\,x}$). The graviphoton and the matter vector fields are collectively denoted by $A^I{}_\mu$, $I, J, \ldots = 0, 1, \cdots, n_v$. The ungauged theory (the couplings between scalars and vector fields dictated by the σ-model metric $g_{xy}(\phi)$, the kinetic matrix $a_{IJ}(\phi)$ and the Chern–Simons couplings) is completely determined by a constant symmetric tensor C_{IJK}.[5]

In the gauged theory, a subset of the vector fields plays the role of gauge field of some non-Abelian group whose structure constants will be denoted by $f_{IJ}{}^K$ in the understanding that they will just vanish for the values of the indices that do not correspond

to the gauge fields. The transformations of the scalars under the gauge group are generated by Killing vectors of the σ-model metric $k_I{}^x(\phi)$ satisfying the Lie algebra of the gauge group. Again, it is assumed that they can be identically zero for the values of $I, J, \cdots$ corresponding to the ungauged directions.

Thus, the bosonic action of $\mathcal{N}=1$, $d=5$ SEYM is given by

$$
S = \int d^5x \sqrt{g} \left\{ R + \tfrac{1}{2}g_{xy}\mathfrak{D}_\mu \phi^x \mathfrak{D}^\mu \phi^y - \tfrac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu} \right.
$$
$$
+ \tfrac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}\left[F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\alpha \right.
$$
$$
\left. - \tfrac{1}{2}\hat{g}f_{LM}{}^I F^J{}_{\mu\nu}A^K{}_\rho A^L{}_\sigma A^M{}_\alpha \right.
$$
$$
\left. \left. + \tfrac{1}{10}\hat{g}^2 f_{LM}{}^I f_{NP}{}^J A^K{}_\mu A^L{}_\nu A^M{}_\rho A^N{}_\sigma A^P{}_\alpha \right] \right\},
$$

(1.1)

where $\hat{g}$ is the gauge coupling constant, $F^I{}_{\mu\nu} = 2\partial_{[\mu}A^I{}_{\nu]} + \hat{g}f_{JK}{}^I A^J{}_\mu A^K{}_\nu$ are the non-Abelian vector field strengths and $\mathfrak{D}_\mu\phi^x = \partial_\mu\phi^x + \hat{g}A^I{}_\mu k_I{}^x$ are the gauge-covariant derivatives of the scalars.

In Ref. [30] we have found a procedure to construct systematically timelike supersymmetric solutions admitting an additional spacelike isometry (with adapted coordinate z) of any $\mathcal{N}=1$, $d=5$ Super-Einstein–Yang–Mills (SEYM):

1. Find a set of t- and z-independent functions M, H, Φ^I, L_I and 1-forms ω, A^I, χ in $\mathbb{E}^3$ satisfying the equations (defined in $\mathbb{E}^3$ as well)[6]

$$
d \star_3 dM = 0, \quad (1.2)
$$

$$
\star_3 dH - d\chi = 0, \quad (1.3)
$$

$$
\star_3 \breve{\mathfrak{D}}\Phi^I - \breve{F}^I = 0, \quad (1.4)
$$

$$
\breve{\mathfrak{D}}^2 L_I - g^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J \Phi^K L_M = 0, \quad (1.5)
$$

$$
\star_3 d\omega - \left\{ HdM - MdH + 3\sqrt{2}(\Phi^I \breve{\mathfrak{D}}L_I - L_I\breve{\mathfrak{D}}\Phi^I) \right\} = 0. \quad (1.6)
$$

The first two equations state that H and M are harmonic functions on $\mathbb{E}^3$. Once H is given, the second equation (which is the Abelian Bogomol'nyi equation on $\mathbb{E}^3$ [33]) can be solved for χ. Eq. (1.4) is the general Bogomol'nyi equation on $\mathbb{E}^3$. In the ungauged (Abelian) directions, it implies that the Φ^I are harmonic functions on $\mathbb{E}^3$ and, once they are chosen, the corresponding vectors $\breve{A}^I$ can be determined. In the non-Abelian directions, the equation becomes non-linear and one has to find simultaneously solutions for the functions Φ^I and gauge fields $\breve{A}^I$ through adequate ansatzs or other methods. Eq. (1.5) is automatically solved if we choose $L_I \propto \Phi^I$ (or zero). Finally, Eq. (1.6) can always be solved if the other equations are solved (because they solve its integrability condition), except, perhaps, at the singularities of the functions where, strictly speaking, the other equations are not solved. In most cases, the integrability condition can be solved by a choice of integration constants in the functions H, M, L_I, Φ^I. Then, of course, one has to integrate explicitly Eq. (1.6) to obtain ω.

2. Using them, reconstruct the solution's 5-dimensional spacetime fields as follows:
   (a) The scalars can be found from this equation for the quotients $h_I(\phi)/\hat{f}$

$$
h_I/\hat{f} = L_I + 8C_{IJK}\Phi^J\Phi^K/H, \quad (1.7)
$$

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because there is always a parametrization of the scalar manifold such that

$$\phi^x \equiv h_x/h_0 . \tag{1.8}$$

With the above equation for the quotients $h_I(\phi)/\hat{f}$ one can also determine the function $\hat{f}$. For the special case of symmetric scalar manifolds, it is given by[7]

$$\hat{f}^{-3} = 3^3 C^{IJK} L_I L_J L_K + 3^4 \cdot 2^3 C^{IJK} C_{KLM} L_I L_J \Phi^L \Phi^M / H$$
$$+ 3 \cdot 2^6 L_I \Phi^I C_{JKL} \Phi^J \Phi^K \Phi^L / H^2$$
$$+ 2^9 \left(C_{IJK} \Phi^I \Phi^J \Phi^K \right)^2 / H^3 . \tag{1.9}$$

(b) The metric has the form

$$ds^2 = \hat{f}^2 (dt + \hat{\omega})^2 - \hat{f}^{-1} d\hat{s}^2 , \tag{1.10}$$

where $\hat{f}$ has been determined above, the 1-form $\hat{\omega}$ is given by[8]

$$\hat{\omega} = \omega_5 (dz + \chi) + \omega , \tag{1.11}$$
$$\omega_5 = M + 16\sqrt{2} H^{-2} C_{IJK} \Phi^I \Phi^J \Phi^K$$
$$+ 3\sqrt{2} H^{-1} L_I \Phi^I , \tag{1.12}$$

and where the 4-dimensional Euclidean metric $d\hat{s}^2$ is given by[9]

$$d\hat{s}^2 = H^{-1} (dz + \chi)^2 + H dx^r dx^r , \quad r = 1, 2, 3 . \tag{1.13}$$

(c) The vector fields and their corresponding field strengths are given by

$$A^I = -\sqrt{3} h^I \hat{f} (dt + \hat{\omega}) + \hat{A}^I ,$$
$$F^I = -\sqrt{3} \hat{\mathfrak{D}} [h^I \hat{f} (dt + \hat{\omega})] + \hat{F}^I , \tag{1.14}$$

where the vector fields $\hat{A}^I$, defined on the 4-dimensional Euclidean space $d\hat{s}^2$, and their field strengths are given by

$$\hat{A}^I = 2\sqrt{6} \left[H^{-1} \Phi^I (dz + \chi) - \check{A}^I \right] ,$$
$$\hat{F}^I = 2\sqrt{6} H^{-1} \left[\check{\mathfrak{D}} \Phi^I \wedge (dz + \chi) - \star_3 H \check{\mathfrak{D}} \Phi^I \right] , \tag{1.15}$$

where $\hat{\mathfrak{D}}$ (resp. $\check{\mathfrak{D}}$) is the exterior gauge-covariant derivative with respect to the connection $\hat{A}^I$ (resp. $\check{A}^I$).

In Ref. [30] we used this recipe to construct black-hole solutions with non-Abelian gauge and scalar fields for the SU(2)-gauged ST[2, 5] model.[10] This model has 4 vector multiplets and, hence, 4 scalar fields that parametrize the symmetric space SO(1, 3)/ SO(3). It is defined by a tensor C_{IJK} with the following non-vanishing components

$$C_{0xy} = \frac{1}{6} \eta_{xy} , \quad \text{where} \quad (\eta_{xy}) = \text{diag}(+ - \cdots -) , \quad \text{and}$$
$$x, y = 1, \cdots, 4 . \tag{1.16}$$

The directions to be gauged are the last three, which we will denote by indices $\alpha, \beta, \ldots = 2, 3, 4$. The ungauged directions will be denoted by indices $i, j, \ldots = 0, 1$.

Being a symmetric space, we can use Eq. (1.9) to write the metric function $\hat{f}$ as a function of the building blocks H, L_I, Φ^I:

$$\hat{f}^{-1} = H^{-1} \left\{ \frac{1}{4} \left(6 H L_0 + 8 \eta_{xy} \Phi^x \Phi^y \right) \right.$$
$$\times \left[9 H^2 \eta^{xy} L_x L_y + 48 H \Phi^0 L_x \Phi^x \right.$$
$$+ \left. 64 (\Phi^0)^2 \eta_{xy} \Phi^x \Phi^y \right]^{1/3} \left. \right\}^{1/3} . \tag{1.17}$$

Now, in order to find solutions of this model, we just need to find building blocks that satisfy Eqs. (1.2)–(1.6). In the next section we will just do this to find a solution that describes a black ring.

2. Non-Abelian black rings

2.1. Construction of the solution

Inspired by Refs. [11,8], we choose a point $\vec{x}_0 \equiv (0, 0, -R^2/4)$ in $\mathbb{E}^3$ and a harmonic function N with a pole at that point,

$$N \equiv \frac{1}{|\vec{x} - \vec{x}_0|} \equiv \frac{1}{r_n} , \tag{2.1}$$

in terms of which we can write the non-vanishing building blocks in the ungauged directions as

$$H = \frac{1}{r} , \qquad M = \frac{3}{4} \lambda_i q^i \left(1 - |\vec{x}_0| N \right) ,$$
$$\Phi^i = -\frac{q^i}{4\sqrt{2}} N , \qquad L_i = \lambda_i + \frac{Q_i - C_{ijk} q^j q^k}{4} N . \tag{2.2}$$

These functions contain the integration constants q^i, Q_i and λ_i. The first two can be interpreted as charges. The latter, whose value will be restricted by requirements such as the normalization of the metric at infinity, are moduli. Eq. (1.2) is satisfied automatically. Eq. (1.3) is satisfied with

$$\chi = \cos\theta d\psi , \tag{2.3}$$

where $r, \theta \in (0, \pi)$ and $\psi \in [0, 2\pi)$ are spherical coordinates centered at $r = |\vec{x}| = 0$ with the definitions and orientation

$$\begin{cases} x^1 = r \sin\theta \sin\psi , \\ x^2 = r \sin\theta \cos\psi , \\ x^3 = -r \cos\theta , \end{cases} \quad \epsilon^{123} = \epsilon^{r\theta\psi} = +1 . \tag{2.4}$$

Eqs. (1.4) are satisfied with

$$\check{A}^i = -\frac{q^i}{4\sqrt{2}} \cos\theta_n d\psi_n , \tag{2.5}$$

where $r_n, \theta_n \in (0, \pi)$ and $\psi_n \in [0, 2\pi)$ are spherical coordinates centered at $r_n = |\vec{x}_n| = 0$ with the definitions

$$\begin{cases} x_n^1 \equiv x^1 - x_0^1 = r_n \sin\theta_n \sin\psi_n , \\ x_n^2 \equiv x^2 - x_0^2 = r_n \sin\theta_n \cos\psi_n , \\ x_n^3 \equiv x^3 - x_0^3 = -r_n \cos\theta_n , \end{cases} \tag{2.6}$$

and the same orientation as the spherical coordinates centered at $r = 0$.

Eqs. (1.5) in the Abelian directions are trivially satisfied because all $f_{ij}{}^k = 0$ and, finally, the integrability condition of Eq. (1.6) is identically satisfied for the chosen integration constants and ω can be found by integration. We will compute ω for the complete solution later.

The above functions are enough to construct an Abelian black ring. Now, we excite the gauged directions of this solution by adding to it a solution of the SU(2) Bogomol'nyi equations on $\mathbb{E}^3$ (1.4)

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[7] In this expression, $C^{IJK} \equiv C_{IJK}$.

[8] The unhatted ω is the one occurring in Eq. (1.6).

[9] With H and χ related by Eq. (1.3), this is a hyperKähler metric admitting a triholomorphic Killing vector, also known as Gibbons–Hawking metric [34,35]. We will also denote the compact coordinate z by φ. It will be assumed to take values in $[0, 4\pi)$.

[10] Actually, this is the name of the model of $\mathcal{N} = 2, d = 4$ supergravity one obtains by dimensional reduction.

$$\Phi^\alpha = \frac{1}{gr_n\left(1+\lambda^2 r_n\right)}\delta^\alpha_{s+1}\frac{x_n^s}{r_n}\,,$$

$$\check{A}^\alpha = \frac{1}{gr_n\left(1+\lambda^2 r_n\right)}\epsilon^\alpha{}_{rs}\frac{x_n^s}{r_n}dx_n^r\,. \tag{2.7}$$

This solution, originally found by Protogenov in Ref. [23], describes a magnetic colored monopole placed at $r_n = 0$. It is singular at $r_n = 0$ as a field configuration in $\mathbb{E}^3$, but this behavior can change when we analyze the whole picture. In fact, we showed in Ref. [28] that the monopole field gives rise to a BPST instanton in $\mathbb{E}^4$ through (1.15), and we used this result in Ref. [30] to construct a regular black hole of the same supergravity theory we consider in this work.

In the present case we obtain a different instanton field configuration from (1.15), which we call *distorted* BPST, because the pole of the harmonic function H is placed in a different point ($r = 0$) than that of the colored monopole ($r_n = 0$). This distorted BPST is singular at $r_n = 0$, which might turn the black ring solution ill-defined. Happily this is not the case. The complete vector field contains the instanton plus an additional term, see (1.14), where the latter cancels precisely this divergence at that critical point

$$\lim_{r_n\to\infty}\left(-\sqrt{3}h^I\hat{f}\omega_5 + 2\sqrt{6}H^{-1}\Phi^I\right)(dz+\chi)=0\,. \tag{2.8}$$

Observe that in the ungauged case the Φ^αs would have been harmonic functions $-q^\alpha N/(4\sqrt{2})$ and the combinations $C_{ijk}q^jq^k$ should have been replaced by $C_{iJK}q^Jq^K$. Here the asymptotic behavior of the non-Abelian gauge field indicates that the "non-Abelian q^αs" do not contribute in the same way the q^is do. However, they have a similar near-horizon behavior.

The above functions define completely the solution. In what follows we are going to analyze its metric to show that it describes a regular black ring and to compute its main properties.

2.2. Analysis of the solution

In this analysis it is convenient to use two set of coordinates: those centered at $r = 0$ (r, θ, ψ, defined in Eq. (2.4)) supplemented by the time coordinate t and the angular coordinate φ, and those centered at $r_n = 0$ (r_n, θ_n, ψ_n, defined in Eq. (2.6)) supplemented by the time coordinate t_n and the angular coordinate φ_n. The relations

$$\begin{aligned} r_n &= \left(r^2 + |\vec{x}_0|^2 - 2|\vec{x}_0|r\cos\theta\right)^{1/2}\,, \\ r &= \left(r_n^2 + |\vec{x}_0|^2 + 2|\vec{x}_0|r_n\cos\theta_n\right)^{1/2}\,, \\ |\vec{x}_0| &= r\cos\theta - r_n\cos\theta_n\,, \end{aligned} \tag{2.9}$$

will be useful in the computations.

The metric function $\hat{f}$ can be obtained by substituting the functions H, L_I, Φ^I in Eq. (1.9). At this moment we just want to impose the standard asymptotic normalization

$$\lim_{r\to\infty}\hat{f} = 1\,, \quad\Rightarrow\quad 3^3 C^{ijk}\lambda_i\lambda_j\lambda_k = \frac{3^3}{2}\lambda_0\lambda_1^2 = 1\,. \tag{2.10}$$

Now let us compute the only missing ingredient in the metric (1.10): the 1-form $\hat{\omega}$. Let us consider Eq. (1.6), which, upon substitution of the chosen functions H, M, L_I, Φ^I, can be written as

$$\begin{aligned} \star_3 d\omega = -\frac{3}{4}\lambda_i q^i \Bigg\{ &-\frac{1}{r^2}\left[1 - \frac{|\vec{x}_0|+r}{r_n}\right.\\ &\left. + \frac{r|\vec{x}_0|(r+|\vec{x}_0|)}{r_n^3}(1-\cos\theta)\right]dr \\ &+ \left[\frac{|\vec{x}_0|\sin\theta}{r_n^3}(r - |\vec{x}_0|)\right]d\theta \Bigg\}\,, \end{aligned} \tag{2.11}$$

and a solution can be readily found assuming ω has only one non-vanishing component, ω_ψ [11]:

$$\omega = -\frac{3}{4}\lambda_i q^i(\cos\theta - 1)\left[1 - \left(r + \frac{R^2}{4}\right)\frac{1}{r_n}\right]d\psi\,. \tag{2.12}$$

Observe that, since $L_\alpha = 0$ the non-Abelian terms do not affect ω. However, they do affect the whole 5-dimensional $\hat{\omega}$ given in Eq. (1.11) via ω_5 in Eq. (1.12):

$$\hat{\omega} = (F - G)d\varphi + (F - G\cos\theta)d\psi\,, \tag{2.13}$$

$$F = \frac{3\lambda_i q^i}{4}\left[1 - \left(r + \frac{R^2}{4}\right)\frac{1}{r_n}\right]\,, \tag{2.14}$$

$$\begin{aligned} G = &\frac{q^i}{16}\left[3\left(Q_i - C_{ijk}q^jq^k\right) + 2C_{ijk}q^jq^k\frac{r}{r_n}\right]\frac{r}{r_n^2}\\ &- \frac{2q^0}{g^2}\frac{r^2}{r_n^3\left(1+\lambda^2 r_n\right)^2}\,. \end{aligned} \tag{2.15}$$

The last term in G has a non-Abelian origin. In the $r \to \infty$ limit in which the metric tends to Minkowski's (so we have an asymptotically flat solution), though, it is subdominant and we do not expect it to contribute to the angular momentum of the solution.

So far we have been working in coordinates in which the hyperKähler metric Eq. (1.13) is of the form

$$\begin{aligned} d\hat{s}^2 = &r\left(d\varphi + \cos\theta d\psi\right)^2 \\ &+ \frac{1}{r}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right]\,, \end{aligned} \tag{2.16}$$

but, in order to compute mass and angular momentum, it is convenient to use a different coordinate system (also centered at $\vec{x} = 0$) $t, \Theta, \phi_1, \phi_2$, related to the former by

$$r = \frac{\rho^2}{4}\,, \qquad \theta = 2\Theta\,, \qquad \psi = \phi_1 - \phi_2\,, \qquad \varphi = \phi_1 + \phi_2\,, \tag{2.17}$$

in which the complete 5-dimensional metric is of the form

$$\begin{aligned} ds^2 = &\hat{f}^2\left(dt + \hat{\omega}\right)^2 - \hat{f}^{-1}\Big[d\rho^2 \\ &+ \rho^2\left(d\Theta^2 + \cos^2\Theta d\phi_1^2 + \sin^2\Theta d\phi_2^2\right)\Big]\,, \end{aligned} \tag{2.18}$$

with

$$\hat{\omega} = \left(2F - 2G\cos^2\Theta\right)d\phi_1 - 2G\sin^2\Theta d\phi_2\,. \tag{2.19}$$

The independent components of the angular momentum are now obtained from the metric behavior in the $\rho \to \infty$ limit [12]

$$J_{\phi_1} = \lim_{\rho\to\infty}\frac{\pi|g_{t\phi_1}|\rho^2}{4G_N\cos^2\Theta} = \frac{1}{2\sqrt{3}}q^i\left(3Q_i - C_{ijk}q^jq^k\right)\,, \tag{2.20}$$

$$J_{\phi_2} = \lim_{\rho\to\infty}\frac{\pi|g_{t\phi_2}|\rho^2}{4G_N\sin^2\Theta} = \frac{1}{2\sqrt{3}}q^i\left(3Q_i - C_{ijk}q^jq^k + 6\lambda_i R^2\right)\,, \tag{2.21}$$

and, from the absence of contribution proportional to g, we see that they coincide with those of the Abelian black ring, as we expected.

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[11] The expression coincides with that of [36] despite we have chosen $\vec{x}_0$ to be on the negative x^3 axis. This is because the coordinate θ has also a relative sign with respect to the used in that reference.

[12] We use units in which $G_N = \sqrt{3}\pi/4$.

Observe that these formulae allow us to identify

$$q^i \lambda_i R^2 = \frac{1}{\sqrt{3}} (J_{\phi_2} - J_{\phi_1}) \,. \tag{2.22}$$

Before we move to study the possible presence of an event horizon, let us point out that the solution does not contain any Dirac–Misner strings.[13] Indeed, the $g_{t\phi_1}$ (resp. $g_{t\phi_2}$) metric component vanishes when the coordinate ϕ_1 (resp. ϕ_2) is not well defined, which happens at $\Theta = \pi/2$ ($\Theta = 0$).

The solution may have an event horizon at $\vec{x} = \vec{x}_0$, where the norm of the timelike Killing vector of the metric vanishes. In order to study the near horizon limit we need to use a different coordinate system because several components of the metric blow up there in the coordinates we have been using so far. Recall the expression for the metric in the original frame centered at $\vec{x} = 0$

$$\begin{aligned} ds^2 =\ & \hat{f}^2 (dt + \omega)^2 + 2\hat{f}^2 \omega_5 (dt + \omega)(d\varphi + \cos\theta d\psi) \\ & - \hat{f}^2 \left(\hat{f}^{-3} r - \omega_5^2 \right) (d\varphi + \cos\theta d\psi)^2 - \hat{f}^{-1} r^{-1} dx^r dx^r \,. \end{aligned} \tag{2.23}$$

We first go to the auxiliary frame centered at the horizon with spherical coordinates and take the $r_n \to 0$ limit. The functions that appear in the metric behave in this limit as follows

$$\hat{f} = \frac{16}{R^2 v^2} r_n^2 + \mathcal{O}(r_n^3) \,, \tag{2.24}$$

$$\omega_{\psi_n} = -\frac{3}{R^2} \lambda_i q^i \sin^2\theta_n r_n + \mathcal{O}(r_n^2) \tag{2.25}$$

$$\hat{f}^{-1} r^{-1} = \frac{v^2}{4} r_n^{-2} + k_1 r_n^{-1} + \mathcal{O}(r_n) \,, \tag{2.26}$$

$$\hat{f}^2 \omega_5 = -\frac{2}{v} r_n + k_2 r_n^2 + \mathcal{O}(r_n^3) \,, \tag{2.27}$$

$$\hat{f}^2 (\hat{f}^{-3} r - \omega_5^2) = \frac{l^2}{4} + k_3 r_n + \mathcal{O}(r_n^2) \,, \tag{2.28}$$

where we have defined the constants

$$v = \left(C_{ijk} q^i q^j q^k - 16 \frac{q^0}{g^2} \right)^{1/3} \,, \tag{2.29}$$

$$\begin{aligned} l = \frac{1}{2v^2} \Big[& 9 \cdot 6^2 C^{ijk} C_{klm} \left(Q_i - C_{ihn} q^h q^n \right) \left(Q_j - C_{jpq} q^p q^q \right) q^l q^m \\ & - 9 \left(q^i Q_i - C_{ijk} q^i q^j q^j \right)^2 - 12 q^i \lambda_i R^2 v^3 \\ & - 9 \left(Q_1 - \frac{q^0 q^1}{3} \right)^2 \left(\frac{32}{g^2} \right) \Big]^{1/2} \,. \end{aligned} \tag{2.30}$$

These expression for the constants v and l resemble those of the Abelian case [11], with an additional non-Abelian term. The precise form of the constants k_1, k_2 and k_3 in terms of the charges are messy. They do not occur in the calculation of any physical quantity, but they play a role in the near horizon analysis,[14] since

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[13] They could have been removed but only at the price of introducing closed timelike curves [37].

[14] We give their form here for the sake of completeness,

$$k_1 = \frac{16\lambda^2 R^2 \frac{q^0}{g^2} + 3 \left(q^i Q_i - C_{ijk} q^i q^j q^k \right)}{3R^2 v} \,, \tag{2.31}$$

$$k_2 = \frac{4k_1}{v} \,, \tag{2.32}$$

$$k_3 = \frac{1}{2g^2 R^2 v^4} \left\{ \frac{3R^2 k_1}{v^3} \left[(q^0 q^1/3)^2 \left(96 - 3g^2 (q^1)^2 \right) \right. \right.$$

they are responsible for the disappearance of $\mathcal{O}(r_n^{-1})$ in the metric after we perform the following coordinate transformation,

$$dt_n = d\tau_n + \left(\frac{b_2}{r_n^2} + \frac{b_1}{r_n} \right) dr_n \,, \qquad d\varphi_n = -d\psi_n + 2d\xi_n + \frac{c_1}{r_n} dr_n \,, \tag{2.34}$$

where the constants b_1, b_2 and c_1 can be chosen such that all divergences in the metric in the $r_n \to 0$ limit disappear:

$$c_1 = \mp \frac{v}{l} \,, \qquad b_2 = \pm \frac{lv^2}{8} \,, \qquad b_1 = \pm \frac{4l^2 k_1 + l^2 v^3 k_2 + 4v^2 k_3}{16l} \,. \tag{2.35}$$

With this choice we find in the $r_n \to 0$ limit that the horizon has the following metric

$$ds_h^2 = -l^2 d\xi_n^2 - \frac{v^2}{4} \left(d\theta_n^2 + \sin^2\theta_n d\psi_n^2 \right) \,, \tag{2.36}$$

with the topology $S^1 \times S^2$, so the solution is a black ring with non-Abelian hair, *i.e.* a non-Abelian black ring. Using this metric we can compute the area of the horizon,[15]

$$\frac{A_h}{2\pi^2} = \frac{1}{2\pi^2} \int d^3 x \sqrt{|g_h|} = lv^2 \,, \tag{2.37}$$

so the entropy of the non-Abelian black ring can be written in terms of the charges and angular momenta using the expressions for the constants v and l Eqs. (2.29) and (2.30) together with Eq. (2.22) as follows:

$$\begin{aligned} S = \pi \Big[& 3 \cdot 6^2 C^{ijk} C_{klm} \left(Q_i - C_{ihn} q^h q^n \right) \left(Q_j - C_{jpq} q^p q^q \right) q^l q^m \\ & - 3 \left(q^i Q_i - C_{ijk} q^i q^j q^k \right)^2 \\ & - \frac{4}{\sqrt{3}} (J_{\phi_2} - J_{\phi_1}) \left(C_{ijk} q^i q^j q^k - 16 \frac{q^0}{g^2} \right) \\ & - 3 \left(Q_1 - \frac{q^0 q^1}{3} \right)^2 \left(\frac{32}{g^2} \right) \Big]^{1/2} \,. \end{aligned} \tag{2.38}$$

$$\begin{aligned} & + 6(q^0 q^1/3) \left(-32 Q_1 + g^2 q^1 (-2(q^1)^2/6q^0 + 2q^0 Q_0 + q^1 Q_1) \right) \\ & + 3 \left(4(q^1)^2/6 g^2 q^0 q^1 Q_1 + 32 Q_1^2 \right. \\ & \left. + g^2 \left(-q^1 Q_1 (4q^0 Q_0 + q^1 Q_1) + (C_{ijk} q^i q^j q^k - q^i Q_i)^2 \right) \right) \\ & + 4\lambda_i q^i \left(-16q^0 + g^2 C_{ijk} q^i q^j q^k \right) R^2 \Big] \\ & - 3 \Big[9 g^2 ((q^1)^2/6 - Q_0)(q^0 q^1/3 - Q_1)^2 \\ & + \left(-24(q^0 q^1/3)^2 \lambda^2 + 6(q^1)^2/6 g^2 \lambda_1 q^0 q^1 \right. \\ & - 6g^2 \lambda_1 q^0 Q_0 q^1 + 96 \lambda_1 Q_1 - 6g^2 \lambda_0 q^0 q^1 Q_1 \\ & - 3g^2 \lambda_1 (q^1)^2 Q_1 - 24\lambda^2 Q_1^2 \\ & + 3q^0 q^1/3 \left(-32\lambda_1 + 2g^2 \lambda_0 q^0 q^1 + g^2 \lambda_1 (q^1)^2 + 16\lambda^2 Q_1 \right) \\ & + 32\lambda_i q^i q^0 - 8C_{ijk} q^i q^j q^k g^2 \lambda_i q^i + 6g^2 \lambda_i q^i Q_j q^j \big) R^2 + 16\lambda^2 \lambda_i q^i q^0 R^4 \Big] \Big\} \,. \end{aligned} \tag{2.33}$$

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[15] Notice that $\xi_n \in [0, 2\pi)$, as can be deduced from expression (2.34) together with $\int d\Omega_{(3)} = 2\pi^2$.

Finally, we would like to compute the mass of the solution. We do so by comparing the asymptotic behavior of the metric component g_{tt} with that of the Schwarzschild solution, $g_{tt} \sim 1 - \frac{8MG}{3\pi\rho^2} + \cdots$. We get

$$M = \frac{3^{5/2}\lambda_1}{2}(\lambda_1 Q_0 + 2\lambda_0 Q_1) \,. \qquad (2.39)$$

The constants λ_i can be expressed in terms of physical constants. If we define the physical scalars of the theory as $\phi^x \equiv h_x/h_0$ we find that the only scalar with a non-vanishing asymptotic value is the Abelian one and this value is $\phi^1_\infty = \lambda_1/\lambda_0$. On the other hand, the asymptotic normalization of the metric Eq. (2.10) implied $\lambda_0\lambda_1^2 = 2/3^3$. Then,

$$\lambda_0 = 2^{1/3}3^{-1}\left(\phi^1_\infty\right)^{-2/3}, \qquad \lambda_1 = 2^{1/3}3^{-1}\left(\phi^1_\infty\right)^{1/3}, \qquad (2.40)$$

and M takes the form

$$M = 2^{-1/3}3^{1/2}\left[\left(\phi^1_\infty\right)^{2/3}Q_0 + 2\left(\phi^1_\infty\right)^{-1/3}Q_1\right], \qquad (2.41)$$

and depends only on the moduli and on the electric charges Q_0, Q_1 while the q^i, which correspond to magnetic dipole momenta do not contribute to it [6]. The non-Abelian field do not contribute, either.

This expression looks identical to that of the non-Abelian black hole solution constructed in Ref. [30], but the charges Q_0 and Q_1 are not the same than the charges q_0 and q_1 that appear in the black-hole mass formula given in that reference. They are, actually, related by $Q_i^{BR} = q_i^{BH} + C_{ijk}q^j_{BR}q^k_{BR}$. This is just reflecting the fact that the conserved electrical charges in the black ring receive contributions from the magnetic dipole momenta via the Chern–Simons term in the action. This effect is commonly described as "charges dissolved in fluxes" [9].

This non-Abelian black-ring mass formula, is, however, identical to that of the Abelian black ring that one would obtain by removing the non-Abelian fields from this solution. In other words: the presence of non-Abelian fields is not observable at spatial infinity. They do contribute to the entropy, though, as in the black-hole case, their entropy being smaller than that of their Abelian siblings.

3. Non-Abelian rotating black holes

In the $R \to 0$ limit, several things happen:

1. All the harmonic functions are now centered at $r = 0$ (except for M which becomes constant):

$$H = N = \frac{1}{r}, \quad M = \frac{3}{4}\lambda_i q^i, \quad \Phi^i = -\frac{q^i}{4\sqrt{2}}N,$$

$$L_i = \lambda_i + \frac{Q_i - C_{ijk}q^j q^k}{4}H. \qquad (3.1)$$

2. The non-Abelian gauge field is also centered at $r = 0$:

$$\Phi^\alpha = \frac{1}{gr\left(1 + \lambda^2 r\right)}\delta^\alpha_{s+1}\frac{x^s}{r}, \qquad \breve{A}^\alpha = \frac{1}{gr\left(1 + \lambda^2 r\right)}\epsilon^\alpha{}_{rs}\frac{x^s}{r}dx^r, \qquad (3.2)$$

and the distorted BPST instanton is not distorted anymore.

3. The metric function $\hat{f}$ is now given by

$$\hat{f}^{-3} = \left[\frac{3}{2}\left(\lambda_0 + \frac{Q_0}{4r}\right) - \frac{2}{g^2}\frac{1}{r(1 + \lambda^2 r)^2}\right]$$
$$\times \left[9\left(\lambda_1 + \frac{Q_1}{4r}\right)^2 - \frac{2(q^0)^2}{g^2}\frac{1}{r^2(1 + \lambda^2 r)^2}\right]. \qquad (3.3)$$

The mass of this object is identical to that of the black ring Eqs. (2.39) and (2.41). It has no non-Abelian contributions. The near-horizon limit, though, includes non-Abelian terms

$$\hat{f}^{-1} \sim \frac{Y}{r}, \quad \text{with} \quad Y^3 = \left(\frac{3}{8}Q_0 - \frac{2}{g^2}\right)\left(\frac{9}{16}Q_1^2 - \frac{2}{g^2}(q^0)^2\right) \qquad (3.4)$$

4. ω vanishes identically and $\hat{\omega}$ is determined only by ω_5, which takes the form

$$\hat{\omega} = \omega_5(d\varphi + \cos\theta d\psi),$$
$$\omega_5 = \frac{q^i}{16}\left(3Q_i - C_{ijk}q^j q^k\right)\frac{1}{r} - \frac{2q^0}{g^2}\frac{1}{r\left(1 + \lambda^2 r\right)^2}. \qquad (3.5)$$

As a result, the two angular momenta become identical

$$J_{\phi_1} = J_{\phi_2} = \frac{1}{2\sqrt{3}}q^i\left(3Q_i - C_{ijk}q^j q^k\right) \equiv J. \qquad (3.6)$$

Observe that the non-Abelian term in ω_5, which does not contribute to the angular momentum, does contribute to the $r \to 0$ limit just as the Abelian terms:

$$\omega_5 \sim Z/r, \quad \text{where} \quad Z = \frac{\sqrt{3}}{8}J - \frac{2q^0}{g^2}. \qquad (3.7)$$

Let us study the near-horizon limit $\to 0$. Using Eqs. (3.4) and (3.7), we find that the metric Eq. (1.10) behaves in this limit as

$$ds^2 \sim \frac{r^2}{Y^2}dt^2 - \frac{Y}{r^2}dr^2 - Yd\Omega^2_{(2)} + \frac{2Z}{Y^2}rdt(d\varphi + \cos\theta d\psi)$$
$$+ \left(\frac{Z^2}{Y^2} - Y\right)(d\varphi + \cos\theta d\psi)^2, \qquad (3.8)$$

which can be rewritten in the form

$$ds^2 \sim Yd\Pi^2_{(2)} - Yd\Omega^2_{(2)} - Y\left[\sin\alpha\rho dt - \cos\alpha(d\varphi + \cos\theta d\psi)\right]^2, \qquad (3.9)$$

where $r = (Y^3 - Z^2)^{1/2}\rho$, $d\Pi^2_{(2)} = \rho^2 dt^2 - \frac{d\rho^2}{\rho^2}$ is the metric of the AdS$_2$ of unit radius and $\sin^2\alpha = Z^2/Y^3$. This space is the near-horizon limit of the BMPV black hole [38], but, due to the non-Abelian contribution to Z (which can be understood as a sort of "near-horizon angular momentum"), now α does not vanish for vanishing asymptotic angular momentum J and we can have a stationary black hole with $J = 0$ whose near-horizon limit is not AdS$_2 \times$ S^3. The converse is also possible: we can make $\alpha = Z = 0$ for $J = \frac{16}{\sqrt{3}}q^0/g^2$ and have a rotating black hole whose near-horizon limit is AdS$_2 \times$ S^3.

The area of the horizon is

$$\frac{A}{2\pi^2} = 8\sqrt{Y^3 - Z^2}. \qquad (3.10)$$

4. Conclusions

The existence of black-hole and black-ring solutions with identical asymptotic behavior but with non-Abelian hair that contributes to the entropy [21,25,24,30] challenges our understanding of black-hole hair and the microscopic interpretation of the black-hole/black-ring entropy, just as the Abelian hair discovered in Ref. [6] did. More research is necessary to gain a better understanding of these solutions. In particular, the stability of these supersymmetric non-Abelian solutions (which are entropically disfavored) needs to be addressed and their possible non-supersymmetric and non-extremal generalizations have to be constructed and studied. Work in these directions is in progress.

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