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A concavity inequality for symmetric norms

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Abstract

We review some recent convexity results for Hermitian matrices and we add a new one to the list: Let A be semidefinite positive, let Z be expansive, $Z^*Z \geqslant I$, and let $f : [0, \infty) \longrightarrow [0, \infty)$ be a concave function. Then, for all symmetric norms

$$\|f(Z^*AZ)\| \leqslant \|Z^*f(A)Z\|.$$

This inequality complements a classical trace inequality of Brown–Kosaki.

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Introduction

A good part of matrix analysis consists in establishing results for Hermitian operators considered as generalized real numbers. In particular several results are matrix versions of inequalities for convex functions f on the real line, such as

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

for all reals a, b and

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$$f(za) \leqslant zf(a) \tag{2}$$

for convex functions f with $f(0) \leqslant 0$ and scalars a and z with $0 < z < 1$.

In this brief note we first review some recent matrix versions of (1), (2) and next we give the matrix version of the companion inequality of (2):

$$f(za) \leqslant zf(a) \tag{3}$$

for concave functions f with $f(0) \geqslant 0$ and scalars a and z with $1 < z$.

Capital letters $A, B, \ldots, Z$ mean n-by-n complex matrices, or operators on a finite dimensional Hilbert space $\mathscr{H}$; I stands for the identity. When A is positive semidefinite, respectively positive definite, we write $A \geqslant 0$, respectively $A > 0$.

1. Some known convexity results

The following are well known trace versions of elementary inequalities (1) and (2).

1.1. von Neumann's trace inequality: For convex functions f and Hermitians A, B,

$$\operatorname{Tr} f\left(\frac{A+B}{2}\right) \leqslant \operatorname{Tr} \frac{f(A)+f(B)}{2} \tag{4}$$

equivalently $\operatorname{Tr} \circ f$ is convex on the set of Hermitians.

1.2. Brown–Kosaki's Trace Inequality [5]: Let f be convex with $f(0) \leqslant 0$ and let A be Hermitian. Then, for all contractions Z,

$$\operatorname{Tr} f(Z^*AZ) \leqslant \operatorname{Tr} Z^* f(A) Z. \tag{5}$$

1.3. Hansen–Pedersen's Trace Inequality [7]: Let f be convex and let $\{A_i\}_{i=1}^n$ be Hermitians. Then, for all isometric columns $\{Z_i\}_{i=1}^n$,

$$\operatorname{Tr} f\left(\sum_i Z_i^* A_i Z_i\right) \leqslant \operatorname{Tr} \sum_i Z_i^* f(A_i) Z_i.$$

Here isometric column means that $\sum_i Z_i^* Z_i = I$. Hansen–Pedersen's result contains (1) and (2).

When f is convex and monotone, we showed [2] that the above trace inequalities can be extended to operator inequalities up to a unitary congruence. Equivalently we have inequalities for eigenvalues. Let us give the precise statements corresponding to von Neumann and Brown–Kosaki trace inequalities.

1.4. Let A, B be Hermitians and let f be a *monotone* convex function. Then, there exists a unitary U such that

$$f\left(\frac{A+B}{2}\right) \leqslant U \cdot \frac{f(A)+f(B)}{2} \cdot U^*. \tag{6}$$

1.5. Let A be a Hermitian, let Z be a contraction and let f be a *monotone* convex function with $f(0) \leqslant 0$. Then, there exists a unitary U such that

$$f(Z^*AZ) \leqslant UZ^* f(A) Z U^*. \tag{7}$$

Statements 1.4 and 1.5 can break down when the monotony assumption is dropped. But we recently obtained [4] substitutes involving the mean of two unitary congruences. Let us recall the precise result corresponding to inequalities (1) and (6).

1.6. Let f be a convex function, let A, B be Hermitians and set $X = f(\{A+B\}/2)$ and $Y = \{f(A)+f(B)\}/2$. Then, there exist unitaries U, V such that

$$X \leqslant \frac{UYU^* + VYV^*}{2}.$$

Another substitute of (6) for general convex functions f would be a positive answer to the following still open problem [2]: Given Hermitians A, B, can we find unitaries U, V such that

$$f\left(\frac{A+B}{2}\right) \leqslant \frac{Uf(A)U^* + Vf(B)V^*}{2}\ ?$$

We turn to a Brown–Kosaki type inequality involving expansive operators Z, that is $Z^*Z \geqslant I$. We showed the following trace version of the elementary inequality (3).

1.7. Let f be convex with $f(0) \leqslant 0$ and let $A \geqslant 0$. Then, for all expansive operators Z,

$$\mathrm{Tr}\ f(Z^*AZ) \geqslant \mathrm{Tr}\ Z^*f(A)Z. \tag{8}$$

It is interesting to note [2] that, contrarily to the contractive case (5), the assumption $A \geqslant 0$ cannot be dropped. Also, still contrarily to (5), this result cannot be extended to eigenvalues inequalities like (7). Nevertheless, we have:

1.8. Let f be nonnegative convex with $f(0) = 0$, let $A \geqslant 0$ and let Z be expansive. Then, for all symmetric norms

$$\|f(Z^*AZ)\| \geqslant \|Z^*f(A)Z\|. \tag{9}$$

Here, by symmetric norm we mean a unitarily invariant one, that is $\|A\| = \|UAV\|$ for all operators A and all unitaries U, V.

2. A new concavity result

Of course if f is concave with $f(0) \geqslant 0$ then inequality (8) is reversed and provides an extension of its scalar version (3). Assuming furthermore f nonnegative we tried to extend it to all symmetric norms but, besides the trace norm, we only got the operator norm case. Here we may state:

Theorem 2.1. *Let $f : [0,\infty) \longrightarrow [0,\infty)$ be a concave function. Let $A \geqslant 0$ and let Z be expansive. Then, for all symmetric norms*

$$\|f(Z^*AZ)\| \leqslant \|Z^*f(A)Z\|.$$

Proof. It suffices to prove the theorem for the Ky Fan k-norms $\|\cdot\|_k$ (cf. [1]). This shows, since Z is expansive, that we may assume that $f(0) = 0$. Note that f is necessarily nondecreasing. Hence, there exists a rank k spectral projection E for Z^*AZ, corresponding to the k-largest eigenvalues $\lambda_1(Z^*AZ), \dots, \lambda_k(Z^*AZ)$ of Z^*AZ, such that

$$\|f(Z^*AZ)\|_k = \sum_{j=1}^{k} \lambda_j(Z^*AZ) = \mathrm{Tr}\, Ef(Z^*AZ)E.$$

Therefore, using a well known property of Ky Fan norms, it suffices to show that

$$\mathrm{Tr}\, Ef(Z^*AZ)E \leqslant \mathrm{Tr} EZ^*f(A)ZE.$$

This is the same as requiring that

$$\mathrm{Tr}\, EZ^*g(A)ZE \leqslant \mathrm{Tr}\, Eg(Z^*AZ)E \tag{10}$$

for all convex functions g on $[0, \infty)$ with $g(0) = 0$. Any such function can be approached by a combination of the type

$$g(t) = \lambda t + \sum_{i=1}^{n} \alpha_i(t - \beta_i)_+ \tag{11}$$

for a scalar λ and some nonnegative scalars α_i and β_i. Here $(x)_+ = \max\{0, x\}$. By using the linearity of the trace it suffices to show that (10) holds for $g_\beta(t) = (t - \beta)_+$, $\beta \geqslant 0$. We claim that there exists a unitary U such that

$$Z^*g_\beta(A)Z \leqslant Ug_\beta(Z^*AZ)U^*. \tag{12}$$

This claim and a basic property of the trace then show that (10) holds for g_β. Indeed, we then have

$$\begin{aligned}
\mathrm{Tr} EZ^*g_\beta(A)ZE &= \sum_{j=1}^{k} \lambda_j(EZ^*g_\beta(A)ZE) \\
&\leqslant \sum_{j=1}^{k} \lambda_j(Z^*g_\beta(A)Z) \\
&\leqslant \sum_{j=1}^{k} \lambda_j(g_\beta(Z^*AZ)) \quad \text{(by (12))} \\
&= \sum_{j=1}^{k} \lambda_j(Eg_\beta(Z^*AZ)E) \\
&= \mathrm{Tr}\, Eg_\beta(Z^*AZ)E,
\end{aligned}$$

where the fourth equality follows from the fact that g_β is nondecreasing and hence E is also a spectral projection of $g_\beta(Z^*AZ)$ corresponding to the k largest eigenvalues.

The inequality (12) has been established in [2] in order to prove (8). Let us recall the proof of (12): We will use the following simple fact. If B is a positive operator with

$\mathrm{Sp}B \subset \{0\} \cup (x, \infty)$, then we also have $\mathrm{Sp}Z^*BZ \subset \{0\} \cup (x, \infty)$. Indeed Z^*BZ and $B^{1/2}ZZ^*B^{1/2}$ (which is greater than B) have the same spectrum.

Let P be the spectral projection of A corresponding to the eigenvalues strictly greater than β and let $A_\beta = AP$. Since $Z^*AZ - \beta I \geqslant Z^*A_\beta Z - \beta I$ and $t \longrightarrow t_+$ is nondecreasing, there exists a unitary operator V such that

$$(Z^*AZ - \beta I)_+ \geqslant V(Z^*A_\beta Z - \beta I)_+V^*.$$

Since $Z^*(A - \beta I)_+Z = Z^*(A_\beta - \beta I)_+Z$ we may then assume that $A = A_\beta$. Now, the above simple fact implies

$$(Z^*A_\beta Z - \beta I)_+ = Z^*A_\beta Z - \beta Q,$$

where $Q = \mathrm{supp}Z^*A_\beta Z$ is the support projection of $Z^*A_\beta Z$. Therefore, using $(A_\beta - \beta I)_+ = A_\beta - \beta P$, it suffices to show the existence of a unitary operator W such that

$$Z^*A_\beta Z - \beta Q \geqslant WZ^*(A_\beta - \beta P)ZW^* = WZ^*A_\beta ZW^* - \beta WZ^*PZW^*.$$

But, here we can take $W = I$. Indeed, we have

$$\mathrm{supp}Z^*PZ = Q \ (*) \quad \text{and} \quad \mathrm{Sp}Z^*PZ \subset \{0\} \cup [1, \infty) \ (**),$$

where $(**)$ follows from the above simple fact and the identity $(*)$ from the observation below with $X = P$ and $Y = A_\beta$.

Observation. If X, Y are two positive operators with $\mathrm{supp}X = \mathrm{supp}Y$, then for every operator Z we also have $\mathrm{supp}Z^*XZ = \mathrm{supp}Z^*YZ$.

To check this, we establish the corresponding equality for the kernels,

$$\ker Z^*XZ = \{h \ : \ Zh \in \ker X^{1/2}\} = \{h \ : \ Zh \in \ker Y^{1/2}\} = \ker Z^*YZ.$$

□

In the above proof, the simple idea of approaching convex functions as in (11) was fruitful. It is also useful to prove (see [2]) the Rotfel'd trace inequality: For concave functions f with $f(0) \geqslant 0$ and $A, B \geqslant 0$,

$$\mathrm{Tr} f(A + B) \leqslant \mathrm{Tr} f(A) + \mathrm{Tr} f(B).$$

If f is convex with $f(0) \leqslant 0$ the reverse inequality holds, in particular we have McCarthy's inequality

$$\mathrm{Tr}(A + B)^p \geqslant \mathrm{Tr}A^p + \mathrm{Tr}B^p$$

for all $p > 1$.

Remark 2.2. Though scalars inequalities (2) and (3) or their concave anologous hold for a more general class than convex or concave functions, the corresponding trace inequalities need the convexity or concavity assumption (cf. [2]). A fortiori, Theorem 2.1 needs the concavity assumption.

Remark 2.3. When f is operator monotone, Theorem 2.1 extends to an operator inequality which can be rephrased for contractions as follows: For nonnegative operator monotone functions f on $[0, \infty)$, contractions Z and $A \geqslant 0$,

$$Z^* f(A)Z \leqslant f(Z^*AZ).$$

This is the famous Hansen's inequality [6]. Similarly when f is operator convex, Hansen–Pedersen's trace inequality can be extended to an operator inequality [7] (see also [3]).

Extensions of Theorem 2.1 to infinite dimensional spaces will be considered in a forthcoming work.

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