Nonzero solutions for generalized variational inequalities by index

Bo Wang

College of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China

Received 10 November 2006
Available online 16 January 2007
Submitted by R. Curto

Abstract

The existence of nonzero solutions for a class of variational inequalities is studied by fixed point index approach for multivalued mappings in finite dimensional spaces and reflexive Banach spaces. © 2007 Elsevier Inc. All rights reserved.

Keywords: Variational inequality; Fixed point index of multivalued mappings; Nonzero solution

1. Preliminaries

Let $X$ be a Banach space, $X^*$ its dual and $(\cdot, \cdot)$ the pair between $X^*$ and $X$. A mapping $A : X \to X^*$ is called hemicontinuous at $x_0 \in X$ if for each $y \in X$, $A(x_0 + t_n y) \overset{w^*}{\to} Ax_0$ when $t_n \to +0$. A multivalued mapping $T : D(T) \subset X \to 2^{X^*}$ is said to be locally bounded in $D(T)$ if there exists a neighbourhood $V$ of $x$ for each $x \in X$ such that the set $T(V \cap D(T))$ is bounded in $X^*$. Suppose that $K$ is a closed convex subset of $X$ with $0 \in K$. For such $K$, the recession cone $rc\, K$ of $K$ is defined by $rc\, K = \{w \in X : v + w \in K, \ \forall v \in K\}$. It is easily seen that the recession cone is indeed a cone and we have that $rc\, K \neq \emptyset$. For a proper lower semicontinuous convex functional $j : X \to \mathbb{R} \cup \{\infty\}$ with $j(0) = 0$ and $j(K) \subset R_+ = [0, +\infty)$, in the virtue of [1], the limit $\lim_{t \to +\infty} \frac{1}{t} j(tw) = j_\infty(w)$ exists in $R \cup \{\infty\}$ for every $w \in X$ and $j_\infty$ is also a lower semicontinuous convex functional with $j_\infty(0) = 0$ and with the property that $j(u + v) \leq j(u) + j_\infty(v)$, $\forall u, v \in X$.

E-mail address: hzgswangbo@163.com.
In the sequel, for mapping \( g : K \to X^* \), we will deal with the following problem:

Find \( u \in K \): \((Au, v - u) + j(v) - j(u) \geq (g(u), v - u) + (f, v - u), \quad \forall v \in K. \) \hspace{1cm} (1)

Suppose that \( K \) is a closed convex subset of \( X \) and \( U \) is an open subset of \( X \) with \( U_K = U \cap K \neq \emptyset \). The closure and boundary of \( U_K \) relative to \( K \) are denoted by \( \bar{U}_K \) and \( \partial(U_K) \), respectively. Assume that \( T : \bar{U}_K \to 2^K \) is an upper semicontinuous mapping with nonempty compact convex values and \( T \) is also condensing, i.e., \( \alpha(T(S)) < \alpha(S) \) where \( \alpha \) is the Kuratowski measure of noncompactness on \( X \). If \( x \notin T(x) \) for \( x \in \partial(U_K) \), then the fixed point index, \( i_K(T, U) \), is well defined (see [2]).

**Proposition 1.** (See [2].) Let \( K \) be a nonempty closed convex subset of real Banach space \( X \) and \( U \) be an open subset of \( X \). Suppose that \( T : \bar{U}_K \to 2^K \) is an upper semicontinuous mapping with nonempty compact convex values and \( x \notin T(x) \) for \( x \in \partial(U_K) \). Then the index, \( i_K(T, U) \), has the following properties:

(i) If \( i_K(T, U) \neq 0 \), then \( T \) has a fixed point.
(ii) For mapping \( \hat{X}_0 \) with constant value \( \{x_0\} \), if \( x_0 \in U_K \), then \( i_K(\hat{X}_0, U) = 1 \).
(iii) Let \( U_1, U_2 \) be two open subsets of \( X \) with \( U_1 \cap U_2 = \emptyset \). If \( x \notin T(x) \) when \( x \in \partial((U_1)_K) \cup \partial((U_2)_K) \), then \( i_K(T, U_1 \cup U_2) = i_K(T, U_1) + i_K(T, U_2) \).
(iv) Let \( H : [0, 1] \times \bar{U}_K \to 2^K \) be an upper semicontinuous mapping with nonempty compact convex values and \( \alpha(H([0, 1] \times Q)) < \alpha(Q) \) whenever \( \alpha(Q) \neq 0, Q \subset \bar{U}_K \). If \( x \notin H(t, x) \) for every \( t \in [0, 1], x \in \partial(U_K) \), then \( i_K(H(1, \cdot), U) = i_K(H(0, \cdot), U) \).

For every \( q \in X^* \), let \( U(q) \) be the set of solutions in \( K \) of the following variational inequality

\[(Au, v - u) + j(v) - j(u) \geq (q, v - u) + (f, v - u), \quad \forall v \in K. \] \hspace{1cm} (2)

Define a mapping \( K_A : X^* \to 2^K \) by

\[ K_A(q) := U(q), \quad q \in X^*. \]

Obviously, \( K_A(q) = \emptyset \) if and only if the variational inequality (2) has no solution in \( K \).

### 2. Nonzero solutions in \( R^n \)

**Theorem 1.** Let \( K \) be a nonempty unbounded closed convex set in \( X = R^n \) with \( 0 \in K \). Suppose that \( A : X \to X^* \) is a monotone hemiconcave mapping with \( (Au, u) \geq 0 \) (\( \forall u \in K \)) and \( j : K \to (-\infty, +\infty] \) is a bounded proper convex lower semicontinuous functional with \( j(0) = 0 \) (i.e., for every bounded subset \( D \) of \( K \), \( j(D) \) is bounded). Give a continuous mapping \( g : K \to X^* \) and a point \( f \in X^* \). Assume

(a) there exists a point \( u_0 \in \text{rel } K \setminus \{0\} \) such that \( (f, u_0) \neq 0 \);
(b) there exist two constants \( C > 0 \) and \( \alpha \geq 0 \) such that \( ||g(u)|| \leq C||u||^\alpha \) for sufficiently large \( ||u|| \);
(c) \( \lim_{||u|| \to 0} \frac{j(u)}{||u||} = +\infty \) and \( \lim_{||u|| \to +\infty} \frac{||A(u)|| + j(u)}{||u||^{\alpha+1}} = +\infty \) (\( u \in K \)) for above \( \alpha \).

Then (1) has a nonzero solution.
Proof. It is easy to see from condition (c) that the variational inequality (2) has a solution in $K$ for every $q \in X^*$ [3,4]. Define a mapping $K_A g : K \to 2^K$ by
\[
(K_A g)(u) := K_A(g(u)), \quad u \in K.
\]
Then $K_A g$ is an upper semi-continuous mapping with nonempty compact convex values by [5, Lemma 1]. Let $K^R = \{x \in K : \|x\| \leq R\}$. We shall verify that $i_K(K_A g, K^R) = 1$ for large enough $R$ and $i_K(K_A g, K^r) = 0$ for small enough $r$.

Firstly, define a mapping by $H : [0, 1] \times K^R \to 2^K$, $H(t, u) = tK_A(g(u))$. It is easily seen that $H(t, u)$ is an upper semicontinuous mapping with nonempty compact convex values. We claim that there exists a small $\epsilon > 0$ such that for all $u \in \partial(K^R)$, there exist two sequences \{tn\}, \{un\}, $t_n \in [0, 1], t_n \neq 0, \|u_n\| \to +\infty$ such that $u_n \in H(t_n, u_n) = t_nK_A(g(u_n))$ or $\frac{u_n}{t_n} \in K_A(g(u_n))$. Thus
\[
(A\left(\frac{u_n}{t_n}\right), v - \frac{u_n}{t_n}) + j(v) + j\left(\frac{u_n}{t_n}\right) \geq \left(g(u_n), v - \frac{u_n}{t_n}\right) + \left(f, v - \frac{u_n}{t_n}\right),
\]
$\forall u \in K$. (3)

Letting $v = 0$ and denoting $z_n = \frac{u_n}{\|u_n\|}$ in (3), we obtain that
\[
\left(\frac{t_n}{\|u_n\|}\right)^{\alpha+1} \left(\frac{u_n}{t_n}, \frac{u_n}{t_n}\right) + \left(\frac{t_n}{\|u_n\|}\right)^{\alpha+1} j\left(\frac{u_n}{t_n}\right) \\
\leq t_n^{\alpha} \left(\frac{g(u_n)}{\|u_n\|^\alpha}, z_n\right) + \left(\frac{t_n}{\|u_n\|}\right)^{\alpha} (f, z_n).
\]
Denote $y_n = \frac{u_n}{\|u_n\|} \in K$. Then $\|y_n\| \to +\infty$. We can obtain from conditions (b), (c) and (4) that
\[
\left(\frac{t_n}{\|y_n\|}\right)^{\alpha+1} (A y_n, y_n) + j(y_n) \leq t_n^{\alpha} \left(\frac{g(u_n)}{\|u_n\|^\alpha}\right) + \|f\| \|y_n\|^{\alpha} \leq C + \|f\| \|y_n\|^{\alpha}
\]
which is a contradiction for the left tends to $+\infty$ and the right tends to $C$. Therefore
\[
i_K(K_A g, K^R) = i_K(H(1, \cdot), K^R) = i_K(H(0, \cdot), K^R) = i_K(\hat{0}, K^R) = 1
\]
by Proposition 1(iv) and (ii).

Secondly, we shall verify that $i_K(K_A g, K^r) = 0$ for small enough $r (r < 1)$. In fact, there exist constants $C_1, C_2, M > 0$ from the boundedness of $j$, locally boundedness of $A$ and condition (b) such that for all $u \in K^1$, we have
\[
|j(u + u_0) - j(u)| \leq C_1, \quad \|g(u)\| \leq C_2, \quad |(g(u), u_0)| \leq C_2\|u_0\|.
\]
\[
\|Au\| \leq M, \quad \|(Au, u_0)\| \leq M\|u_0\|.\]
Since $(f, u_0) \neq 0$, let $(f, u_0) < 0$. Take $N$ large enough such that
\[
(1 - N)(f, u_0) > C_1 + (C_2 + M)\|u_0\|.
\]
Define a mapping by $H\{0, 1\} \times K^R \to 2^K$, $H(t, u) = K_A(g(u) - tNf)$. Then $H$ is an upper semicontinuous mapping with nonempty compact convex values. We claim that there exists a small enough $r$ such that $u \notin H(t, u)$ for all $u \in \partial(K^r)$, $t \in [0, 1]$. Otherwise, there exist sequences \{tn\}, \{un\}, $t_n \in [0, 1], u_n \in \partial(K^r), \|u_n\| \to 0$ such that $u_n \in H(t_n, u_n) = K_A(g(u_n) - t_nNf)$. Thus
\[
(Au_n, v - u_n) + j(v) - j(u_n) \geq (g(u_n) - Nt_n f, v - u_n) + (f, v - u_n), \forall v \in K.
\]
Taking \( v = 0, z_n = \frac{u_n}{\|u_n\|} \), we have
\[
\frac{1}{\|u_n\|} (Au_n, u_n) + \frac{j(u_n)}{\|u_n\|} \leq (g(u_n), z_n) + (1 - t_n N)(f, z_n).
\]
Since \( \frac{1}{\|u_n\|} (Au_n, u_n) + \frac{j(u_n)}{\|u_n\|} \geq \frac{j(u_n)}{\|u_n\|} \to +\infty \) and
\[
(g(u_n), z_n) + (1 - t_n N)(f, z_n) \leq \|g(u_n)\| + (1 + N)\|f\| \leq C_2 + (1 + N)\|f\|,
\]
we obtain a contradiction. Therefore \( i_K(K Ag, K r) = i_K(H(0, \cdot), K r') \) by Proposition 1(iv). If \( i_K(H(1, \cdot), K r') \neq 0 \), then the mapping \( H(1, \cdot) : K \to 2^K \) has a fixed point \( u \) in \( K r' \) by Proposition 1(i), i.e., \( u \in H(1, u) = K_A(g(u) - Nf) \). Thus
\[
(Au, v - u) + j(v) - j(u) \geq (g(u) - Nf, v - u) + (f, v - u), \quad \forall v \in K.
\]
Taking \( v = u + u_0 \), we have
\[
(Au, u_0) + j(u + u_0) - j(u) \geq (g(u), u_0) + (1 - N)(f, u_0).
\]
Hence
\[
(1 - N)(f, u_0) \leq (Au, u_0) + j(u + u_0) - j(u) - (g(u), u_0)
\]
\[
\leq M\|u_0\| + C_1 + C_2\|u_0\| = (C_2 + M)\|u_0\| + C_1
\]
by (7) and (9). That contradicts to (8). Therefore, \( i_K(H(1, \cdot), K r') = 0 \) and then \( K(K Ag, K r') = 0 \).

It follows from Proposition 1(iii) that \( i_K(K Ag, K F) = 0 \) and \( K(K Ag, K F) = 1 \). Therefore there exists a fixed point \( u \in K F \setminus K r' \) which is a nonzero solution of (1). \( \Box \)

3. Nonzero solutions in reflexive Banach spaces

**Theorem 2.** Let \( X \) be a reflexive Banach space and \( K \subset X \) a nonempty unbounded closed convex set with \( 0 \in K \). Suppose that \( A : X \to X^* \) is a monotone hemicontinuous mapping with \( (Au, u) \geq 0 \) for \( u \in K \) and \( j : K \to (-\infty, +\infty] \) is a bounded convex lower semicontinuous functional with \( j(0) = 0 \). Assume that \( g : K \to X^* \) is continuous from the weak topology on \( X \) to the strong topology on \( X^* \). Give \( f \in X^* \). The following conditions are assumed to be satisfied

(a) \( (f, u_0) \neq 0 \) for some \( u_0 \in rc K \setminus \{0\} \);

(b) there exist two constants \( C > 0 \) and \( \alpha \geq 0 \) such that \( \|g(u)\| \leq C\|u\|^\alpha \) for sufficiently large \( \|u\| \);

(c) \( \lim_{\|u\| \to +\infty} \frac{(Au, u) + j(u)}{\|u\|^2 + 1} = +\infty (u \in K) \) for above \( \alpha \);

(d) \( \lim_{u \to 0} j(u) > 0 \).

Then (1) has a nonzero solution.

**Remark.** It is easily seen that \( \lim_{\|u\| \to 0} \frac{j(u)}{\|u\|} = +\infty \) by the condition (d).

**Proof.** Let \( F \subset X \) be a finite dimensional subspace containing \( u_0 \). We shall show that all conditions in Theorem 1 are satisfied on space \( F \). Denote \( K_F = K \cap F \) which is a nonempty unbounded closed convex set. Let \( j_F : F \to X \) be an injective mapping and \( j_F^* : X^* \to F^* \) its dual mapping. Denote \( A_F = j_F^*(A|F) : F \to F^* \), \( g_F = j_F^*(g|K_F) : K_F \to F^* \). We know that
$A_F = j_F^* A F, \ g_F = j_F^* g J F$. Then $A_F, g_F$ are hemicontinuous and continuous, respectively. For $x_1, x_2 \in K_F$, we have

$$(A_F(x_1) - A_F(x_2), x_1 - x_2) = \left(j_F^* A(x_1) - j_F^* A(x_2), x_1 - x_2\right) = (A x_1 - A x_2, j_F(x_1 - x_2)) = (A x_1 - A x_2, x_1 - x_2) \geq 0$$

by the monotony of $A$. This means that $A_F$ is monotone. On the other hand, for $u \in K_F$ with $\|u\|$ large enough and $w \in F$, we have

$$\left\| (g_F(u), w) \right\| = \left\| (j_F^* g F(u), w) \right\| = \left\| (g(u), j F(w)) \right\| = \left\| (g(u), w) \right\| \leq \|g(u)\| \|w\| \leq C \|u\|^\alpha \|w\|$$

and then $\|g_F(u)\| \leq C \|u\|^\alpha$. Furthermore, $j_F^* f \in F^*$ and $(j_F^* f, u_0) = (f, j F u_0) = (f, u_0) \neq 0$. Therefore there exists $u_F \in K_F, u_F \neq 0$ such that

$$(A_F(u_F), v - u_F) + j(v) - j(u_F) \geq (g_F(u_F), v - u_F) + (j_F^* f, v - u_F), \quad \forall v \in K_F,$$

by Theorem 1. It yields that

$$(A(u_F), v - u_F) + j(v) - j(u_F) \geq (g(u_F), v - u_F) + (f, v - u_F), \quad \forall v \in K_F.$$  

Taking $v = 0$, we get $(A u_F, u_F) + j(u_F) \leq (g(u_F), u_F) + (f, u_F)$. Hence

$$\frac{(A u_F, u_F) + j(u_F)}{\|u_F\|^\alpha + 1} \leq \frac{\|g(u_F)\|}{\|u_F\|^\alpha} \frac{\|f\|}{\|u_F\|^\alpha} \leq C + \frac{\|f\|}{\|u_F\|^\alpha}.$$

If $\|u_F\| \to \infty$, then the right side of the above inequality is finite. This conduces a contradiction by condition (c). There exists a constant $M > 0$ such that $\|u_F\| \leq M$ for all finite dimensional subspace $F$ containing $u_0$. Since $X$ is reflexive and $K$ is weakly closed, with a similar argument to that in the proof of Theorem 2 in Ref. [5] (also see [6]), we shall show there exists $u' \in K$ such that for every finite dimensional subspace $F$ containing $u_0$, $u'$ is in the weak closure of the set $V_F = \bigcup_{F \subset F_1} \{u_F\}$, where $F_1$ is a finite dimensional subspace in $X$.

In fact, since $V_F$ is bounded, we know that $(V_F)^w$ (the weak closure of the set $V_F$) is weakly compact. On the other hand, let $F^1, F^2, \ldots, F^m$ be finite dimensional subspaces containing $u_0$. Define $F^{(m)} := \text{span}(F^1, F^2, \ldots, F^m)$. Then $F^{(m)}$ containing $u_0$ is a finite dimensional subspace. Hence, $\bigcap_{i=1}^m V_{F^i} = \bigcap_{i=1}^m \left(\bigcup_{F \subset F_1} \{u_F\}\right) = \bigcup_{F^{(m)} \subset F_1} \{u_F\} \neq \emptyset$ and then $\bigcap_{F} (V_F)^w \neq \emptyset$, that is to say, there exists $u' \in K$ such that for every finite dimensional subspace $F$ containing $u_0$, $u'$ is in the weak closure of the set $V_F = \bigcup_{F \subset F_1} \{u_F\}$.

Now let $v \in K$ and $F'$ a finite dimensional subspace of $X$ which contains $u_0$ and $v$. Since $u'$ belongs to the weak closure of the set $V_{F'} = \bigcup_{F' \subset F_1} \{u_F\}$, we may find a sequence $\{u_{F_{a}}\}$ in $V_{F'}$ such that $u_{F_{a}} \rightharpoonup u'$. However, $u_{F_{a}}$ satisfies the following inequality

$$(A u_{F_{a}}, v - u_{F_{a}}) + j(v) - j(u_{F_{a}}) \geq (g(u_{F_{a}}), v - u_{F_{a}}) + (f, v - u_{F_{a}}).$$  

(10)

The monotony of $A$ implies that

$$(A v, v - u_{F_{a}}) + j(v) - j(u_{F_{a}}) \geq (g(u_{F_{a}}), v - u_{F_{a}}) + (f, v - u_{F_{a}}).$$

Letting $u_{F_{a}} \rightharpoonup u'$ yields that

$$0 \geq (A v, v - u') + j(v) - j(u') \geq (g(u'), v - u') + (f, v - u'), \quad \forall v \in K.$$ 

Thus

$$(Au', v - u') + j(v) - j(u') \geq (g(u'), v - u') + (f, v - u'), \quad \forall v \in K,$$
by Minty’s Theorem [3,4]. We claim that $u' \neq 0$. Otherwise, $u_{F_a} \xrightarrow{w} 0$. Taking $v = 0$ in (10) yields that

$$j(u_{F_a}) \leq -(Au_{F_a}, u_{F_a}) + (g(u_{F_a}), u_{F_a}) + (f, u_{F_a}) \leq (g(u_{F_a}), u_{F_a}) + (f, u_{F_a}).$$

The right side of the above inequality tends to 0, which contradicts to the condition (d). Therefore $u'$ is a nonzero solution of (1). □

References