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Unshellable Triangulations of Spheres

W. B. R. Lickorish

A direct proof is given of the existence of non-shellable triangulations of spheres which, in higher dimensions, yields new examples of such triangulations.

This paper produces a way of constructing triangulations of the n-sphere S^n, for each $n \geqslant 3$, that are not shellable. A triangulation is said to be *shellable* if its n-simplexes can be ordered $A_0, A_1, \ldots, A_N$ and each A_r can be expressed as the join B_rC_r of two of its faces (either of which may be empty) so that, for each $r \geqslant 0$, $A_r \cap \bigcup_{i=r+1}^{N} A_i = (\partial B_r)C_r$. Most of the concepts that will be used here were familiar to those working in combinatorial topology some years ago [1, 5], but more recently the possibility of unshellable triangulations of spheres has been of some interest to others [2, 7]. It seems that a proof of the existence of such triangulations has never been explicitly recorded. A proof is implicit in [5] but it is overgrown with so much irrelevant material as to render it inaccessible. The proof given here is a generalisation of ideas of Bing [1]. It consists of a distinct simplification of some of the ideas of [5] and is intended to be short and direct. For $n \geqslant 4$ it does produce, via suspensions, some triangulations of S^n not previously known to be unshellable (because in [5] interesting submanifolds were required to be locally unknotted). Note that a triangulation of S^n is just a simplicial complex the geometric realisation of which is homeomorphic to S^n; it does not necessarily have the property that it is isomorphic to a subdivision of the boundary of an $(n+1)$-simplex. The triangulations considered here do have subdivisions with this latter property, but those subdivisions might be shellable.

The notation '$A \leqslant B$' will mean that the simplex A is a *face* of the simplex B; '$A < B$' will mean that A is a *proper* face of B. If B and C are disjoint faces of A, BC will denote the *join* of B and C, namely the face having vertices the union of those of B and those of C. If K is a finite simplicial complex, $|K|$ will denote the *geometric realisation* of K. An *elementary collapse* of K consists of the removal of two simplexes A and B from K, where $A < C \in K$ if and only if $C = B$. K is *collapsible* if a finite sequence of elementary collapses reduces K to just a single vertex. Note that if K triangulates S^n and K is shellable, then for some n-simplex A_0 of K the complex $K - A_0$ is collapsible. It will be shown that this is not so for some triangulations of S^n.

The *first derived* subdivision $K^{(1)}$ of a simplicial complex K is the complex consisting of all simplexes $\hat{A}_0\hat{A}_1 \cdots \hat{A}_r$, where $A_0 < A_1 < \cdots < A_r \in K$ and $\hat{A}_i$ is the barycentre of A_i. If L is a subcomplex of K let $U(L^{(1)}, K^{(1)})$ be defined by

$$U(L^{(1)}, K^{(1)}) = \{\hat{A}_0\hat{A}_1 \cdots \hat{A}_r : A_0 < A_1 < \cdots < A_r \in K \text{ and } A_0 \in L\}.$$

This is not a simplicial complex (it is not closed under facing) but L^c, its complement in $K^{(1)}$, is the subcomplex of $K^{(1)}$ consisting of all simplexes that have no face in $L^{(1)}$.

Lemma 1. *If L is a subcomplex of a finite simplicial complex K, then $|L^c|$ is a deformation retract of (and hence is homotopy equivalent to) $|K| - |L|$.*

Proof. If $A \in K^{(1)}$, A can be written as a join $A = BC$, where $B \in L^{(1)}$ and $C \in L^c$ (sometimes this is expressed as '$L^{(1)}$ is a *full* subcomplex of $K^{(1)}$'); either of B or C may

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be empty. The required deformation retraction consists of a deformation retraction of every $|A| - |B|$ to $|C|$, shrinking along the lines of the join. □

If $A \in K$ the *dual* A^* of A is the subcomplex of $K^{(1)}$ defined by

$$A^* = \{\hat{A}_0\hat{A}_1 \cdots \hat{A}_r : A \leqslant A_0 < A_1 < \cdots < A_r \in K\}.$$

This is a cone with vertex $\hat{A}$; the base of the cone will be denoted ∂A^*. Note that $\partial A^* = \cup\{B^* : A < B\}$. Intuitively, A^* is the intersection of $|K|$ with a small cut through $\hat{A}$ perpendicular to A. A complex is a *P.L.* $(n-1)$*-sphere* if it has some subdivision isomorphic to a subdivision of the boundary of an n-simplex. Any triangulation of S^3 is a P.L. 3-sphere and the suspension of a P.L. $(n-1)$-sphere is a P.L. n-sphere. A closed *combinatorial n-manifold* is a complex in which the link of every vertex is a P.L. $(n-1)$-sphere. Any subdivision of such a complex has the same property, and the link of every r-simplex is a P.L. $(n-r-1)$-sphere. These standard results of combinatorial topology are not difficult and can be found in [4]. If A is an r-simplex in a combinatorial n-manifold then ∂A^* is a P.L. $(n-r-1)$-sphere, for it is the link of A in the subdivision of K consisting of

$$\{B\hat{A}_{s+1}\hat{A}_{s+2} \cdots \hat{A}_t : B < A_{s+1} < A_{s+2} < \cdots < A_t \in K, \dim B \leqslant r < \dim A_{s+1}\}.$$

Thus A^* is a P.L. $(n-r)$-ball, being the cone on ∂A^*. (The correspondence between A and A^* is the genesis of the Poincaré duality isomorphisms between the homology and cohomology groups of a manifold.)

Lemma 2. *Suppose that a non-empty subcomplex L of a closed combinatorial n-manifold K has v_r r-simplexes. Suppose that $K - A_0$ is collapsible, where A_0 is some n-simplex in $K - L$. Then $|L^c|$ is homotopy equivalent to a cell complex having at most $1 + v_{n-1}$ 0-cells and at most v_r $(n-r-1)$-cells for $0 < r+1 < n$.*

Proof. (The collapsing property implies that K is a P.L. n-sphere, but that will not be used.) The simplexes of $K - A_0$ can be ordered $A_1, A_2, \ldots, A_{2N+1}$, where A_{2N+1} is a vertex, $A_{2i-1} < A_{2i}$ and $A_{2i-1} \leqslant A_j$ implies that $j \leqslant 2i$. Define X_j, a subcomplex of $K^{(1)}$, by

$$X_j = \cup\{A_i^* : 0 \leqslant i \leqslant 2j, A_i \notin L\}.$$

The ordering can be chosen so that A_{2N+1} is in L. Then X_0 is the single vertex A_0^* and $X_N = L^c$. Consider how X_j differs from X_{j-1}. There are three cases to investigate.

(i) If $A_{2j-1} \notin L$ then A_{2j-1}^* and A_{2j}^* are both in X_j. If $\dim A_{2j-1} = r$ then A_{2j-1}^* is an $(n-r)$-cell with A_{2j}^* an $(n-r-1)$-cell in its boundary. But $|X_{j-1} \cap A_{2j-1}^*|$ is the closure of $|\partial A_{2j-1}^* - A_{2j}^*|$ and so is also an $(n-r-1)$-cell by Newman's theorem [4]. Thus $|X_j|$ is just $|X_{j-1}|$ with an $(n-r)$-cell attached via a cell in its boundary, and such an attaching does not change the homotopy type.

(ii) If $A_{2j-1} \in L$ but $A_{2j} \notin L$ then, if $\dim A_{2j-1} = r$, $|X_j|$ is $|X_{j-1}|$ with the $(n-r-1)$-cell A_{2j}^* attached via its whole boundary.

(iii) If $A_{2j} \in L$ then $|X_j| = |X_{j-1}|$.

Thus only in the second case does the homotopy type of $|X_j|$ change at all, and it is by the adding of an $(n-r-1)$-cell, one $(n-r-1)$-cell occurs for each relevant r-simplex. □

A few remarks about knot theory are now in order. A (classical) knot is just a simple closed curve in the 3-sphere, and a *tame* knot is one that can be regarded as some subcomplex of a triangulation of the 3-sphere. The *group* G of a knot k is the fundamental group of the knot's complement, $G = \Pi_1(S^3 - k)$. The easiest knot to envisage is the trefoil knot which has a well-known diagram with three crossings, but knots can be made to be very complicated. In particular, consider the simple closed curve that is the sum of m copies of the trefoil knot (obtained by placing m trefoils one after the other in the same loop of string). This, of course, is a tame knot. It is known that the group of this knot has no group presentation with fewer than $(m + 1)$ generators. A proof appears in [3]. The idea of that proof is as follows. The Alexander module of the knot is the first homology of the universal abelian cover of the knot's complement viewed as a module over the Laurent polynomial ring $\mathbb{Z}[t^{-1}, t]$. Direct calculation shows that, for the sum of m trefoils, the mth elementary ideal of this module is not the whole ring. However, that ideal would be the whole ring if there were a group presentation with fewer than $(m + 1)$ generators.

The next result (probably originally due to Bing [1]) shows that a tame knot in S^3 can be complicated without having many simplexes.

LEMMA 3. *If k is a tame knot in S^3 there is a simplicial complex T, containing a subcomplex κ, such that κ has just three vertices and three 1-simplexes and $(|T|, |\kappa|)$ is P.L. homeomorphic to (S^3, k).*

PROOF. Let L be a subcomplex of K such that $(|K|, |L|)$ is homeomorphic to (S^3, k). The closed simplicial neighbourhood of $L^{(2)}$ (the second derived subdivision of L) in $K^{(2)}$ is a solid torus. If A_1, A_2 and A_3 are distinct 1-simplexes of $L^{(1)}$ then $|A_1^*|$, $|A_2^*|$ and $|A_3^*|$ are (meridian) discs that divide the solid torus into three cylinders. Because a cylinder is a convex subset of $\mathbb{R}^3$, it is easy to re-triangulate the cylinders as cones with vertices $\hat{A}_1$, $\hat{A}_2$ and $\hat{A}_3$ respectively, without changing the triangulation on the boundary of any cylinder. Let T be $K^{(2)}$ with the triangulation of the solid torus changed in this way. The solid torus now has a core that is a simple closed curve triangulated with $\hat{A}_1$, $\hat{A}_2$ and $\hat{A}_3$ as its only vertices. □

These results can now be assembled to give the promised theorem.

THEOREM. *There exists, for each $n \geq 3$, a triangulation K of the n-sphere such that for no n-simplex $A_0 \in K$ is $K - A_0$ collapsible. K can be chosen to be a P.L. n-sphere.*

PROOF. Let T be a simplicial complex containing a subcomplex κ that is a simple closed curve with just three vertices and three 1-simplexes, such that T triangulates S^3 and κ corresponds to a knot for which the knot group has no presentation with fewer than $3(2^{n-3}) + 1$ generators. This is possible by Lemma 3 and the remarks preceding that lemma. Let K be $\Sigma^{n-3}T$, the $(n - 3)$-fold suspension of T, obtained by starting with T and performing $(n - 3)$ times the operation of joining to a pair of points. Let L be the subcomplex $\Sigma^{n-3}\kappa$. Now, $|K| - |L|$ deformation retracts to $|T| - |\kappa|$. Hence $\Pi_1(|K| - |L|)$ is isomorphic to the chosen knot group and so has no presentation with fewer than $3(2^{n-3}) + 1$ generators; thus $\Pi_1(|L^c|)$ also has this property, by Lemma 1. If there were an n-simplex $A_0 \in K$ with $K - A_0$ collapsible then, by Lemma 2, $|L^c|$ would be homotopy equivalent to a cell complex with at most $3(2^{n-3})$ 1-cells (that being the number of $(n - 2)$-simplexes in L). But this leads to a contradiction, because homotopy equivalent spaces have isomorphic fundamental groups, and the algorithm that presents the fundamental group of a connected complex gives one generator for each 1-cell not in a fixed maximal tree, and a relator for each 2-cell. □

One might note that the complexity of the required knot can be reduced if, rather than using suspensions, K is taken to be the join of T to the boundary of an $(n-3)$-simplex. For $n=3$ the above procedure shows that there is a triangulation of S^3 that is not shellable because it contains a subcomplex, of only three vertices and three 1-simplexes, that is knotted in the sum of three trefoil knots. The proof shows that if one 3-simplex is removed the remainder does not collapse. Contrary to popular belief, a triangulation is known that contains a simple closed curve of three vertices and three 1-simplexes knotted in a *single* trefoil, such that a 3-simplex can be removed leaving a collapsible remainder. The triangulation is a cone added to the boundary of the example given in [6] of a collapsible triangulation of the 3-ball with a knotted spanning 1-simplex. Using that example and a little ingenuity, a similarly collapsible triangulation can be constructed involving the sum of two trefoils. It is not clear whether either of these two triangulations of S^3 is shellable.

The referee has pointed out that, for $n \geqslant 4$, the join of any triangulation of S^{n-4} to a non-shellable triangulation of S^3 produces a non-shellable triangulation of S^n (because in a shellable triangulation the link of any simplex is shellable); this economises on the number of simplexes required for a non-shellable S^n.

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W. B. R. LICKORISH
Department of Pure Mathematics,
16, Mill Lane,
Cambridge, CB2 1SB, U.K.