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A New Measure of Consistency for Positive Reciprocal Matrices

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Abstract—The analytic hierarchy process (AHP) provides a decision maker with a way of examining the consistency of entries in a pairwise comparison matrix and the hierarchy as a whole through the consistency ratio measure. It has always seemed to us that this commonly used measure could be improved upon. The purpose of this paper is to present an alternative consistency measure and demonstrate how it might be applied in different types of matrices.

Keywords—Reciprocal matrices, Consistency index, Eigenvalues, Preferences.

1. INTRODUCTION

The traditional eigenvector method for estimating weights in the analytic hierarchy process (AHP) (see [1]) yields a way of measuring the consistency of a decision maker's preferences arranged in the form of a reciprocal pairwise comparison matrix. The *consistency index* (CI) is given by

$$\mathrm{CI} \equiv \frac{\lambda_{\max} - n}{n - 1}, \tag{1}$$

where $\lambda_{\max}$ is the largest eigenvalue of the $n \times n$ reciprocal pairwise comparison matrix.

In [1], Saaty showed that if a decision maker is perfectly consistent (i.e., $a_{ik} = a_{ij}\, a_{jk}$ for all $i, j, k = 1, \dots, n$), $\lambda_{\max} = n$ ($\mathrm{CI} = 0$) and if the decision maker is not perfectly consistent, then $\lambda_{\max} > n$. To measure this consistency, Saaty proposed a consistency ratio defined as

$$\mathrm{CR} \equiv \frac{\mathrm{CI}}{\mathrm{RI}}, \tag{2}$$

where RI is the average value of CI obtained from 500 positive reciprocal pairwise comparison matrices whose entries were randomly generated using the 1 to 9 scale. Saaty considers that a

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Table 1. Values of the random index for different matrix orders.

N	1–2	3	4	5	6	7
RI	0	0.58	0.90	1.12	1.24	1.32

value of CR under 0.10 indicates that the decision maker is sufficiently consistent. Table 1 gives values of the average RI for different values of n.

This consistency measure is a reasonable measure but, at the same time, somewhat arbitrary [2–7]. Several questions come to mind.

(1) Why ten percent?
(2) Should the cut-off rule be a function of the matrix size?
(3) It is possible to use the CI in other types of reciprocal matrices, e.g., $a_{ik} = a_{ij} \oplus a_{jk}$?

In an attempt to answer these questions, we have developed a measure of consistency [4] that is

(1) easy to use,
(2) a function of the matrix size,
(3) applicable to other types of reciprocal matrices.

The purpose of this work is to develop and demonstrate an alternative measure of consistency. The paper is laid out as follows. In Section 2, we present the new consistency index (CI*); in Section 3, we recommend a critical value for the new consistency index (RC*); in Section 4, we applied the alternative consistency measure to other types of reciprocal matrices; in Section 5, we illustrate it with some examples, and finally in Section 6, we show the conclusion.

2. A NEW MEASURE OF CONSISTENCY

So that inconsistency exists in the judgments, we need to at least compare three alternatives, because when we compare two alternatives the judgments are always perfect (for $n \times n$ matrices with $n < 3$ there is no inconsistency). Comparing three alternatives, it is possible that inconsistency exists when

(i) there exists a cycle between the alternatives ($a_i > a_j > a_k > a_i$),
(ii) the intensity with value a_{ik} is different to the product of the arc a_{ij}, a_{jk} (see Figure 1).

Therefore, the existent relationship among three alternatives (Figure 1) is defined as transitivity (Γ) [8], for us being the minimal element of consistency [6].

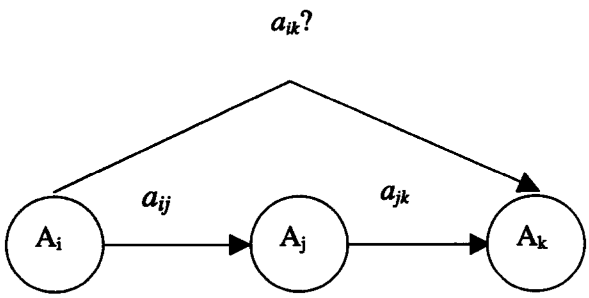

Figure 1. Transitivity (Γ).

DEFINITION 1. *(See [8].) A preference structure on a set A is a triplet $\{P, I, R\}$ where*

- *P is a preference binary relation (asymmetric);*
- *I is an indifference binary relation (reflexive and symmetric);*
- *R is a binary relation representing no preference (irreflexive and symmetric);*
- *$P \cup I \cup R$ is a strongly complete binary relation;*
- *$P \cap I = \emptyset$, $I \cap R = \emptyset$, $P \cap R = \emptyset$.*

DEFINITION 2. *(See [8].) A preference structure $\{P, I, R\}$ is a weak order if and only if*

$$R = \emptyset,$$
$$P \text{ is transitive,}$$
$$I \text{ is transitive.}$$

DEFINITION 3. *(See [8].) A transitivity Γ is a weak order preference structure on a set of three alternatives $A = \{A_i, A_j, A_k\}$.*

DEFINITION 4. *Two transitivities, Γ_i and Γ_j, are different, if they have at least one different element.*

By them, in the AHP the $M_{3\times3}$ reciprocal matrix *is the minimal element of consistency* [4].

C	A_i	A_j	A_k
A_i	1	a_{ij}	a_{ik}
A_j	$\frac{1}{a_{ij}}$	1	a_{jk}
A_k	$\frac{1}{a_{ik}}$	$\frac{1}{a_{jk}}$	1

Another way to measure consistency is by using the determinant of the matrix.

THEOREM 1. *The reciprocal pairwise comparison matrix $M_{3\times3}$ is perfect if and only if* $\det(M_{3\times3}) = 0$.

PROOF.

$$\det(M_{3\times3}) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2.$$

If the judgments are perfect, then $a_{ik} = a_{ij}a_{jk}$ and $\det(M_{3\times3}) = 0$. ∎

COROLLARY 1. *The judgments in a pairwise comparison matrices $M_{3\times3}$ are not perfect if* $\det(M) > 0$.

Since a number and its inverse sum is higher or equal to 2, then $\det(M_{3\times3}) > 0$.

Then, to measure the consistency of an $n \times n$ matrix, we measure all the different transitivities. The number of different transitivities (NT) of an $n \times n$ matrix is given by

$$\mathrm{NT}(M_{n\times n}) = \begin{cases} 0, & \text{if } n < 3, \\ \dfrac{n!}{(n-3)!3!}, & \text{otherwise.} \end{cases}$$

We defined the consistency index, and we denoted it as CI* to distinguish it from Saaty's index [1].

DEFINITION 5. *The consistency index CI* of an $M_{n\times n}$ matrix is given by the average of the consistency index of the matrix transitivities.*

$$\mathrm{CI}^*(M_{n\times n}) = \begin{cases} 0, & \text{if } n < 3, \\ \det(M_{n\times n}), & \text{if } n = 3, \\ \dfrac{1}{\mathrm{NT}(M_{n\times n})} \displaystyle\sum_{i=1}^{\mathrm{NT}(M_{n\times n})} \mathrm{CI}^*(\Gamma_i), & \text{if } n > 3, \end{cases}$$

where $\mathrm{NT}(M_{n\times n})$ is the number of different transitivities.

3. A CRITICAL VALUE

In the first section, we asked if the ten percent rule proposed by Saaty [1] to accept or reject judgments is reasonable and whether or not it should be a function of the matrix size. To answer these questions, we first study the relationship between the consistency index CI* and Saaty's consistency ratio. Table 2 shows the value of the new consistency index CI* corresponding to matrices with a value of Saaty's consistency ratio less than or equal to ten percent. As has already been pointed out by other authors [2], there are more than 25 percent of the 3-by-3 reciprocal matrices with a consistency ratio less than or equal to ten percent. As the matrix size increases, this percentage decreases dramatically. This shows that to uniformly accept or reject paired comparison matrices, the critical value should be a function of the matrix size.

Table 2. Values of the average CI* and percent of reciprocal matrices with CR ≤ 0.10.

n	3	4	5	6	7	8	9
CI*	1.132	1.208	1.258	1.284	1.329	1.354	1.379
Percent	25.88	4.62	4.60	0.20	0*	0*	0*

*Less than 0.1 percent.

In Table 2, 25.88 percent corresponds to the ten percent rule, and it is the 25.88 percentile of the distribution of Saaty's consistency ratio. The equivalent value of the new consistency index for 3-by-3 matrices would be 1.132 also given in Table 2. To develop a critical value for the new consistency index CI* we use a percentile of the distribution of CI*. These values will allow us to accept the same percentage of matrices for all values of n. Table 3 provides the 25.88 percentiles of CI* and Figure 2 gives the corresponding graphical plots for samples of size 100,000. These plots show that there appears to be a linear relationship between CI* and n for each corresponding percentile.

Table 3. 25.88 percentiles of CI*.

n	3	4	5	6	7	8	9
CI*	1.132	5.239	10.234	16.329	19.699	23.755	27.223

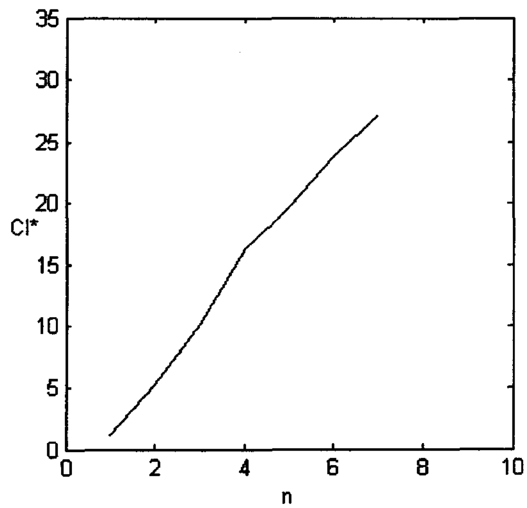

Figure 2. 25.88 percentile.

Table 4 *gives different values of* CI* for matrices of size three to nine and for different percentiles.

Figure 3 shows that the consistency index CI* is a function of the size of the matrix. This suggests that we should select a percentile of the distribution as the critical value that in turn yields the corresponding value of CI* for each value of n.

Table 4. CI* values for different matrices and percentiles (100,000 simulations).

Percentiles	Matrix Size						
	3	4	5	6	7	8	9
0.01	0.00	0.406	1.697	3.238	4.698	6.158	7.618
0.05	0.05	1.301	3.350	5.842	8.641	11.44	14.239
0.10	0.166	2.098	4.857	8.303	11.668	15.028	18.388
0.15	0.355	2.885	6.387	10.571	14.117	17.663	21.209
0.20	0.694	3.679	7.915	12.816	16.496	20.176	23.856
0.25	1.033	4.918	10.099	16.002	19.389	23.662	26.996
0.30	1.432	5.695	11.701	17.305	20.648	23.991	27.331
0.35	2.25	6.908	13.874	19.541	22.778	26.015	29.25
0.40	2.722	8.514	16.629	22.040	24.927	27.814	30.701
0.45	3.52	10.453	19.535	24.654	26.791	28.928	31.065
0.50	4.28	12.954	22.886	27.063	28.962	30.860	32.758

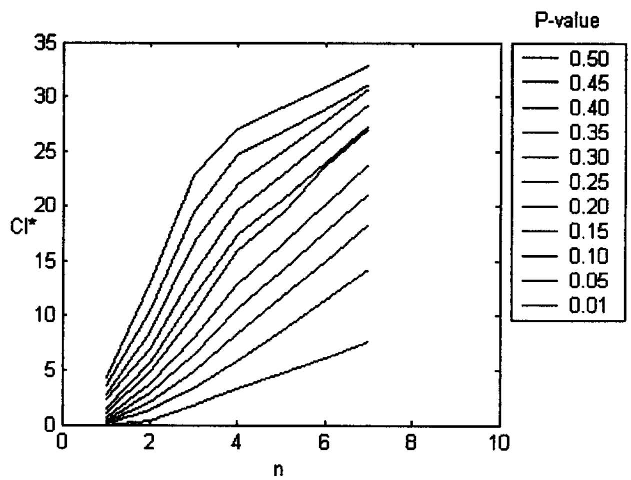

Figure 3. Percentiles of CI* as a function of n from Table 3.

4. USE OF THE NEW CONSISTENCY INDEX IN OTHER TYPES OF RECIPROCAL MATRICES

The attractiveness of the new consistency index is due to its potential use in fuzzy set theory [9–11]. Fuzzy sets are used in decision theory where the preference relation among the alternatives is additive instead of multiplicative. In the AHP, the preference relations satisfy the condition $a_{ij}a_{ji} = 1$, while in fuzzy set theory we have $a_{ji} = 1 - a_{ij}$, where $a_{ij} \in [0, 1]$, and indifference corresponds to the value 0.5. Table 5 gives a 3-by-3 additive reciprocal matrix and its corresponding consistency index.

For these types of matrices we cannot use $\lambda_{\max}$ to measure inconsistency, and hence define a consistency index, because it is not a monotone function of the entries of the matrix. However, we

Table 5. A 3-by-3 additive reciprocal matrix.

C	A_i	A_j	A_k
A_i	0.5	0.6	0.8
A_j	0.4	0.5	0.8
A_k	0.2	0.2	0.5

$CI^* = 0.005$

can use CI*. In this context, a consistent matrix is a matrix whose entries satisfy the condition for additive preference matrices defined by Lamata-Peláez [4]

$$a_{ik} = (a_{ij} - 0.5) + a_{jk}, \qquad \text{for all } i, j, k,$$

where a_{ij} ($a_{ij} \in [0, 1]$) represents how much more preferred alternative A_i is than alternative A_j, and the indifference value is given by $a_{ii} = 0.5$. This definition of consistency is consistent with the definition of the new consistency measure given. We have the following.

THEOREM 2. *An additive reciprocal pairwise comparison matrix $M_{3\times3}$ is consistent if and only if* $\det(M_{3\times3}) = 0$. *If the matrix is inconsistent,* $\det(M_{3\times3}) > 0$.

5. EXAMPLES

We illustrate the behavior of the new consistency measure with some 4-by-4 multiplicative reciprocal matrices. Table 6a shows a matrix considered inconsistent according to the CR criterion, but the new measure of inconsistency considers it consistent (see Table 2). The matrix in Table 6b is still inconsistent according to CR but not with the index. The matrices in Tables 6c and 6d are consistent under both criteria.

Table 6. Preference matrices with their consistency measures.

(a)

C	A_1	A_2	A_3	A_4	
A_1	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{5}$	
A_2	7	1	$\frac{1}{2}$	$\frac{1}{3}$	$\lambda_{max} = 4.828$
A_3	7	2	1	$\frac{1}{9}$	CR = 0.306
A_4	5	3	9	1	$CI^* = 4.446$

(b)

C	A_1	A_2	A_3	A_4	
A_1	1	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{9}$	
A_2	5	1	4	$\frac{1}{8}$	$\lambda_{max} = 4.38$
A_3	3	$\frac{1}{4}$	1	$\frac{1}{9}$	CR = 0.14
A_4	9	8	9	1	$CI^* = 1.664$

(c)

C	A_1	A_2	A_3	A_4	
A_1	1	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{9}$	
A_2	3	1	$\frac{1}{2}$	$\frac{1}{5}$	$\lambda_{\max} = 4.275$
A_3	7	2	1	$\frac{1}{7}$	$\mathrm{CR} = 0.102$
A_4	9	5	7	1	$\mathrm{CI}^* = 1.268$

(d)

C	A_1	A_2	A_3	A_4	
A_1	1	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{9}$	
A_2	3	1	$\frac{1}{2}$	$\frac{1}{5}$	$\lambda_{\max} = 4.167$
A_3	7	2	1	$\frac{1}{5}$	$\mathrm{CR} = 0.061$
A_4	9	5	5	1	$\mathrm{CI}^* = 0.741$

6. CONCLUSIONS

In this paper, we have presented a alternative measure to examine the consistency of entries in a pairwise comparison matrix. This measure is based in the determinant of the matrix and it measure the minimal element of consistency: the transitivities.

The advantages of this measure of consistency are:

(a) It is easy to use;
(b) it is a function of the matrix size; and finally,
(c) it is applicable to other types of reciprocal matrices.

Also in this work, we have proposed a new critical value that it is based in this measure and we have applied this measure in several examples.

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