Game chromatic number of lexicographic product graphs

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Abstract

In this paper, we determine the exact values of the game chromatic number of lexicographic product of path \( P_2 \) with path \( P_n \), star \( K_{1,n} \) and wheel \( W_n \). Also we give an upper bound for the game chromatic number of lexicographic product of any two simple graphs \( G \) and \( H \).

\( \text{Keywords: Graphs; Game chromatic number; Lexicographic product} \)

1. Introduction

Let \( G = (V, E) \) be a finite graph and \( X \) be a set of colors. The game chromatic number of \( G \) is defined through a two person game. Two players, say Alice and Bob, with Alice starting first, alternately color a vertex of \( G \) with a color from the color set \( X \) so that no two adjacent vertices receive the same color. Alice wins the game if all the vertices of \( G \) are colored. Bob wins the game if at any stage of the game, there is an uncolored vertex which is adjacent to vertices of all colors from \( X \). The game chromatic number, \( \chi_g(G) \), of \( G \) is the least number of colors in the color set \( X \) for which Alice has a winning strategy in the coloring game on \( G \). This parameter is well defined since Alice always wins if \( |X| = |V| \). It is obvious that \( \chi(G) \leq \chi_g(G) \leq \Delta(G) + 1 \), where \( \chi(G) \) is the usual chromatic number of \( G \) and \( \Delta(G) \) is the maximum degree of \( G \).

The game coloring number of \( G \) is also defined through a two person game, say Alice and Bob. The players fix a positive integer \( k \) and instead of coloring the vertices of \( G \), in each turn, they mark an unmarked vertex starting with Alice. Bob wins if at some point of time some unmarked vertex has \( k \) marked neighbors, while Alice wins if this never occurs. The game coloring number of \( G \), denoted by \( \text{col}_g(G) \), is defined as the least number \( k \) for which Alice has a winning strategy on the graph \( G \). Clearly, if Alice can win the marking game for some integer \( k \), then she can also win the coloring game with \( k \) colors. Thus \( \chi_g(G) \leq \text{col}_g(G) \).

We assume that, both the players use their optimal strategy in each of their moves. We say a color \( i \) is an available color for an uncolored vertex \( x \) if no neighbors of \( x \) have been colored by color \( i \). As given by Raspaud and Wu [1], an uncolored vertex \( x \) is called color \( i \)-critical if the following hold:

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Definition 2.1. The lexicographic product of two simple graphs $G$ and $H$, denoted by $G[H]$, has vertex set $V(G) \times V(H)$ and edge set
\[ \{(a, x)(b, y) : ab \in E(G) \text{ or } (a = b \text{ and } xy \in E(H))\}. \]

Notation. The $|V(G)|$ copies of $H$ in $G[H]$ are denoted by $F_i$, $1 \leq i \leq |V(G)|$. In the figures given below, a vertex $v$ is labeled $i^{(j)}$ if vertex $v$ is given color $i$ in the $j$th move.

In this paper, we determine the exact values of $\chi_g(P_2[P_n])$, $\chi_g(P_2[K_{1,n}])$ and $\chi_g(P_2[W_n])$.

**Theorem 2.2.** For any positive integer $n$, $\chi_g(P_2[P_n]) = \begin{cases} 4 & n = 2 \\ 5 & n = 3 \\ 6 & n = 4, 5, 6 \\ 7 & n \geq 7. \end{cases}$

**Proof.** Denote the vertices of $F_1$ and $F_2$ by $v_1, v_2, v_3, \ldots, v_n$ and $v_1', v_2', v_3', \ldots, v_n'$ respectively. We discuss the game chromatic number of $P_2[P_n]$ for different values of $n$ in the following cases.

**Case 1:** $n = 2$
As $P_2[P_2]$ is isomorphic to the complete graph $K_4$, $\chi_g(P_2[P_2]) = 4$.

**Case 2:** $n = 3$
First we show that $\chi_g(P_2[P_3]) > 4$, by giving a winning strategy for Bob using four colors.

**Trap I:** When Alice colors a vertex of degree five
Without loss of generality, Alice colors a vertex say $v_2$ with color 1 in her first move. Bob replies with the vertex $v_2'$ with color 2. This forces Alice to use a third color, say color 3, for any uncolored vertex which makes at least one uncolored vertex color 4-critical. Hence Bob wins.

**Trap II:** When Alice colors a vertex of degree four
Without loss of generality, Alice colors a vertex say $v_1$ with color 1 in her first move. Bob replies with the vertex $v_3$ with color 2. Now in the third move,

- if Alice colors the vertex $v_2$ with color 3, then this makes the vertices of $F_2$ color 4-critical.
- If Alice colors any uncolored vertex of $F_2$ with color 3, then this makes the vertex $v_2$ color 4-critical. Hence Bob wins.

Thus $\chi_g(P_2[P_3]) \geq 5$.

Note that in $P_2[P_3]$, there are only two vertices of degree five and rest of the vertices are of degree four. In the first two moves of Alice, she colors the vertices of degree five with two new colors. Hence, five colors are enough for Alice to win. Thus $\chi_g(P_2[P_3]) = 5$.

**Case 3:** $4 \leq n \leq 6$
Consider $P_2[P_4]$. First we show that Bob has a winning strategy using five colors.

In the first move, Alice colors any vertex of $F_1$ or $F_2$, say $v_2$, with color 1. In the second move, Bob colors a vertex with color 2 which is at a distance two from the vertex chosen by Alice in the first move. Now in the third move, Alice either plays in $F_1$ or $F_2$. 
• If Alice repeats either of the colors 1 or 2 for a vertex of $F_1$, then Bob colors a vertex of $F_1$ with color 3. In the next turn, Alice has to use a new color, say color 4, for a vertex of $F_2$, which makes one of its neighbors color 5-critical.

• If Alice colors a vertex of $F_1$ with a new color, say color 3, in her third move, then Bob colors an uncolored vertex of $F_1$ with color 4 which makes all the vertices of $F_2$ color 5-critical.

• If Alice colors any vertex of $F_2$, say $v_1'$ with color 3 in her third move, then Bob colors a vertex of $F_2$ which is at a distance two from $v_1'$ with color 4. This makes the vertices $v_3$ and $v_2'$ color 5-critical. Hence Bob wins.

Thus $\chi_s(P_2[P_4]) \geq 6$.

As $P_2[P_4]$ is an induced subgraph of $P_2[P_n]$, $n \geq 4$, we have $\chi_s(P_2[P_n]) \geq 6$. Now we show that $\chi_s(P_2[P_n]) = 6$, $4 \leq n \leq 6$. For $n = 4$, Fig. 2.1 shows that Alice can win with six colors, however Bob plays.

For $n = 5, 6$, we give a winning strategy for Alice using six colors. The strategy is as follows. Alice colors $v_2$ with color 1 in her first move. Now Bob can play either in a vertex of $F_1$ or in a vertex of $F_2$.

**Move b1:** Bob plays in a vertex of $F_1$

Bob colors a vertex of $F_1$ with a color different from $v_2$. Now Alice replies with $v_2'$. Again Bob can either play in a vertex of $F_1$ or in a vertex of $F_2$.

**Move b1.1:** Bob replies in a vertex of $F_1$

Bob plays in a vertex of $F_1$ with a color different from the colors used in $F_1$. Alice replies with $v_3'$ with a color different from that of $v_2'$. Observe that if Bob uses the sixth color in $F_2$, then Alice wins the game. So Bob plays in a vertex of $F_1$ with the sixth color. Alice replies with $v_5'$ with the color of $v_3'$. Hence Alice wins.

**Move b1.2:** Bob replies in a vertex of $F_2$

Bob plays in a vertex of $F_2$ with color 4. If Bob replies in $v_3'$ in the last move, then Alice colors $v_2'$ with a color of $v_3'$. Otherwise Alice chooses one of the neighbors of a vertex chosen by Bob in the last move for which color of $v_3'$ is available and colors it with a color of $v_3'$. Again Bob either repeats the color or uses a new color for a vertex of $F_1$ or $F_2$.

• If Bob repeats the color for a vertex of $F_1(F_2)$, then Alice replies with a vertex of $F_2(F_1)$ for which one of the colors used in $F_2(F_1)$ respectively is available and colors it with this available color. Now Bob either repeats the color or uses a new color.

  – If Bob uses a new color for a vertex of $F_1$ or a vertex of $F_2$, say color 5, then Alice colors a vertex of $F_2$ or $F_1$ respectively with color 6. Hence Alice wins.

  – If Bob repeats the color for a vertex of $F_1(F_2)$, then there are two possibilities for Alice. (i) If there is exactly one uncolored vertex in $F_2(F_1)$ respectively, then Alice colors that vertex. Hence Alice wins. (ii) If not, she colors a vertex for which one of the colors used in $F_2(F_1)$ is available and colors it with this available color.

Now, Bob either plays in a vertex of $F_1$ or $F_2$. If Bob repeats the color, then Alice replies in the same way as she had done in her previous move. Otherwise, if Bob uses a new color for a vertex of $F_1$ or $F_2$, then Alice responds with a vertex of $F_2$ or $F_1$ respectively with a new color. Hence Alice wins.

• If Bob plays in a vertex of $F_1$ or $F_2$ with a new color, then Alice replies in a vertex of $F_2$ or $F_1$ respectively with a new color. Hence Alice wins.

**Move b2:** Bob plays in a vertex of $F_2$

Bob plays in a vertex of $F_2$ with color 2 in the second move. Alice replies with a vertex $v_3$ with a new color. Now Bob either plays in a vertex of $F_1$ or $F_2$, then this situation is same as Move b1.1 and Move b1.2. Hence Alice will play accordingly and wins.

Thus $\chi_s(P_2[P_n]) = 6$, $n = 5, 6$.
Case 4: \( n \geq 7 \)

As \( P_2[P_3] \) is an induced subgraph of \( P_2[P_n] \), \( n \geq 7 \), we have \( \chi_g(P_2[P_n]) \geq 6 \). First we show that Bob has a winning strategy using six colors.

Without loss of generality, assume that Alice colors a vertex of \( F_1 \) in her first move. Now in the next three turns of Bob’s, irrespective of Alice’s move, he colors three vertices of \( F_1 \) with three new distinct colors. Note that at most two different colors are used for the vertices of \( F_2 \). Thus Bob easily makes at least one vertex of \( F_2 \) color \( j \)-critical, for some \( j \). Thus \( \chi_g(P_2[P_n]) \geq 7 \), \( n \geq 7 \). Now we show that Alice has a winning strategy using seven colors. The strategy is as follows.

We fix the first two moves of Alice. Alice colors a vertex of \( F_1 \) in her first move and she colors a vertex of \( F_2 \) with a new color in her second move. Alice uses a new color in each of her moves till both \( F_i \)’s receive at least three distinct colors. In the following moves, whenever

- Bob colors any vertex with a new color then Alice replies with a vertex in the other \( F_i \) with a new color.
- Bob colors a vertex of any \( F_i \) with an already used color then before responding, Alice will consider the \( F_i \)’s.
  - If at least one of the \( F_i \) has got at most two distinctly colored vertices, then she counts the number of distinctly colored vertices in both the \( F_i \)’s and chooses whichever is minimum and colors any vertex of the chosen \( F_i \) with a new color. If both \( F_i \)’s have the same number of distinctly colored vertices, then Alice randomly chooses any \( F_i \) and colors a vertex of it with a new color.
  - Otherwise Alice colors any vertex with an available color.

Observe that the above strategy is a winning strategy since at any stage, any uncolored vertex is adjacent to at most six distinctly colored vertices. Note that, if each \( F_i \) has got at least three distinct colors, then Alice wins.

**Theorem 2.3.** For any positive integer \( n \), \( \chi_g(P_2[K_{1,n}]) = \begin{cases} 4 & n = 1 \\ \geq 5 & n \geq 2 \end{cases} \)

**Proof.** Denote the vertices of \( F_i \) by \( v_{ik} \), \( 1 \leq i \leq 2 \), \( 0 \leq k \leq n \), with \( v_{i0} \) as the center vertex of \( F_i \). When \( n = 1 \), \( P_2[K_{1,1}] \) is isomorphic to \( K_4 \), its game chromatic number is four. Consider \( n \geq 2 \).

As \( P_2[P_3] \) is an induced subgraph of \( P_2[K_{1,n}] \), \( n \geq 2 \), by Theorem 2.2, we have \( \chi_g(P_2[K_{1,n}]) \geq 5 \). Now we show that Alice has a winning strategy using five colors. The strategy is as follows.

Alice colors \( v_{10} \) in her first move. In the following moves, whenever

- Bob colors \( v_{20} \) then Alice colors a vertex of \( F_1 \) with a new color. Now Bob responds in any vertex. Alice replies with a new color for a vertex of \( F_2 \).
- Bob colors a vertex of \( F_1 \) or \( F_2 \) except \( v_{20} \). Alice replies with \( v_{20} \). In the next turn, Bob colors a vertex of \( F_1 \) or \( F_2 \) then Alice replies in a vertex of \( F_2 \) or \( F_i \) respectively.

Observe that, using this strategy, at any stage, any uncolored vertex is adjacent to at most four distinctly colored vertices and hence at least one color is always available. Thus \( \chi_g(P_2[K_{1,n}]) = 5 \), \( n \geq 2 \).

**Theorem 2.4.** For any positive integer \( n \geq 8 \), \( \chi_g(P_2[W_n]) = 9 \), where \( W_n \) is a wheel on \( n + 1 \) vertices.

**Proof.** Denote the vertices of \( F_i \) by \( v_{ik} \), \( 1 \leq i \leq 2 \), \( 0 \leq k \leq n \), with \( v_{i0} \) as the center vertex of \( F_i \). First we show that Bob has a winning strategy using eight colors. The strategy of Bob is as follows.

Without loss of generality, Alice colors \( v_{10} \) with color 1. Bob replies with \( v_{11} \) with color 2. Now Alice either plays in \( F_1 \) or \( F_2 \) which is discussed in the following two cases.

**Case 1:** Alice plays in \( F_1 \)

- Alice plays in \( F_1 \) say \( v_{1k} \), \( 2 \leq k \leq n - 1 \), with a color of \( v_{11} \). Bob replies with \( v_{1n-1} \) or \( v_{13} \) with a new color say color 3. Now Alice either plays in \( F_1 \) or \( F_2 \).

**Case 1.1:** Alice plays in \( F_1 \)

- Alice plays in a vertex of \( F_1 \) with a color used in \( F_1 \). Irrespective of Alice’s move, in the next two moves of Bob’s, Bob uses two new distinct colors to color the vertices of \( F_1 \). Observe that with at most three distinct colors Alice cannot color all the vertices of \( F_2 \). Hence Bob wins.
– Alice colors a vertex of $F_1$ with a new color. Then in the next two consecutive moves of Bob, irrespective of Alice’s move, he colors a vertex of $F_1$ with a new color each time. Observe that with at most two distinct colors Alice cannot color all the vertices of $F_2$. Hence Bob wins.

**Case 1.2:** Alice plays in $F_2$

Alice colors a vertex of $F_2$. In the next two moves of Bob, irrespective of Alice’s move, he colors a vertex of $F_1$ with a new color each time. Observe that with at most three distinct colors Alice cannot color all the vertices of $F_2$. Hence Bob wins.

- Alice plays in $F_1$ with a new color. Again irrespective of Alice’s move, in the next three moves of Bob, he uses three distinct new colors for any three vertices of $F_1$. Observe that with at most two distinct colors Alice cannot color all the vertices of $F_2$. Hence Bob wins.

**Case 2:** Alice plays in $F_2$

Alice colors a vertex of $F_2$. In the next three moves of Bob’s, irrespective of Alice’s move, he colors a vertex of $F_1$ with a new color each time. Observe that Alice cannot color all the vertices of $F_2$ with at most three distinct colors. Hence Bob wins.

Thus $\chi^g(P_2[W_n]) \geq 9$, $n \geq 8$. Now we show that Alice has a winning strategy using nine colors. The strategy is as follows.

We fix the first two moves of Alice. Alice colors a vertex $v_{10}$ in her first move and $v_{20}$ in her second move. Alice uses a new color in each of her moves till both $F'_i$s receive at least four distinct colors. In the following moves, whenever

- Bob colors any vertex with a new color then Alice replies with a vertex in the other $F_i$ with a new color.
- Bob colors a vertex of any $F_i$ with an already used color then before responding, Alice will consider the $F'_i$s.
  - If one of the $F_i$ has got at most three distinctly colored vertices, then she counts the number of distinctly colored vertices in both the $F'_i$s and chooses whichever is minimum and colors any vertex of the chosen $F_i$ with a new color. If both $F'_i$s have the same number of distinctly colored vertices, then Alice randomly chooses any $F_i$ and colors a vertex of it with a new color.
  - Otherwise Alice colors any vertex with an available color.

Observe that the above strategy is a winning strategy since at any stage, any uncolored vertex is adjacent to at most eight distinctly colored vertices. Note that if each $F_i$ has got at least four distinct colors, then Alice wins.

Hence $\chi^g(P_2[W_n]) = 9$, $n \geq 8$.

**Proposition 2.5 ([2])**. Suppose that $G = (V, E)$ is a graph with $E = E_1 \cup E_2$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ then $\chi^g(G) \leq \text{col}_g(G) \leq \text{col}_g(G_1) + \Delta(G_2)$.

**Corollary.** For any two simple graphs $G$ and $H$, $\chi^g(G[H]) \leq \text{col}_g(G[H]) \leq \text{col}_g(\bigcup_{|V(G)|}(H)) + |V(H)| \Delta(G)$.

**Proof.** In Proposition 2.5, set $G_1$ to be the union of all $F'_i$s (copies of $H$) and $G_2 = (V, E_2)$ where $E_2 = E - E_1$.

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**References**
