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# On values in relatively normal lattices

Alexandru Filipoiu[a,*], George Georgescu[b]

[a] *Department of Mathematics 1, Univ. 'Politechnica' Bucharest, Str. Splaiul Independenpei 313, Bucharest, Romania*

[b] *Institute of Mathematics, Str. Academiei 14, Bucharest, Romania*

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**Abstract**

In [8, 11, 12] the class IRN was introduced in order to obtain the lattice-theoretic analogues of some results of Conrad (see e.g. [4]). The aim of these paper is to provide other useful constructions in the study of the structure of relatively normal lattices. The introduced notions and results are purely lattice-theoretic extensions of notions and results for lattice-ordered groups [2, 4, 5]. In the second section, the notion of plenary set of a member of the class IRN is introduced and the characterization of maximal plenary sets is given, extending a well-known theorem in *l*-groups. In the third section with any lattice in IRN is associated a tree and we investigate how the properties of this tree are reflected in the structure of the lattice. For the case of *l*-groups, one gets some of Conrad's results in [5].

## 1. Preliminaries

In this section we review some relevant concepts. For notions not defined here, we refer the reader to [6, 11].

Let $A$ be an algebraic, distributive lattice with least element 0 and greatest element 1 and $\mathrm{Com}(A)$ the join-subsemilattice of compact elements of $A$.

An element $p < 1$ is *meet-irreducible* if $p = x \wedge y$ implies $p = x$ or $p = y$; an element $p < 1$ is *meet-prime* if $x \wedge y \leqslant p$ implies $x \leqslant p$ or $y \leqslant p$. These definitions can be extended to arbitrary meets and we obtain the concepts of *completely meet-irreducible* and *meet completely-prime* elements. The dual notions of *join-irreducible*, *join-prime*, *completely join-irreducible* and *completely join-prime* elements are defined dually.

In an algebraic lattice every element is the meet of a set of completely meet-irreducible elements [6].

A *value* of a compact element $c$ of $A$ is an element $p \in A$ which is maximal with respect to not exceeding $c$ [11]. For any $c \in \mathrm{Com}(A)$ we shall denote by $\mathrm{Val}(c)$ the set

* Corresponding author.


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of values of $c$. $V(A)$ will denote the set of the values in $A$. Thus, $V(A)$ is exactly the set of completely meet-irreducible elements in $A$, so every element of $A$ is a meet of a set of values.

In [11] there was proved that an element is completely join-prime iff it is compact and has a unique value; an element is completely meet-prime iff it is the unique value of a compact element.

A *root-system* is a poset $(P, \leqslant)$ for which the principal order filter $[p) = \{x \in P \mid p \leqslant x\}$ is a chain for all $p \in P$. A root is a maximal chain in $(P, \leqslant)$. We shall denote by IRN the class of the algebraic, distributive lattices $A$ such that $\mathrm{Com}(A)$ is a sublattice of $A$ and the meet-primes element in $A$ form a root system (see [11,12]).

**Lemma 1.1** (Snodgrass and Tsinakis [11]). *For an algebraic, distributive lattice $A$ such that $\mathrm{Com}(A)$ is a sublattice of $A$, the following are equivalent*:

(1) *$A$ is a member of* IRN;

(2) *For all $c$, $d \in \mathrm{Com}(A)$ there exist $c', d' \in \mathrm{Com}(A)$ such that $c \vee d' = c' \vee d = c \vee d$ and $c' \wedge d' = 0$.*

We remark that if $c, d, c', d'$ are as in (2), then $c' \leqslant c$, $d' \leqslant d$. Further if $c$ and $d$ are incomparable, then $0 < c'$, $d'$.

**Remark.** If $G$ is an $l$-group then the set $C(G)$ of all its convex $l$-subgroups is a member of IRN (see [2,4]). The set of the ideals in a relatively normal lattice, in an MV-algebra [3] or in a bounded commutative BCK-algebra [9] is also a member of IRN. Thus, the results of this paper can be applied for all these cases.

*Throughout this paper $A$ will denote a member of* IRN.

An element $x \neq 0$ is linear if $(x] = \{y \in A \mid y \leqslant x\}$ is a chain. If $a \leqslant b$, then we shall say that $b$ is an *ordinal extension* of $a$ if $(b] - (a]$ is a chain and every element of $(b] - (a]$ exceeds $a$. An *ordinal element* is a proper ordinal extension of some element. All these notions are defined in [12] as natural extensions of some notions in $l$-groups [1,2,4]. We remark that a linear element is also an ordinal element and a non-zero compact element below a linear element is completely join-prime [12].

For $x \in A$ we shall denote by $x^*$ the pseudo-complement of $x$ in $A$. The set of meet-prime elements of $A$ will be denoted by $\mathrm{Spec}\, A$ and the set of minimal meet-prime elements of $A$ by $\mathrm{Min}\, A$.

For any $a \in A$, $(a] = \{x \in A \mid x \leqslant a\}$ is a member of IRN and $\mathrm{Com}(a] = \mathrm{Com}(A) \cap (a]$ (see [12]). It is easy to prove that for $a \in A$ the mapping $\varphi : \{p \in \mathrm{Spec}\, A \mid a \nleqslant p\} \to \mathrm{Spec}(a]$ defined by $\varphi(p) = p \wedge a$ is an order preserving bijection and its inverse, given by $\varphi^{-1}(q) = a \to q$, is also order preserving.

**Lemma 1.2.** *Let $c \in \mathrm{Com}(a]$. Then $m \in \mathrm{Val}(c)$ iff $\varphi(m) \in \mathrm{Val}_{(a]}(c)$, where $\mathrm{Val}_{(a]}(c)$ is the set of values of $c$ in $(a]$.*

**Proof.** Suppose $m \in \mathrm{Val}(c)$; then $m \in \mathrm{Spec}\, A$ and $a \nleqslant m$, so we can compute $\varphi(m)$. Since, $c \nleqslant m \wedge a$ there exists $k \in \mathrm{Val}_{(a]}(c)$ such that $m \wedge a \leqslant k$. First note that $c \nleqslant \varphi^{-1}(k)$ since otherwise $c \wedge a \leqslant k$, $c \nleqslant k$ imply $a \leqslant k$, contradicting $c \nleqslant k$. Thus, $m \leqslant \varphi^{-1}(k)$, $c \nleqslant \varphi^{-1}(k)$ and $m \in \mathrm{Val}(c)$ imply $m = \varphi^{-1}(k)$, that is $\varphi(m) = k \in \mathrm{Val}_{(a]}(c)$.

We next assume $k \in \mathrm{Val}_{(a]}(c)$ and we shall establish that $\varphi^{-1}(k) \in \mathrm{Val}(c)$. It follows that $c \nleqslant \varphi^{-1}(k)$ since otherwise $c \wedge a \leqslant k$ and $c \nleqslant k$ imply $a \leqslant k$, contradiction. Hence there exist $m \in \mathrm{Val}(c)$ such that $\varphi^{-1}(k) \leqslant m$; then $k \leqslant \varphi(m)$, $c \nleqslant \varphi(m)$ and $k \in \mathrm{Val}_{(a]}(c)$ imply $k = \varphi(m)$, so $\varphi^{-1}(k) = m \in \mathrm{Val}(c)$. □

**Remark.** The mapping $\varphi$ realizes an order preserving bijection between $\mathrm{Val}(c)$ and $\mathrm{Val}_{(a]}(c)$, for any $c \in \mathrm{Com}(a]$.

**Lemma 1.3.** *The mapping $\varphi$ induces an order preserving bijection between $\{p \in V(A) \mid a \nleqslant p\}$ and $V((a])$.*

**Proof.** Suppose $m \in V(A)$, $a \nleqslant m$ and we establish that $\varphi(m) \in V((a])$. Since $a \nleqslant m$ there exist a compact $y$ such that $y \leqslant a$, $y \nleqslant m$ and because $m \in V(A)$ there exists a compact $x$ such that $m \in \mathrm{Val}(x)$. Then $x \wedge y$ is a compact, $x \wedge y \nleqslant m$ and $m \in \mathrm{Val}(x \wedge y)$, since for $m < n$ we obtain $x \leqslant n$, using $m \in \mathrm{Val}(x)$, so $x \wedge y \leqslant n$. Using Lemma 1.2 it follows that $\varphi(m) \in \mathrm{Val}_{(a]}(x \wedge y) \subseteq V((a])$.

Conversely, for $k \in V((a])$ there exist a compact $x \leqslant a$ such that $k \in \mathrm{Val}_{(a]}(x)$, hence $\varphi^{-1}(k) \in \mathrm{Val}(x)$ using Lemma 1.2. □

**Lemma 1.4.** *Let $a \in A$ and $p \in V(A)$. Then $a \nleqslant p$ iff there exists $c \in \mathrm{Com}(A)$ such that $c \leqslant a$ and $p \in \mathrm{Val}(c)$.*

**Proof.** Using Lemmas 1.2 and 1.3, for any $p \in V(A)$, the following equivalences hold: $a \nleqslant p$ iff $\varphi(p) \in V((a])$ iff $\varphi(p) \in \mathrm{Val}_{(a]}(c)$ for some $c \in \mathrm{Com}(a]$ iff $p \in \mathrm{Val}(c)$ for some compact element $c \leqslant a$. □

## 2. Minimal plenary sets

**Definition 2.1.** A nonempty subset $D$ of $V(A)$ will be called *a plenary set* of $A$ if the following conditions are satisfied:

(1) $\bigwedge D = 0$,

(2) If $p \in D$, $q \in V(A)$, $p \leqslant q$ then $q \in D$.

This notion extends a notion in $l$-groups (see [2,4]). Condition (2) of the previous definition can be stated: $D$ is an increasing subset (or an order-filter) of $V(A)$. The main result of this section is the characterization of minimal plenary sets of $A$, extending a well-known theorem in $l$-groups [2,4].

By Zorn's lemma, for any nonzero compact element $c \in A$, the set $D(c) = D \cap \mathrm{Val}(c)$ is nonempty.

If $a, b \in A$ are incomparable we shall write $a \| b$.

**Lemma 2.2.** *If $q, q_1, \ldots, q_n \in A$ are such that $q$ is meet-prime and $q \| q_i$ for $i = 1, \ldots, n$, then there exist two compact elements $c, d$ such that $c \leqslant q$, $c \nleqslant q_i$ for $i = 1, \ldots, n$, $d \nleqslant q$, $d \leqslant \bigwedge\{q_i / i = 1, \ldots, n\}$ and $c \wedge d = 0$.*

**Proof.** We can find compact elements $x_i, y_i$ such that $x_i \leqslant q, x_i \nleqslant q_i$ and $y_i \leqslant q_i, y_i \nleqslant q$ for $i = 1, \ldots, n$. Denoting $x = \bigvee\{x_i / i = 1, \ldots, n\}$ and $y = \bigwedge\{y_i / i = 1, \ldots, n\}$ we have $x \leqslant q, y \nleqslant q$ and $x \nleqslant q_i, y \leqslant q_i$ for any $i = 1, \ldots, n$. Since, $x$ and $y$ are incomparable, using Lemma 1.1, there exist two compact elements $c$ and $d$ such that $c \vee y = d \vee x = x \vee y$, $c \wedge d = 0$ and $0 \leqslant c \leqslant x$, $0 < d \leqslant y$. It is obvious that $c \leqslant q$ and $d \leqslant \bigwedge\{q_i / i = 1, \ldots, n\}$. If $c \leqslant q_i$ for some $i$, then $x \leqslant d \vee x = c \vee y \leqslant q_i$, contradiction. Similarly, one can verify that $d \nleqslant q$. □

**Lemma 2.3.** *If $D$ is a plenary set in $A$, $x \in \mathrm{Com}(A)$ and $D(x)$ is finite then $D(x) = \mathrm{Val}(x)$.*

**Proof.** Let $D(x) = \{q_1, \ldots, q_n\}$ and $q \in \mathrm{Val}(x)$. Suppose $q \notin D$, then $q, q_1, \ldots, q_n$ are pairwise incomparable meet-prime elements in $A$, so one can find two compact elements $c, d$ as in the previous lemma. Since $d \nleqslant q, x \nleqslant q$ it follows that $d \wedge x \nleqslant q$, so $d \wedge x \neq 0$, hence there exists an element $p_0 \in D(d \wedge x)$. Thus $x \nleqslant p_0$, so there is $p \in D(x)$ such that $p_0 \leqslant p$, therefore $p = q_i$ for some $i \in \{1, \ldots, n\}$. Hence, $c \nleqslant p_0$ and $d \wedge x \nleqslant p_0$. But $p_0$ is meet-prime and $c \wedge d \wedge x = 0$, contradiction; so $q \in D$. □

**Lemma 2.4.** *Let $D$ be a plenary set in $A$, $x \in \mathrm{Com}(A) - \{0\}$ and $p \in V(A)$ such that $\bigvee D(x) \leqslant p$. Then $p \in D$ and $\bigvee \mathrm{Val}(x) \leqslant p$.*

**Proof.** It is obvious that $p \in D$, because $D(x) \neq \emptyset$ and $D$ is plenary set. If $p \in \mathrm{Val}(x)$ then $p \in D(x)$ therefore, using the hypothesis $\bigvee D(x) \leqslant p$, it follows that $D(x) = \{p\}$ and by Lemma 2.3, $\mathrm{Val}(x) = \{p\}$.

Suppose $p \notin \mathrm{Val}(x)$ and $q \nleqslant p$ for some $q \in \mathrm{Val}(x)$. If $x \nleqslant p$ there is $p_0 \in D(x)$ such that $p < p_0$, hence $p = p_0$, which contradicts $p \notin \mathrm{Val}(x)$. Thus, $x \leqslant p$. Since, $q \nleqslant p$ there exists $f \in \mathrm{Com}(A)$, $f \leqslant q$ and $f \nleqslant p$. By Lemma 1.1, $c \vee f = d \vee x = f \vee x$ and $c \wedge d = 0$ for two compact elements $c \leqslant x$, $d \leqslant f$. If $c = 0$ then $f = d \vee x$ so $x \leqslant f \leqslant q$ which contradicts $q \in \mathrm{Val}(x)$. Thus, $c \neq 0$ and there is $m \in D(c)$. We shall consider two cases:

(a) $m \leqslant p$; since $c \nleqslant m$ and $c \wedge d = 0$, one gets $d \leqslant m \leqslant p$, so $c \vee f = d \vee x \leqslant p$, which contradicts $f \nleqslant p$.

(b) $m \nleqslant p$. If $x \nleqslant m$ then there is $m_0 \in D(x)$, $m \leqslant m_0$, so by the hypothesis $\bigvee D(x) \leqslant p$, we have $m_0 \leqslant p$, which contradicts $m \nleqslant p$. If $x \leqslant m$, then $c \leqslant m$, which contradicts $m \in \mathrm{Val}(c)$.

In both cases we have obtained a contradiction, therefore $q \leqslant p$ for any $q \in \mathrm{Val}(x)$. □

For any nonzero compact element $x$ let us denote $r_x = \bigvee \mathrm{Val}(x)$. We shall say that $p \in V(A)$ is essential if there is $x \in \mathrm{Com}(A) - \{0\}$ such that $r_x \leqslant p$. If $p$ is completely meet-prime then there is $c \in \mathrm{Com}(A) - \{0\}$ such that $\mathrm{Val}(c) = \{p\}$, therefore $p$ is essential.

Let us denote by $E(A)$ the set of essential values in $A$ and $r(A) = \bigwedge \{r_x \mid x \in \mathrm{Com}(A) - \{0\}\}$.

**Lemma 2.5.** *If $D$ is a plenary set in $A$ then $E(A) \subseteq D$.*

**Proof.** If $p \in E(A)$ then $r_x \leqslant p$ for some $x \in \mathrm{Com}(A)$-$\{0\}$. But $\bigwedge D = 0$ so $x \nleqslant q$ for some $q \in D$. Thus $q \leqslant q'$ for some $q' \in \mathrm{Val}(x)$, hence $q' \leqslant p$, so $p \in D$. □

**Theorem 2.6.** *For any plenary set $D$ in $A$ the following are equivalent*:

(1) *$D$ is a minimal plenary set in $A$*;

(2) *$D = E(A)$*;

(3) *$D$ is the least plenary set in $A$.*

**Proof.** (1) $\Rightarrow$ (2). By Lemma 2.4, it suffices to prove that for any $d \in D$ there is $x \in \mathrm{Com}(A) - \{0\}$ such that $\bigvee D(x) \leqslant d$. Suppose, for a proof by contradiction, there is $d \in D$ such that for any $x \in \mathrm{Com}(A) - \{0\}$ there exists $q_x \in D(x)$, $q_x \nleqslant d$. Consider the following set: $D' = D - \{p \in V(A) \mid p \leqslant d\}$. Thus, $q_x \in D'$ for any $x \in \mathrm{Com}(A) - \{0\}$. If $\bigwedge D' \neq 0$ then there is $a \in \mathrm{Com}(A) - \{0\}$ such that $a \leqslant p$ for any $p \in D'$, in particular, $a \leqslant q_a$ which contradicts $q_a \in \mathrm{Val}(a)$. It follows that $\bigwedge D' = 0$ and $D'$ being increasing one gets that $D'$ is a plenary set in $A, D' \subseteq D$ and $D' \neq D$, because $d \in D - D'$, which contradicts the minimality of $D$.

(2) $\Rightarrow$ (1). By Lemma 2.5.

(1) $\Leftrightarrow$ (3). By (1) $\Rightarrow$ (2) and Lemma 2.5. □

We shall say that $A$ is *finite-valued* [11] if $\mathrm{Val}(x)$ is finite for any $x \in \mathrm{Com}(A)$. □

**Proposition 2.7.** (1) $r(A) = \bigwedge E(A)$.

(2) *For $c \in \mathrm{Com}(A)$, $c \leqslant r(A)$ iff $c$ has no essential values.*

(3) *If $A$ is finite-valued then $r(A) = 0$.*

**Proof.** (1) Let us consider $c \in \mathrm{Com}(A)$ such that $c \nleqslant \bigwedge \{r_x \mid x \in \mathrm{Com}(A) - \{0\}\}$ so there is $x \in \mathrm{Com}(A) - \{0\}$ such that $c \nleqslant r_x = \bigvee \mathrm{Val}(x)$. Hence there is $m \in \mathrm{Val}(c)$ such that $r_x \leqslant m$, so $c \nleqslant m$ and $m \in E(A)$. This yields $r(A) \geqslant \bigwedge E(A)$. On the other hand, for any $p \in E(A)$ there is $x(p) \in \mathrm{Com}(A) - \{0\}$ such that $r_{x(p)} \leqslant p$, therefore $r(A) \leqslant \bigwedge \{r_{x(p)} \mid p \in E(A)\} \leqslant \bigwedge E(A)$.

(2) In the light of (1), for $c \in \mathrm{Com}(A)$ we have the following equivalences: $c \leqslant r(A) \Leftrightarrow c \leqslant p$, for any $p \in E(A) \Leftrightarrow c$ has no essential values.

(3) In accordance with Lemma 2.3 [11], the values of any nonzero compact element are completely meet-prime, so they are essential. By (2) one gets $r(A) = 0$. □

**Corollary 2.8.** *The following conditions are equivalent:*

(1) *$E(A)$ is a plenary set in $A$;*
(2) *There exists a minimal plenary set in $A$;*
(3) *There exists the least plenary set in $A$;*
(4) *$r(A) = 0$.*

**Proof.** By Theorem 2.6 and Proposition 2.7, since $E(A)$ is an increasing set in $V(A)$. □

**Proposition 2.9.** *The following assertions are equivalent:*

(1) *$A$ is finite-valued;*
(2) *$A$ is completely distributive.*

**Proof.** (1) ⇒ (2). Denoting $a = \bigvee\{\bigwedge(x_{ij} \mid j \in J) \mid i \in I\}$ and $b = \bigwedge\{\bigvee(x_{if(i)} \mid i \in I) \mid f \in J^I\}$ we always have $a \leqslant b$. For the converse inequality it suffices to prove that if $p \in V(A)$, $a \leqslant p$ then $b \leqslant p$, because any element in $A$ is a meet of values. By Lemma 2.3 [11] $p$ is completely meet-prime, hence for $i \in I$ there exists $f(i) \in J$ such that $x_{if(i)} \leqslant p$, so $b \leqslant \bigvee(x_{if(i)} \mid i \in I) \leqslant p$.

(2) ⇒ (1). By Lemma 2.3 [11] it suffices to prove that any $p \in V(A)$ is completely meet-prime. If $\bigwedge\{x_i \mid i \in I\} \leqslant p$ then, since $A$ is completely distributive, we obtain $p = p \vee (\bigwedge\{x_i \mid i \in I\} = \bigwedge\{p \vee x_i \mid i \in I\}$. But $p$ is completely meet-irreducible, hence $p = p \vee x_i$ for some $i \in I$, that is $x_i \leqslant p$ for some $i \in I$. □

**Corollary 2.10.** *If $A$ is completely distributive then $r(A) = 0$.*

**Proof.** By Propositions 2.9 and 2.7. □

**Proposition 2.11.** $r(A) \leqslant \bigwedge\{c^* \mid c \in \mathrm{Com}(A)$, *$c$ linear element*$\}$.

**Proof.** Let $x \in \mathrm{Com}(A)$ such that $x \leqslant r(A)$. For any linear compact element $c$ we have $\mathrm{Val}(c) = \{m\}$ for some $m \in V(A)$, using Lemma 3.1 [12]. Then $x \leqslant m$ and $c \nleqslant x$; but any linear element is an ordinal element hence, by Lemma 3.9 [12], one gets $x \leqslant c \vee c^*$. Since, $x \in \mathrm{Com}(A)$, one can find $y, z \in \mathrm{Com}(A)$ such that $x = y \vee z$, $y \leqslant c$ and $z \leqslant c^*$, so $y \wedge z = 0$. If $y \neq 0$ then there is $q \in V(A)$, $\mathrm{Val}(y) = \{q\}$, because $y$ is linear and compact. Thus, $y \nleqslant q$ and $y \leqslant x$ so $x \nleqslant q$, a contradiction, because $q \in E(A)$ and $x \leqslant \bigwedge E(A)$. Hence, $y = 0$, therefore $x \leqslant c^*$. □

A subset $B$ of $A$ is *a basis* of $A$ [12] if it is a maximal orthogonal set in $A$ and every element of $B$ is linear. Thus, $A$ has a basis iff every nonzero element of $A$ exceeds a linear element (see [12, Proposition 4.3]).

**Corollary 2.12.** *If $A$ has a basis than $r(A) = 0$.*

**Proof.** By Proposition 2.11, if $d \in \mathrm{Com}(A) - \{0\}$ and $d \leqslant r(A)$ then $d$ does not exceed any linear compact element. But $A$ has a basis so, by the previous remark, $r(A) = 0$. □

## 3. The tree $S$

Let $A$ be a member of IRN. We shall associate with $A$ a tree $S$ and we shall investigate how the properties of $S$ are reflected in the structure of $A$. For the case of $l$-groups one gets some of Conrad's results [5].

For $p \in V(A)$ consider $\bar{p} = \bigwedge\{x \in A \mid p < x\}$. Thus, for any $c \in \mathrm{Com}(A)$, $p \in \mathrm{Val}(c)$ implies $c \leqslant \bar{p}$. For any $p \in V(A)$ we shall denote $s_p = \bigvee\{c \in \mathrm{Com}(A) \mid q \in \mathrm{Val}(c) \Rightarrow p \| q\}$.

**Lemma 3.1.** *For $x \in \mathrm{Com}(A)$ and $p \in V(A)$ the following are equivalent:*
(1) $x \leqslant s_p$;
(2) *for any $q \in \mathrm{Val}(x)$, $p \| q$.*

**Proof.** (1) $\Rightarrow$ (2). If $x \leqslant s_p$ then $x \leqslant c_1 \vee \cdots \vee c_n$ for some compact elements $c_1, \ldots, c_n$ such that $p \| q$ for all $q \in \mathrm{Val}(c_i)$, $i = 1, \ldots, n$. Let $q \in \mathrm{Val}(x)$; we have $x \nleqslant q$, so $c_i \nleqslant q$ for some $i \in \{1, \ldots, n\}$. Thus, $q \leqslant q'$ for some $q' \in \mathrm{Val}(c_i)$, so $p \| q'$. If $p \leqslant q$ then $p \leqslant q'$. If $q < p$ then $q \leqslant p, q'$ so $p, q'$ are comparable since $A$ is a member of IRN. Both cases are impossible, therefore $p \| q$.

(2) $\Rightarrow$ (1). Obvious. □

**Lemma 3.2.** *Let $q, q_1, \ldots, q_n \in V(A)$ be such that $q \| q_i$, $i = 1, \ldots, n$. Then there exists $c \in \mathrm{Com}(A)$ such that $c \leqslant \bar{q}$, $c \nleqslant q$ and $c \leqslant \bigwedge\{s_{q_i}/i = 1, \ldots, n\}$.*

**Proof.** For $p_1, p_2 \in V(A)$, $p_1 \| p_2$ there exist $x, y \in \mathrm{Com}(A)$ such that $x \leqslant \bar{p}_1$, $x \nleqslant p_1$ and $y \leqslant p_2$, $y \nleqslant p_1$. Hence, $z = x \wedge y \in \mathrm{Com}(A)$, $z \leqslant \bar{p}_1 \wedge p_2$ and $z \nleqslant p_1$.

Using this remark, in our case there exist $a_i, b_i \in \mathrm{Com}(A)$, such that $a_i \leqslant \bar{q} \wedge q_i$, $a_i \nleqslant q$ and $b_i \leqslant \bar{q}_i \wedge q$, $b_i \nleqslant q_i$ for $i = 1, \ldots, n$. Denoting $a = \bigwedge\{a_i/i = 1, \ldots, n\}$, $b = \bigvee\{b_i/i = 1, \ldots, n\}$ we have $a \leqslant \bar{q} \wedge q_i$, $a \nleqslant q$, $b \leqslant q$ and $b \nleqslant q_i$ for $i = 1, \ldots, n$. It is obvious that $a, b$ are incomparable compact elements, so there exist compact elements $0 < c \leqslant a$, $0 < d \leqslant b$ such that $a \vee d = b \vee c = a \vee b$ and $c \wedge d = 0$. We have $c \leqslant \bar{q} \wedge q_i$, $i = 1, \ldots, n$ and $c \nleqslant q$, because $c \leqslant q$ implies $a \leqslant a \vee b = c \vee b \leqslant q$. Now we shall prove that $c \leqslant s_{q_i}$, $i = 1, \ldots, n$, using Lemma 3.1. Suppose the contrary: there is $i$ and $p \in \mathrm{Val}(c)$ such that $p, q_i$ are comparable. Two cases are possible:
(1) $q_i \leqslant p$, hence $c \leqslant a \leqslant q_i \leqslant p$, which contradicts $p \in \mathrm{Val}(c)$;
(2) $p < q_i$, hence $d \leqslant p < q_i$, because $c \wedge d = 0$ and $c \nleqslant p$. Thus, $b \leqslant a \vee b = a \vee d \leqslant q_i$, which is impossible.

In this way we have proved that $c$ satisfies all the conditions of the lemma. □

**Theorem 3.3.** *If $p \in V(A)$ then $s_p = \bigvee\{c^* \mid c \in \mathrm{Com}(A),\ c \not\leqslant p\} = \bigwedge\{m \mid m \in \mathrm{Min}\, A,\ m \leqslant p\}$.*

**Proof.** The second equality was proved in [9] (see Proposition 4.4). Consider $x \in \mathrm{Com}(A)$ such that $x \leqslant \bigvee\{c^* \mid c \in \mathrm{Com}(A),\ c \not\leqslant p\}$. Thus there exist compact elements $c_1, \dots, c_n$ such that $x \leqslant \bigvee\{c_i^*/i = 1, \dots, n\}$ and $c_i \not\leqslant p$, $i = 1, \dots, n$, hence $d = \bigwedge\{c_i/i = 1, \dots, n\} \not\leqslant p$ and $x \leqslant \bigvee c_i^* \leqslant d^*$. It follows that $x \wedge d = 0$, so $x \leqslant p$ because $d \not\leqslant p$. Now we shall prove that $x \leqslant s_p$ using Lemma 3.1. Let $q \in \mathrm{Val}(x)$, hence $p \not\leqslant q$, because $p \leqslant q$ implies $x \leqslant p \leqslant q$. If $q < p$ then $d \not\leqslant q$, because $d \not\leqslant p$; but $x \wedge d = 0$ and $x \not\leqslant q$ which contradicts $q \in \mathrm{Spec}\, A$. Hence, $p \| q$ and we have proved that $\bigvee\{c^* \mid c \in \mathrm{Com}(A),\ c \not\leqslant p\} \leqslant s_p$.

Let $x \in \mathrm{Com}(A)$ be such that $q \| p$ for all $q \in \mathrm{Val}(x)$. For any $m \in \mathrm{Min}\, A$, $m \leqslant p$ implies $x \leqslant m$. Indeed, suppose $m \leqslant p$ and $x \not\leqslant m$, then there is $q \in \mathrm{Val}(x)$, $m \leqslant q$. Thus, $m \leqslant p$, $q$ so $p, q$ are comparable, contradiction. Therefore, one gets $s_p \leqslant \bigwedge\{m \mid m \in \mathrm{Min}\, A,\ m \leqslant p\}$. □

**Corollary 3.4.** *For any $p \in V(A)$, $s_p^* = \bigwedge\{c^{**} \mid c \in \mathrm{Com}(A),\ c \not\leqslant p\}$.*

**Proposition 3.5.** *Let $p, q \in V(A)$. Then $s_p \leqslant q$ iff $p, q$ are comparable.*

**Proof.** If $s_p \not\leqslant q$ then there is $x \in \mathrm{Com}(A)$ such that $x \leqslant s_p$ and $x \not\leqslant q$. In accordance with Lemma 3.1, $p \| q'$ for any $q' \in \mathrm{Val}(x)$. From $x \not\leqslant q$ one gets $q \leqslant q'$ for some $q' \in \mathrm{Val}(x)$. It follows that $p \| q$, because if $p \leqslant q$ then $p \leqslant q'$ and if $q < p$ then $q \leqslant p, q'$. In both cases one contradicts the assumption $p \| q'$.

For the converse implication, suppose $p \| q$. By Lemma 3.2 there exists a compact element $x$ such that $x \leqslant \bar{q} \wedge s_p$, $x \not\leqslant q$ so $s_p \not\leqslant q$. □

**Corollary 3.6.** *Let $p, q \in V(A)$. Then $s_p \leqslant q$ iff $s_q \leqslant p$.*

**Corollary 3.7.** *For any $p \in V(A)$, $s_p = \bigwedge\{q \in V(A) \mid p, q$ are comparable$\}$.*

**Proof.** Denote by $u$ the second member of this equality. If $u \not\leqslant s_p$ there is $c \in \mathrm{Com}(A)$, $c \leqslant u$, $c \not\leqslant s_p$ so by Lemma 3.1 there is $q \in \mathrm{Val}(c)$ such that $p, q$ are comparable, hence $c \leqslant u \leqslant q$ which contradicts $q \in \mathrm{Val}(c)$. This contradiction shows that $u \leqslant s_p$. The converse inequality follows by Proposition 3.5. □

**Remark.** If $p \in V(A)$ then $s_p \leqslant p$.

**Corollary 3.8.** *For $p, q \in V(A)$, $p \| q$ iff $s_p \| s_q$.*

**Proof.** Suppose $s_p \leqslant s_q$; then using $s_q \leqslant q$ we obtain $s_p \leqslant q$ and by Proposition 3.5 $p, q$ are comparable. If $p \leqslant q$, by Theorem 3.3, one gets $s_q = \bigwedge\{m \in \operatorname{Min} A \mid m \leqslant q\} \leqslant \bigwedge\{m \in \operatorname{Min} A \mid m \leqslant p\} = s_p$. □

**Remark.** By the previous corollary, $s_p < s_q$ implies $q < p$.

Let us denote $S = \{s_p \mid p \in V(A)\}$. From the previous results and the fact that $V(A)$ is a root system it follows that $S$ is a tree.

**Proposition 3.9.** *For $p, q \in V(A)$ the following are equivalent*:
(1) $s_p = s_q$;
(2) *For any $r \in V(A)$, $r \| p$ iff $r \| q$*;
(3) *$p, q$ belong to the same roots of $V(A)$.*

**Proof.** (1) ⇒ (2). By Corollary 3.8 we have the following equivalences:

$$r \| p \text{ iff } s_r \| s_p \text{ iff } s_r \| s_q \text{ iff } r \| q.$$

(2) ⇒ (1). By Corollary 3.6.
(2) ⇒ (3). Assume there is a root $U$ in $V(A)$ such that $p \in U$, $q \notin U$; so $r \| q$ for some $r \in U$, therefore $r \| p$. This contradicts $p, r \in U$.
(3) ⇒ (2). We remark that $p \| r$ iff for any root $U$ in $V(A)$, $p \in U$ implies $r \notin U$. □

**Proposition 3.10.** For $a \in A$, $a^* = \bigwedge\{s_p \mid p \in V(A),\ a \nleqslant p\}$.

**Proof.** Suppose $x \in \operatorname{Com}(A)$, $x \leqslant a^*$ and $p \in V(A)$, $a \nleqslant p$. By Lemma 1.4 there is $c \in \operatorname{Com}(A)$, $c \leqslant a$ and $p \in \operatorname{Val}(c)$. Thus $x \wedge c = 0$ and $c \nleqslant p$, so $x \leqslant p$. Assume there is $q \in V(A)$ such that $p, q$ are comparable and $x \nleqslant q$, hence $c \leqslant q$. We remark that $p \leqslant q$ implies $x \leqslant q$ and $q < p$ implies $c < p$. In both cases we obtain a contradiction, so if $p, q$ are comparable then $x \leqslant q$. This yields $x \leqslant \bigwedge\{q \in V(A) \mid p, q \text{ are comparable}\} = s_p$, therefore $x \leqslant \bigwedge\{s_p \mid p \in V(A),\ a \nleqslant p\}$. We have proved that $a^* \leqslant \bigwedge\{s_p \mid p \in V(A),\ a \nleqslant p\}$.

For the converse inequality, consider a compact element $x$ such that $x \leqslant \bigwedge\{s_p \mid p \in V(A),\ a \nleqslant p\}$. Note first that if $c$ is a compact element satisfying $c \leqslant a$, $p \in \operatorname{Val}(c)$ and $q \in \operatorname{Val}(x)$, then $a \nleqslant p$ because $c \nleqslant p$, so $x \leqslant s_p$, hence in accordance with Lemma 3.1 one gets $p \| q$. Now if $x \wedge a \neq 0$, then $c \leqslant x \wedge a$ for some $c \in \operatorname{Com}(A) - \{0\}$, hence there is $p \in \operatorname{Val}(c)$, so $x \wedge a \nleqslant p$. It follows that $x \nleqslant p$, so there is $q \in \operatorname{Val}(x)$, $p \leqslant q$. This contradicts the previous remark, so $x \wedge a = 0$, hence $x \leqslant a^*$. □

**Corollary 3.11.** *For any $p \in V(A)$ we have $s_p^* = \bigwedge\{s_q \mid q \in V(A),\ q \| p\}$.*

**Proof.** From Propositions 3.10 and 3.5, one can infer

$$s_p^* = \bigwedge\{s_q \mid q \in V(A),\ s_p \nleqslant q\} = \bigwedge\{s_q \mid q \in V(A),\ q \| p\}. \quad \square$$

**Corollary 3.12.** *If* $\mathrm{Val}(c) = \{p\}$ *then* $s_p = c^*$.

**Proof.** By Proposition 3.10 we have $c^* = \bigwedge\{s_q \mid q \in V(A),\ c \nleqslant q\}$. If $q \in V(A)$, $c \nleqslant q$, since $\mathrm{Val}(c) = \{p\}$, then $q \leqslant p$. In accordance with Theorem 3.3, $s_p \leqslant s_q$, therefore $c^* = s_p$. □

**Proposition 3.13.** *If* $p \in V(A)$ *then the following equality holds*:

$$\bar{p} \wedge s_p^* = \bigvee\{c \in \mathrm{Com}(A) \mid \mathrm{Val}(c) \subseteq (p]\}.$$

**Proof.** Assume $c \in \mathrm{Com}(A)$ such that $\mathrm{Val}(c) \subseteq (p]$. Consider $q \in V(A)$ such that $q \| p$. We shall prove that $r \| q$ for any $r \in \mathrm{Val}(c)$. Suppose there is $r \in \mathrm{Val}(c)$ such that $r, q$ are comparable. Thus $r \leqslant p$ and two cases are possible:

(a) $r \leqslant q$; then $r \leqslant p, q$ so $p, q$ are comparable;

(b) $q < r$; then $q < p$.

It follows a contradiction in both cases, so $r \in \mathrm{Val}(c)$ implies $r \| q$, hence $c \leqslant s_q$. This yields $c \leqslant \bigwedge\{s_q \mid q \in V(A),\ q \| p\} = s_p^*$. If $c \neq 0$ then there is $q \in \mathrm{Val}(c)$, so $q \leqslant p$, hence $c \leqslant \bar{q} \leqslant \bar{p}$.

For the converse inequality let $c$ be a compact element such that $c \leqslant \bar{p} \wedge s_p^*$. Suppose there is $q \in \mathrm{Val}(c)$, $q \nleqslant p$. If $p < q$, then since $c \leqslant \bar{p}$ it follows that $c \leqslant q$, which contradicts $q \in \mathrm{Val}(c)$. Hence, $p \nleqslant q$, so $p \| q$. From $c \leqslant s_p^* = \bigwedge\{s_r \mid r \in V(A),\ r \| p\}$ and $p \| q$ one deduces, via the remark after Corollary 3.7, that $c \leqslant s_q \leqslant q$. This contradiction shows that $\mathrm{Val}(c) \subseteq (p]$. □

**Lemma 3.14.** *For* $p, q \in V(A)$, $s_p < s_q$ *iff* $q < p$ *and there exists* $r \in V(A)$ *such that* $r < p$ *and* $r \| q$.

**Proof.** Suppose $s_p < s_q$, hence $q < p$ by the remark after Corollary 3.8, and there is $x \in \mathrm{Com}(A)$ such that $x \leqslant s_q$ and $x \nleqslant s_p$. Using Lemma 3.1, $p$ is comparable with some value $r$ of $x$. Applying again that lemma, for $r \in \mathrm{Val}(x)$ we infer that $r \| q$. If $p \leqslant r$ then $q < r$ which is not possible, therefore $r < p$.

Assume $q, r < p$ and $r \| q$, so $s_p \leqslant s_q \wedge s_r \leqslant s_q$. If $s_q \wedge s_r = s_q$ then we obtain $s_q \leqslant s_r$, so, by Corollary 3.7, $q$ and $r$ are comparable. This contradiction shows that $s_p < s_q$. □

**Theorem 3.15.** *For* $p \in V(A)$ *the following are equivalent*:

(1) $s_p$ *is maximal in* $S$;

(2) $s_p \in \mathrm{Spec}\, A$;

(3) $s_p \in \mathrm{Min}\, A$;

(4) $(p] \cap V(A)$ *is a chain*;

(5) $p$ *is contained in a unique root of* $V(A)$;

(6) $p$ *exceeds a unique element of* $\mathrm{Min}\, A$.

**Proof.** It is obvious that (3) $\Rightarrow$ (2) and (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6).

(1) $\Rightarrow$ (4). Assume there exists $q, r \in (p] \cap V(A)$ such that $q \| r$. In accordance with Lemma 3.14, $s_p < s_q$ which contradicts the maximality of $s_p$ in $S$.

(4) $\Rightarrow$ (3). If $m_0$ is the unique element of $\operatorname{Min} A$ such that $m_0 \leqslant p$ then, by Theorem 3.3, $s_p = \bigwedge\{m \in \operatorname{Min} A \mid m \leqslant p\} = m_0$.

(2) $\Rightarrow$ (1). Suppose $s_p < s_q$, then $q, r < p$ for some $r \| q$. By Lemma 3.2 there exist compact elements $x, y$ such that $x \leqslant \bar{q} \wedge r, x \nleqslant q$ and $y \leq \bar{r} \wedge q, y \nleqslant r$. It is obvious that $x \| y$; then by Lemma 1.1 there exist compact elements $0 < c \leqslant x, 0 < d \leqslant y$ such that $x \vee d = y \vee c = x \vee y$ and $c \wedge d = 0$. From this one gets $c \leqslant \bar{q}$, $c \nleqslant q$, $d \leqslant \bar{r}$, $d \nleqslant r$ (by example, $c \leqslant q$ implies $x \vee y = y \vee c \leqslant q$ so $x \leqslant q$, contradiction). Then $q \in \operatorname{Val}(c)$ and $r \in \operatorname{Val}(d)$. Since, $q \in \operatorname{Val}(c)$ and $q < p$ we have by Lemma 3.1, $c \nleqslant s_p$ and similarly $d \nleqslant s_p$. But this contradicts $s_p \in \operatorname{Spec} A$ and $c \wedge d = 0$. Hence, $s_p$ is maximal in $S$. □

**Corollary 3.16.** *Let $c \in \operatorname{Com} A$ such that $\operatorname{Val}(c) = \{p\}$. Then $s_p$ is maximal in $S$ iff $c$ is linear.*

**Proof.** By Proposition 4.2 [12], $c$ is linear iff $c^* \in \operatorname{Min} A$. In accordance with Corollary 3.12, $s_p = c^*$ so, by the previous theorem, $s_p$ is maximal in $S$ iff $c$ is linear. □

**Lemma 3.17.** *For $p, q_1, \dots, q_n \in V(A)$, $\bigwedge\{s_{q_i}/i = 1, \dots, n\} \leqslant p$ iff $p$ is comparable with some $q_i$.*

**Proof.** If $p$ is comparable with some $q_i$ then, by Proposition 3.5, $s_{q_i} \leqslant p$, hence $\bigwedge\{s_{q_i} \mid i = 1, \dots, n\} \leqslant p$. In order to prove the converse implication one can assume that $q_1, \dots, q_n$ are pairwise incomparable. Suppose that $p \| q_i$ for $i = 1, \dots, n$. By Lemma 3.2 there is a compact element $c$ such that $c \nleqslant p$ and $c \leqslant \bar{p} \wedge \bigwedge\{s_{q_i} \mid i = 1, \dots, n\}$, so $\bigwedge\{s_{q_i}/i = 1, \dots, n\} \nleqslant p$. □

**Remark.** $\bigwedge V(A) = 0$, otherwise $c \leqslant \bigwedge V(A)$ for some $c \in \operatorname{Com}(A) - \{0\}$, implying $c \leqslant p$, for any $p \in V(A)$, which contradicts the fact that $c$ does have values. It follows easily that $\bigwedge \operatorname{Min} A = 0$.

**Theorem 3.18.** *If $q_1, \dots, q_n \in V(A)$ are pairwise incomparable then the following are equivalent*:

(1) $\bigwedge\{s_{q_i}/i = 1, \dots, n\} = 0$;

(2) *$\{q_1, \dots, q_n\}$ is a maximal set of pairwise incomparable elements in $V(A)$*;

(3) *If $U$ is a root in $V(A)$ then $q_i \in U$ for some $i \in \{1, \dots, n\}$*;

(4) *If $m \in \operatorname{Min} A$ then $s_{q_i} \leqslant m$ for some $i \in \{1, \dots, n\}$.*

**Proof.** (1) $\Rightarrow$ (2). By the previous lemma.

(2) $\Rightarrow$ (3). Assume that $U$ is a root of $V(A)$ and $q_i \notin U$ for any $i = 1, \dots, n$. Consider $p_1 \in U$; so there is $q_{i1}$ comparable with $p_1$, hence $q_{i1} < p_1$, because $p_1 \leqslant q_{i1}$ implies

$q_{i1} \in U$. But $U$ is a root so there is $p_2 \in U$ such that $p_2 < p_1$ and $q_{i1} \| p_2$. In this way, one can obtain a sequence $q_{i1}, q_{i2}, \dots, q_{ik}, \dots$ and $p_1 > p_2 > \cdots > p_k > \cdots$ such that $q_{ik} \| p_{k+1}$ and $q_{ik} < p_k$ for each $k$. Since $\{q_1, \dots, q_n\}$ is finite there is $l < k$ such that $q_{ik} = q_{il}$. Thus, $q_{ik} < p_k < \cdots < p_{l+1}$, hence $q_{il} < p_{l+1}$. We have obtained a contradiction, so $U$ contains one of the $q_i$.

(3) $\Rightarrow$ (4). If $m \in \mathrm{Min}\, A$ then $U = \{q \in V(A) \mid m \leqslant q\}$ is a root of $V(A)$. By hypothesis, there exists $q_i \in U$ so $m \leqslant q_i$, therefore $s_{q_i} = \bigwedge\{k \in \mathrm{Min}\, A \mid k \leqslant q_i\} \leqslant m$.

(4) $\Rightarrow$ (1). For any $m \in \mathrm{Min}\, A$ there exists $q_i$ such that $s_{q_i} \leqslant m$ so $\bigwedge\{s_{q_i} / i = 1, \dots, n\} \leqslant \bigwedge \mathrm{Min}\, A = 0$. □

**Remarks.** (1) An element $c \in A$ is completely join-prime iff $c$ has a unique value (see [10]).

(2) Let $\{c_1, \dots, c_n\}$ be a finite set of completely join-prime elements in $A$, $\mathrm{Val}(c_i) = \{q_i\}$ $i = 1, \dots, n$. If $c_1, \dots, c_n$ are pairwise orthogonal then $q_1, \dots, q_n$ are pairwise incomparable. Indeed, $c_i \wedge c_j = 0$ and $c_j \not\leqslant q_j$ imply $c_i \leqslant q_j$, hence if $q_j < q_i$ then $c_i < q_i$, a contradiction.

**Theorem 3.19.** *Let $A$ be a member of* IRN. *The following statements are equivalent:*

(1) *There exists a maximal finite set of pairwise incomparable elements of $V(A)$;*

(2) *There exists a finite subset $S'$ of $S$ such that $\bigwedge S' = 0$;*

(3) *There exists a finite set of completely join-prime elements of $A$, which is maximal orthogonal;*

(4) *Any set of pairwise disjoint roots of $V(A)$ is finite.*

**Proof.** (1) $\Rightarrow$ (3). Let $\{q_1, \dots, q_n\}$ be a maximal set of pairwise incomparable elements in $V(A)$. By Lemma 3.2, there is a compact element $c_1 \not\leqslant q_1$, and $c_1 \leqslant \bar{q}_1 \wedge s_{q_i}$, $i = 2, \dots, n$, so $q_1 \in \mathrm{Val}(c_1)$. Suppose there is a $p \in \mathrm{Val}(c_1)$, $p \neq q_1$ so $p \| q_1$. By the maximality of $\{q_1, \dots, q_n\}$, $p$ is comparable with $q_i$ for some $i \in \{2, \dots, n\}$, so $c_1 \not\leqslant s_{q_i}$, using Lemma 3.1. This contradiction shows that $\mathrm{Val}(c_1) = \{q_1\}$. In this way one can obtain the completely join-prime elements $c_1, \dots, c_n$ such that $\mathrm{Val}(c_i) = \{q_i\}$, $i = 1, \dots, n$. We shall prove that $\{c_1, \dots, c_n\}$ is a maximal orthogonal set. Note first that $c_1, \dots, c_n$ are pairwise incomparable: by example if $c_1 \leqslant c_2$ then $c_1 \leqslant \bar{q}_2 \wedge s_{q_1} \leqslant s_{q_1} \leqslant q_1$, contradiction. By [10, Lemma 2.3] it follows that $\{c_1, \dots, c_n\}$ is orthogonal. Assume $c \wedge c_i = 0$ for any $i = 1, \dots, n$, where $c \in \mathrm{Com}(A)$. If $c \neq 0$ there is $p \in \mathrm{Val}(c)$, so $c_i \leqslant p$ for any $i = 1, \dots, n$. This yields $p \| q_i$, $i = 1, \dots, n$. Indeed, $p \leqslant q_i$ implies $c_i \leqslant q_i$ and $q_i < p$ implies $c \not\leqslant q_i$, so $c_i \leqslant q_i$, because $c \wedge c_i = 0$ and $q_i$ is meet-prime. Our conclusion that $p \| q_i$, $i = 1, \dots, n$ contradicts the maximality of $\{q_1, \dots, q_n\}$, hence $c = 0$.

(3) $\Rightarrow$ (1). Suppose $\{c_1, \dots, c_n\}$ is maximal orthogonal and $\mathrm{Val}(c_i) = \{q_i\}$, $i = 1, \dots, n$. Using the previous remark $q_1, \dots, q_n$ are pairwise incomparable. Assume there exists $p \in V(A)$ such that the elements of the set $\{p, q_1, \dots, q_n\}$ are pairwise incomparable. By Lemma 3.2 there exists a compact element $c \leqslant \bar{p} \wedge s_{q_i}$, $i = 1, \dots, n$ and $c \not\leqslant p$, so $p \in \mathrm{Val}(c)$. We shall prove that the set $\{c, c_1, \dots, c_n\}$ is orthogonal. If $c \wedge c_i \neq 0$ for

some $i$, there is $q \in \mathrm{Val}(c \wedge c_i)$, so $c \not\leqslant q$ and $c_i \not\leqslant q$. But $\mathrm{Val}(c_i) = \{q_i\}$, therefore $c \leqslant s_{q_i} = \bigwedge\{r \in V(A) \mid r,\ q_i$ are comparable$\} \leqslant q$. This contradiction shows that $\{c, c_1, \ldots, c_n\}$ is orthogonal and this contradicts the hypothesis. It follows that $\{q_1, \ldots, q_n\}$ is a maximal set of pairwise incomparable elements in $V(A)$.

The rest of the proof follows by Theorem 3.8. □

**Corollary 3.20.** *Let $A$ be a member of* IRN. *The following are equivalent*:

(1) *$A$ has a finite basis*;

(2) *There exists a finite set $S' \subseteq S$ of maximal elements in $S$, such that $\bigwedge S' = 0$*;

(3) *$S$ is finite*;

(4) *$V(A)$ has a finite number of roots.*

**Proof.** (1) ⇒ (2). Let $\{c_1, \ldots, c_n\}$ be a finite basis of $A$. One can suppose $c_1, \ldots, c_n$ are completely join-prime (see [12], Corollary 4.4). If $\mathrm{Val}(c_i) = \{q_i\}$, $i = 1, \ldots, n$ then by Corollary 3.16, $s_{q_i}$ are maximal in $S$ for $i = 1, \ldots, n$. In accordance with Theorem 3.18 we have $\bigwedge\{s_{q_i} / i = 1, \ldots, n\} = 0$.

(2) ⇒ (4). If $S' = \{s_{q_1}, \ldots, s_{q_n}\}$ then, by Theorem 3.18, for any root $U$ of $V(A)$ there exists some $q_i \in U$. By Theorem 3.15, every $q_i$ is contained in a unique root, so $V(A)$ has a finite number of roots.

(4) ⇒ (3). Denote by $U_1, \ldots, U_n$ the roots in $V(A)$. For any subset $I$ of $\{1, \ldots, n\}$ denote by $\sum_I$ the set of $p \in V(A)$ such that $\{U_i \mid i \in I\}$ is the set of roots in $V(A)$ which contain $p$. In accordance with Proposition 3.9, $s_p = s_q$ for any $p, q \in \sum_I$, so one can denote by $s_I$ the common value of $s_p$ for $p \in \sum_I$. Thus $S = \{s_I \mid I \subseteq \{1, \ldots, n\}\}$ and $S$ is finite.

(3) ⇒ (1). Let $S'$ be the maximal elements in $S$, say $S' = \{s_{q_1}, \ldots, s_{q_n}\}$. For $p \in V(A)$ we have two cases:

(i) $s_p \in S'$; hence $\bigwedge S' \leqslant s_p \leqslant p$;

(ii) $s_p \in S - S'$; $S$ being finite there is $s_{q_i} \in S'$ such that $s_p < s_{q_i}$, hence $q_i < p$. We obtain $\bigwedge S' \leqslant s_{q_i} \leqslant q_i < p$. So, for every $p \in V(A)$, $\bigwedge S' \leqslant p$, then $\bigwedge S' \leqslant \bigwedge V(A) = 0$, i.e. $\bigwedge S' = 0$. Because $\{q_1, \ldots, q_n\}$ are pairwise incomparable elements of $V(A)$, using Theorem 3.18, $\{q_1, \ldots, q_n\}$ is a maximal set of pairwise incomparable elements of $V(A)$. By Theorem 3.19, there is the set $\{c_1, \ldots, c_n\}$ which is maximal orthogonal and $c_i, i = 1, \ldots, n$ are completely join-prime. Because $\mathrm{Val}(c_i) = \{q_i\}$, using Corollary 3.16, $c_i$ are linear elements, so $A$ has a finite basis. □

**Remark.** The equivalence of (1) and (4) there was proved firstly in [12].

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