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Geodetic Graphs of Diameter Two

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In this paper, geodetic graphs of diameter two are studied. A structural description of them is given in terms of the number of vertices, valences, and complete subgraphs.

1. INTRODUCTION

In this paper, G will denote a connected undirected graph without loops or multiple edges.

G is *geodetic* if every pair of vertices is connected by a unique shortest arc. The problem of characterizing all geodetic graphs was first proposed by Ore [2, p. 105]. A characterization of planar geodetic graphs was found by Watkins [5], who together with this writer changed and shortened the proof and presented it in [4].

A partial characterization of geodetic graphs of diameter two, along with some of the work in [4], first appeared in the doctoral dissertation of this writer [3]. These results, together with a number of improvements, are presented here.

2. DEFINITIONS AND NOTATION

Except where otherwise indicated, the notation and terminology in this paper are the same as that used in [2] and [4].

We shall denote the vertices of a graph G by $V(G)$. If H is a subgraph of G, $G - H$ denotes the subgraph of G obtained by deleting $V(H)$ from $V(G)$ and removing all edges from G that have an endpoint in $V(H)$. A subset S of $V(G)$ is said to *generate* a subgraph if H is the section subgraph

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on S (i.e., $V(H) = S$ and H contains all edges of G connecting two vertices of S).

An arc A whose terminal vertices are x and y will be denoted $A[x, y]$. If $a, b \in V(A)$ then $A[a, b]$ is the subarc whose terminal vertices are a, b. An edge between a and b is denoted $[a, b]$.

If S is a set, then $| S |$ will denote its cardinality. In the case where S is an arc, $| S |$ will mean the length of S. A circuit C is *even* or *odd* if its length is even or odd respectively. We can assign an orientation to each circuit C. If C is an even circuit and $x, y \in V(C)$, we shall say that x, y are *C-opposite* if $| C[x, y]| = | C[y, x]|$. If C is an odd circuit and $x \in V(C)$, then there are vertices $y, z \in V(C)$ such that y, z are adjacent on C and $| C[x, y]| = | C[z, x]|$; we then say that y, z are each *C-opposite* to x.

The *distance* between $x, y \in V(G)$, denoted $d(x, y)$, is the length of a shortest arc connecting these vertices. The *diameter* of G is the maximum of the distances between two vertices. If there is a unique shortest arc joining x, y, we call it a *geodesic* and denote it $\Gamma[x, y]$. Clearly, if $\Gamma[x, y]$ exists then $d(x, y) = | \Gamma[x, y]|$. Any subarc of a geodesic is also a geodesic. If every pair of vertices of G is connected by a unique shortest arc, then G is *geodetic*.

Any vertex v with valence $\rho(v) \geqslant 3$ is called a *node*; v is an *antinode* if $\rho(v) \leqslant 2$. If $A[x, y]$ is an arc in which x, y are the only nodes, then A is called a *suspended arc*. Let C be a circuit and $S[x, y]$ be an arc which intersects C only in x, y. We say that S is a *secant* of C if it is a geodesic (i.e., $S[x, y] = \Gamma[x, y]$). Clearly, in a geodetic graph every even circuit must contain a secant. The subgraph of G consisting of a circuit C and all secants of C is called the *wheel* induced by C.

We shall let ρ_0 denote the maximum valence in G. If H is a subgraph of G, then $\rho_H(v)$ is the valence of v in H. If $x \in V(G)$ we define the vertex distance sets $A_i(x)$ by

$$A_i(x) = \{v \in V(G) | \ d(x, v) = i\}.$$

If $v \in V(G)$, we shall then use the notation $\rho_{A_i(x)}(v)$ to mean the number of edges from v to vertices in $A_i(x)$.

A *clique* is defined as a maximal complete subgraph U_k, $k \geqslant 3$; that is, a complete subgraph on at least three vertices which is contained in no larger complete subgraph. A *clique vertex* is a vertex belonging to some clique; *clique edge* is similarly defined. If a vertex lies on no clique, it is called a *nonclique vertex*.

G is a *Moore graph with diameter k* if G is a regular graph with diameter k and $| V(G)| = 1 + \rho \sum_{i=1}^{k} (\rho - 1)^{i-1}$ where ρ is the constant valence. For a Moore graph of diameter 2, this condition becomes $| V(G)| = 1 + \rho^2$.

A graph P is a *pyramid* if it has a decomposition $P = H \cup B$ where B is a complete graph on n vertices for some n (let $a_1, a_2, a_3, \ldots, a_n$ be these vertices) and $H = \bigcup_{i=1}^{n} A_i[v, a_i]$ where the A_i are arcs (called the *sides* of the pyramid), $A_i \cap A_j = \{v\}$ for $i \neq j$, and $A_i \cap B = \{a_i\}$ for $i = 1, 2, 3, \ldots, n$. B is called the *base* of the pyramid and v is the *apex* of the pyramid. We say that P is a *regular pyramid* if $|A_i| = |A_j|$ for all $A_i, A_j \in H$; then $\alpha = |A_i|$ is called the *altitude* of the pyramid.

3. General Results on Geodetic Graphs

The following two theorems are proven in [4].

THEOREM 3.1. *G is geodetic if and only if every lobe graph of G is geodetic.*

THEOREM 3.2. *In a geodetic graph, a suspended arc is a geodesic.*

The appearance of circuits of even length in a geodetic graph often induces certain types of configurations in the graph. In our study of diameter two geodetic graphs, the 4- and 6-circuits play important roles. We thus obtain the following two theorems, the first being obvious.

THEOREM 3.3. *If a geodetic graph contains a circuit of length four, then it contains a complete graph on the vertices of this circuit.*

THEOREM 3.4. *If a geodetic graph contains a circuit C of length six, then C and its secants form one of the following three configurations:*

(I) *A complete six-graph.*

(II) *Labeling the vertices of C cyclically as $v_1, v_2, \ldots, v_6$, an edge $[v_1, v_3]$ and an arc $A[v_2, v_5]$ of length two whose intermediate vertex is not on C.*

(III) *Labeling as in (II), three arcs of length two, $A_1[v_1, v_4]$, $A_2[v_2, v_5]$, $A_3[v_3, v_6]$, whose intermediate vertices are all different and not on C.*

Proof. Assume the wheel induced by C is not U_6.

If there is an edge connecting a pair of C-opposite vertices, repeated application of Theorem 3.3 will result in U_6. Similarly if two secants of C are edges joining nonadjacent vertices on C, a 4-circuit will be formed and U_6 will result.

Thus, the secants of C can be either one edge such as $[v_1, v_3]$ or arcs of length two connecting C-opposite vertices. The only way to put in secants so as to avoid creating 4-circuits (which will result in U_6) and to still guarantee that unique shortest arcs exist between all vertices of C, is by configurations (II) and (III).

THEOREM 3.5. *If $H = U_k \subset G$ where G is any geodetic graph and if there is a vertex $v \in V(G) - V(H)$ such that v is adjacent to two vertices of H, then $V(H) \cup \{v\}$ generates a complete subgraph $H' = U_{k+1}$.*

Proof. Assume $a, b \in V(H)$ are adjacent to v. Let $c \in V(H)$, $c \neq a, b$. Then a, c, b, v determine a 4-circuit, thus generating a complete 4-graph. So v is adjacent to each vertex $c \in V(H)$. Thus $V(H) \cup \{v\}$ generates a complete graph $H' = U_{k+1}$.

COROLLARY 3.5.1. *If $H_1 = U_k$, $H_2 = U_j$ are subgraphs of a geodetic graph G, and if H_1 and H_2 have an edge in common, then $V(H_1) \cup V(H_2)$ generates a complete subgraph $H \subset G$.*

Proof. Let $[a, b]$ be the common edge and v be any vertex of H_1. Then v is adjacent to $a, b \in V(H_1) \cap V(H_2)$. So, by Theorem 3.5, v is adjacent to every vertex of H_2. Thus, every pair of vertices, one from H_1 and the other from H_2, are adjacent. Hence $V(H_1) \cup V(H_2)$ generates a complete graph H.

COROLLARY 3.5.2. *Let G be a geodetic lobe graph and let $x \in V(G)$. If there are vertices $a_1, a_2 \in A_1(x)$ and $b_1, b_2 \in A_2(x)$ where $[a_1, b_1]$ and $[a_2, b_2]$ are edges, then at least one of the edges $[a_1, a_2]$ or $[b_1, b_2]$ cannot exist.*

Proof. If both edges exist, then by Theorem 3.3 $\{a_1, a_2, b_2, b_1\}$ generates a complete 4-graph. Since x is adjacent to two of the vertices of the U_4, Theorem 3.5 implies that x is adjacent to b_1, contrary to assumption.

4. PRINCIPAL RESULTS

The following two theorems can be proven in a straightforward manner.

THEOREM 4.1. *If G is a geodetic graph of diameter two that is not a lobe graph, then it consists of a set of complete graphs, all attached at a single vertex.*

THEOREM 4.2. *G is a geodetic lobe graph of diameter two if and only if the minimal circuit containing any pair of vertices is of length three or five.*

While the preceding theorem gives a necessary and sufficient condition for a lobe graph to be geodetic of diameter two, it does not adequately describe the types of graphs involved. The characterization we are aiming at gives specific results on the structure of such graphs, with relationships involving the number of vertices, clique sizes, and valences.

From here on, we shall only be discussing geodetic lobe graphs of diameter two. The letter G will denote such a graph. We will let n represent the number of vertices in G.

The major results of this paper, which will be proven in section 5, are summarized here.

I. If G contains a vertex of degree two, then G is a regular pyramid with altitude two and base U_m, $m \geqslant 2$.

II. If G contains no cliques then G is a Moore graph of diameter two.

For the rest of this summary, assume that G contains a clique.

III. All cliques in G are of the same order; call it k.

IV. If an edge has each endpoint on a clique, then the edge itself lies in a clique.

V. If x is a nonclique vertex and $v \in A_1(x)$ then $\rho(v)\, \rho(x) = n - 1$.

VI. $\rho_0 = \rho(x)$ if and only if either x is a clique vertex or x is at distance two from all clique vertices.

VII. If x is a nonclique vertex at distance one from some clique then $\rho(x) = \rho_0 - (k - 2)$.

VIII. $\rho_0 \geqslant 2(k - 1)$.

IX. If all vertices are clique vertices then $\rho_0 \geqslant k(k - 1)$.

X. $n = \rho_0{}^2 - \rho_0(k - 2) + 1$.

XI. If G contains a clique $H = U_k$ with the property that, for each vertex $a_i \in V(H)$ there exists a clique H_i where $V(H) \cap V(H_i) = \{a_i\}$ and $A_1(a_i) \subset V(H) \cup V(H_i)$ then $G_1 = G - H$ is a geodetic lobe graph of diameter two. If $k = 3$, then G_1 is the Petersen graph. If $k \geqslant 4$ then G_1 has clique size $k - 1$ and no two distinct cliques have a common vertex (by IV there cannot be an edge connecting them either).

The graphs in Figs. 1 and 2 are graphs of the type described in XI. The graph in Fig. 1 is for the case $k = 3$ (the outer triangle is H). This graph is the smallest (in terms of number of vertices) nonregular geodetic lobe graph of diameter two in which all vertices are nodes. The graph in Fig. 2 has $k = 4$ and the complete 4-graph with darkened edges is H.

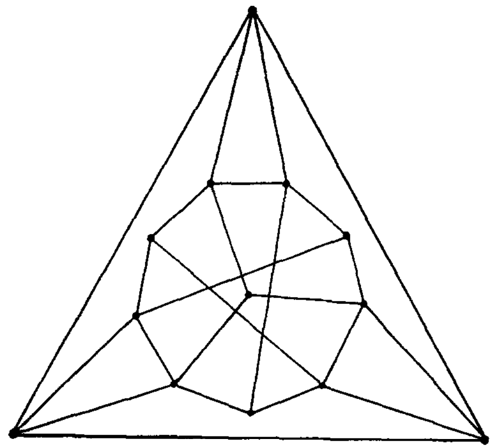

FIGURE 1

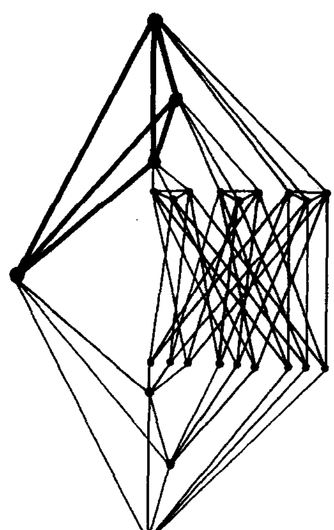

FIGURE 2

5. PROOFS OF RESULTS

In this section we shall continue to let G denote a geodetic lobe graph of diameter two and to let n represent the number of vertices in G.

LEMMA 5.1. *For every vertex* $v \in V(G)$ *there exists at least one vertex* $w \in V(G)$ *such that* $d(v, w) = 2$.

Proof. Assume that v is of distance one from all other vertices. Let $x, y \in V(G)$ be distinct from v and let $B[x, y]$ be the minimal arc joining x and y which doesn't pass through v. If $|B[x, y]| > 1$ let $x = x_0, x_1, x_2, \ldots, x_n = y$ be the vertices, in order, on B. v is adjacent to each x_i and so x, v, x_2, x_1 are on a 4-circuit, thus giving the edge $[x, x_2]$, contrary to the minimality of B. Hence $|B[x, y]| = 1$. Therefore all vertices of G are joined by an edge and so G is a complete graph, contrary to the fact that G is of diameter two.

LEMMA 5.2. *For any $x \in V(G)$, $A_1(x)$ is contained in no clique.*

Proof. Assume $A_1(x)$ is contained in a clique. Let $b_1 \in A_2(x)$ where b_1 is adjacent to $a_1 \in A_1(x)$. Since, by Theorem 3.3, b_1 is not adjacent to any other vertex in $A_1(x)$, there is a vertex $b_2 \in A_2(x)$ such that b_1 and b_2 are adjacent. Let $a_2 \in A_1(x)$ be adjacent to b_2. By Corollary 3.5.2, a_1 and a_2 cannot be adjacent. However, since $A_1(x)$ is contained in a clique, this only leaves the possibility $a_1 = a_2$. But then all paths from b_1 to x pass through a_1, contrary to the assumption that G is a lobe graph.

LEMMA 5.3. *If $a_1, a_2 \in A_1(x)$ for some $x \in V(G)$, then if a_1, a_2 are not adjacent and if $b_1 \in A_2(x)$ is adjacent to a_1, there exists exactly one vertex $b_2 \in A_2(x)$ which is adjacent to a_2 and b_1.*

Proof. In Theorem 3.3, b_1 has at most one neighbor in $A_1(x)$. Hence $d(a_2, b_1) = 2$, and the unique common neighbor of a_2 and b_1 must necessarily lie in $A_2(x)$.

THEOREM 5.1. *If $a_1, a_2 \in A_1(x)$ for some $x \in V(G)$, then*

$$\rho_{A_2(x)}(a_1) = \rho_{A_2(x)}(a_2).$$

Proof. Assume a_1, a_2 are not adjacent.

Let $S(a_i) = \{y \in A_2(x) | \, y \text{ is adjacent to } a_i\}$, $i = 1, 2$.

If $x_1 \in S(a_1)$ then there is a vertex $y_1 \in S(a_2)$ such that x_1, y_1 are adjacent (Lemma 5.3). Let $x_2 \in S(a_1)$ with $x_2 \neq x_1$. If x_2 is adjacent to y_1 then y_1, x_1, a_1, x_2 are on a 4-circuit, giving $[y_1, a_1]$. Then x, a_1, y_1, a_2 are on a 4-circuit, giving $[x, y_1]$ contrary to the fact that $y_1 \in A_2(x)$.

Continuing in this manner, for each $x_i \in S(a_1)$ there is a distinct $y_i \in S(a_2)$ such that $[x_i, y_i]$ exists. So we have $|S(a_1)| \leqslant |S(a_2)|$. Similarly, we can reverse the roles of a_1 and a_2 to get $|S(a_2)| \leqslant |S(a_1)|$.

Hence $\rho_{A_2(x)}(a_1) = |S(a_1)| = |S(a_2)| = \rho_{A_2(x)}(a_2)$.

If a_1, a_2 are adjacent, let $a_3 \in A_1(x)$ be a vertex not adjacent to either of them. The existence of such a vertex is guaranteed by Theorem 3.5 and Lemma 5.2. Then by the above,

$$\rho_{A_2(x)}(a_1) = \rho_{A_2(x)}(a_3) = \rho_{A_2(x)}(a_2).$$

COROLLARY 5.1.1. *If $x \in V(G)$ is not a clique vertex and if a_1, $a_2 \in A_1(x)$, then $\rho(a_1) = \rho(a_2)$.*

Proof. $\rho(a_1) = \rho_{A_2(x)}(a_1) + 1 = \rho_{A_2(x)}(a_2) + 1 = \rho(a_2)$.

THEOREM 5.2. *If x is a nonclique vertex in G and if $v \in A_1(x)$, then $\rho(v) \cdot \rho(x) = n - 1$.*

Proof. Since x is not on a clique, no two vertices of $A_1(x)$ are adjacent. Hence, if $w \in A_1(x)$ we have $\rho_{A_2(x)}(w) = \rho_{A_2(x)}(v) = \rho(v) - 1$. Also, if $b \in A_2(x)$ is adjacent to $a_1 \in A_1(x)$, then it is adjacent to no other vertex in $A_1(x)$.

We know that $n = | A_0(x)| + | A_1(x)| + | A_2(x)|$.

It follows that $n = 1 + \rho(x) + \rho(x) [\rho(v) - 1]$.

Thus $\rho(v) \cdot \rho(x) = n - 1$

THEOREM 5.3. *If G has no cliques then G is regular and $n = 1 + \rho_0{}^2$.*

Proof. Let $x \in V(G)$.

By Corollary 5.1.1, we know that all vertices in $A_1(x)$ have the same valence, call it ρ_1.

If $a \in A_2(x)$, let $b \in A_1(x)$ be adjacent to a. Then $\rho(x) = (n - 1)/\rho(b) = \rho(a)$. Hence all vertices in $A_2(x)$ have the same valence as each other and as x; call this valence ρ_2. Let $c \in A_1(x)$ be a vertex not adjacent to $a \in A_2(x)$. Then $d(a, c) = 2$ and hence $c \in A_2(a)$. So by the preceding argument, $\rho(a) = \rho(c)$. Hence, $\rho_1 = \rho_2$ and so G is regular.

That $n = 1 + \rho_0{}^2$ follows from Theorem 5.2.

It is clear that all geodetic lobe graphs of diameter two without cliques are Moore graphs with diameter two. Conversely, given a Moore graph with diameter two, the graph is clearly a lobe graph and contains no cliques. It is easy to verify that every pair of vertices in this graph lies on a 5-circuit. So, by Theorem 4.2, the graph is geodetic. Hence, the set of geodetic lobe graphs of diameter two without cliques is the same as the set of Moore graphs with diameter two.

We then have the following theorem, which stated for Moore graphs, is proven in [1].

THEOREM 5.4 (Hoffman and Singleton). *If G has no cliques then the only possibilities for* ρ_0 *are 2, 3, 7, and 57. If* $\rho_0 = 2$ *we have a 5-circuit; for* $\rho_0 = 3$ *we get the Petersen graph. If* $\rho_0 = 7$ *the graph obtained is unique (its incidence matrix is given in* [1]).

The existence of a graph as described in the preceding theorem with $\rho_0 = 57$ is still undecided.

LEMMA 5.4. *If* $H = U_k$ *is a clique of G, then all vertices of H have the same valence.*

Proof. Let $x, y \in V(H)$ and take z as any other vertex of H. Then $x, y \in A_1(z)$.
By Theorem 5.1, $\rho_{A_2(z)}(x) = \rho_{A_2(z)}(y)$.
But

$$\rho(x) = \rho_{A_2(z)}(x) + (k - 1),$$

$$\rho(y) = \rho_{A_2(z)}(y) + (k - 1).$$

Hence, $\rho(x) = \rho(y)$.

LEMMA 5.5. *Let* $H = U_k$ *be a clique of G. If* $a \in V(H)$, $b \notin V(H)$, *and the edge* $[a, b]$ *exists and is not on any clique, then* $\rho(b) = \rho(a) - (k - 2)$.

Proof. Let $c \in V(H)$, $c \neq a$. Then $c, b \in A_1(a)$.
By Theorem 5.1, $\rho_{A_2(a)}(b) = \rho_{A_2(a)}(c)$.
But

$$\rho(b) = \rho_{A_2(a)}(b) + 1,$$

$$\rho(c) = \rho_{A_2(a)}(c) + (k - 1).$$

Hence, $\rho(b) = \rho(c) - (k - 2)$. Since $\rho(a) = \rho(c)$, the result follows.

LEMMA 5.6. *If* $H_1 = U_k$, $H_2 = U_j$ *are cliques of G with a vertex in common, then* $k = j$ *and all vertices on* H_1 *and* H_2 *have the same valence.*

Proof. Let x be the common vertex and let $b \in V(H_1)$, $c \in V(H_2)$ with $b \neq x$, $c \neq x$.
Then, by Lemma 5.4, $\rho(b) = \rho(x) = \rho(c)$.
But

$$\rho(b) = \rho_{A_2(x)}(b) + (k - 1),$$

$$\rho(c) = \rho_{A_2(x)}(c) + (j - 1).$$

Since $\rho_{A_2(x)}(b) = \rho_{A_2(x)}(c)$, it follows that $k = j$.

DEFINITION. Given two cliques H_1 and H_2, the distance between them is defined as:

$$d(H_1, H_2) = \min\{d(a, b) \mid a \in V(H_1), b \in V(H_2)\}.$$

LEMMA 5.7. *Let $H_1 = U_k$, $H_2 = U_j$ be cliques of G. If $d(H_1, H_2) = 1$ then $k = j$ and, if $x \in V(H_1)$, $y \in V(H_2)$ are vertices such that $d(x, y) = 1$, then there exists a clique $H_3 = U_k$ containing $[x, y]$. The valences of the vertices of H_1, H_2, and H_3 are all the same.*

Proof. If $[x, y]$ does not lie on a clique then by Lemma 5.5:

$$\rho(y) = \rho(x) - (k - 2),$$
$$\rho(x) = \rho(y) - (j - 2).$$

Hence $k + j = 4$ which is impossible since $k, j \geqslant 3$.

So $[x, y]$ is an edge of some clique having a vertex in common with H_1 and H_2. The result now follows from Lemma 5.6.

LEMMA 5.8. *Let $H_1 = U_k$, $H_2 = U_j$ be cliques in G and let ρ_i be the valence of the vertices of H_i, $i = 1, 2$. If $d(H_1, H_2) = 2$ then $\rho_1 = \rho_2$ and $k = j$.*

Proof. Let $x \in V(H_1)$, $y \in V(H_2)$. Then there is a vertex b adjacent to x and y. If there is a clique containing b then the desired results follow from Lemma 5.7.

If b is not on a clique, then by Lemma 5.5:

$$\rho(b) = \rho(x) - (k - 2) = \rho_1 - (k - 2),$$
$$\rho(b) = \rho(y) - (j - 2) = \rho_2 - (j - 2).$$

So $\rho_1 - k = \rho_2 - j$.

Theorem 5.2 tells us that:

$$\rho_1 = \rho(x) = (n - 1)/\rho(b) = \rho(y) = \rho_2 .$$

Hence, $\rho_1 = \rho_2$ and $k = j$.

In a geodetic lobe graph of diameter two, the distance between any two cliques is zero, one, or two. The results of the preceding three lemmas can then be summarized as the following theorem.

THEOREM 5.5. *If $H_1 = U_k$, $H_2 = U_j$ are cliques of G then $k = j$ and all clique vertices of G have the same valence.*

LEMMA 5.9. *Assume that G contains a clique H. If $x \in V(H)$ then $\rho(x) = \rho_0$. If there is a vertex $y \in V(G)$ such that y is distance two from all cliques, then $\rho(y) = \rho_0$.*

Proof. By Theorem 5.5, all clique vertices have the same valence, call it ρ'. If b is a nonclique vertex at distance one from some clique, then, by Lemma 5.5, $\rho(b) < \rho'$.

If there is a vertex $y \in V(G)$ which is distance two from all cliques and if x is a clique vertex, then there is a nonclique vertex b adjacent to both x and y. By Theorem 5.2:

$$\rho(x) = (n - 1)/\rho(b) = \rho(y).$$

Hence, $\rho(y) = \rho' = \rho_0$.

DEFINITION. If G contains a clique U_k we call k the *clique size* of G. If G contains no clique, we let $k = 2$ be the *clique size*.

THEOREM 5.6. *If k is the clique size of G then for any $x \in V(G)$ either $\rho(x) = \rho_0$ or $\rho(x) = \rho_0 - (k - 2)$. If G contains a clique then $\rho(x) = \rho_0 - (k - 2)$ if and only if x is a nonclique vertex at distance one from some clique.*

Proof. This follows from Lemmas 5.5 and 5.9, and from Theorem 5.3.

THEOREM 5.7. *If G contains a vertex of valence two, then G is a regular pyramid with altitude two and base U_m, $m \geqslant 2$.*

Proof. If G is an odd circuit then it is a 5-circuit, which is a regular pyramid with altitude two and base U_2.

If G is not an odd circuit, then the suspended arc S containing the vertex of valence two must be of length two because suspended arcs are geodesics. Let the vertices of S be a, b, c where $\rho(b) = 2$ and $\rho_0 = \rho(a) = \rho(c) > 2$.

By Theorem 5.2, $2\rho_0 = \rho(b)\,\rho(a) = n - 1$ or $n = 2\rho_0 + 1$.

Since G is not regular it has a clique U_k. By Theorem 5.6, $2 = \rho(b) = \rho_0 - (k - 2)$ giving $k = \rho_0$ and $n = 2k + 1$.

Let $H = U_k$ be a clique in G and let the vertices of H be $v_1, v_2, \ldots, v_k$. Since $\rho_H(v_i) = k - 1$ and $\rho(v_i) = k$, every vertex in H is adjacent to a single edge not in H and these edges cannot be clique edges. Let the other end points of these edges be $a_1, a_2, \ldots, a_k$.

The a_i are nonclique vertices at distance one from a clique, so $\rho(a_i) = 2$, $i = 1, 2, \ldots, k$.

Let x be the vertex other than v_1 which is adjacent to a_1. Then $\rho(x) = \rho_0 = k$.

Since $n = 2k + 1$ and we already have this number of vertices, the graph can only be completed by x being adjacent to $a_2, a_3, \ldots, a_k$. Thus we have a regular pyramid with altitude two, apex x, and base $H = U_k$.

THEOREM 5.8. *If k is the clique size of G then $n = \rho_0^2 - \rho_0(k - 2) + 1$.*

Proof. If $k = 2$, the result follows from Theorem 5.4.

If G contains a clique and also has a nonclique vertex, then Theorem 5.2 and Lemma 5.5 give $n - 1 = \rho_0(\rho_0 - (k - 2))$ and the result follows. If all vertices of G are clique vertices then for any $x \in V(G)$ we have $n = |A_0(x)| + |A_1(x)| + |A_2(x)|$. Hence, $n = 1 + \rho_0 + \rho_0[\rho_0 - (k - 1)]$ and the desired result follows.

THEOREM 5.9. *Assume that G has no vertex of valence two and that G contains a clique U_k. Then $\rho_0 \geqslant 2(k - 1)$.*

Proof. If all vertices are clique vertices, then Lemma 5.7 implies that every edge lies on a clique. Lemma 5.2 then says that each vertex lies on at least two cliques, so $\rho_0 \geqslant 2(k - 1)$.

If G contains both clique and nonclique vertices, then assume $\rho_0 = k + r$ where $k + r < 2(k - 1)$, hence $r + 2 < k$. Let x be a clique vertex. Then x lies on one clique $H = U_k$ and $\rho(x) = k + r$. Let $a_1, a_2, \ldots, a_{k+r}$ be the vertices of $A_1(x)$, labeled so that $a_1, a_2, \ldots, a_{k-1}$ are vertices of H. It follows from Lemma 5.7 that $a_k, \ldots, a_{k+r}$ are not clique vertices. If $a \in A_1(x)$ then $\rho_{A_2(x)}(a) = r + 1$ by Theorem 5.1. Note that $r \geqslant 1$, for if $r = 0$ then $\rho(a_k) = \rho_0 - (k - 2) = k - (k - 2) = 2$.

For each $a_i \in A_1(x)$, let $\{a_{i,1}, a_{i,2}, \ldots, a_{i,r+1}\}$ be the set of vertices in $A_2(x)$ adjacent to a_i. We may assume by Lemma 5.3, that $a_{k,1}$ is adjacent to each of $a_{1,1}, a_{2,1}, \ldots, a_{k-1,1}$. By the same lemma, each of these vertices must also be adjacent to a vertex in the set $\{a_{k+1,j}\}_{j=1}^{r+1}$. This set has $r + 1$ members and $\{a_{i,1}\}_{i=1}^{k-1}$ has $k - 1$ members. Since $r + 1 < k - 1$, there is a member of the former set, say $a_{k+1,1}$, which is adjacent to two members of the latter set; we may assume these to be $a_{1,1}$ and $a_{2,1}$. But then $a_{1,1}$, $a_{k,1}$, $a_{2,1}$, $a_{k+1,1}$ lie on a 4-circuit, thus generating a U_4. Since all cliques of G are the same size, this U_4 lies on a clique $H_1 = U_k$. But $a_{1,1} \in V(H_1)$ and so $d(H, H_1) \leqslant 1$. If $d(H, H_1) = 0$ then $\rho_0 = \rho(a_{1,1}) \geqslant 2(k - 1)$. If $d(H, H_1) = 1$, then Lemma 5.7 implies that there is another clique H_2 adjacent to both of these. So $d(H, H_2) = 0$ and again $\rho_0 \geqslant 2(k - 1)$.

THEOREM 5.10. *Assume all vertices of G are clique vertices and let k be the clique size. Then $\rho_0 \geqslant k(k - 1)$.*

Proof. It follows from Lemma 5.7 that every edge of G lies on a clique.

Theorem 5.6 says that $\rho_0 = \rho(v)$ for all $v \in V(G)$. Let $x \in V(G)$ and let m be the number of cliques containing x. Then $\rho(x) = m(k-1)$ since every edge containing x lies on a clique and all cliques are edge disjoint (by Corollary 3.5.1).

Let $a_1 \in A_1(x)$ and $a_{11} \in A_2(x)$ be adjacent to a_1. Let H_1 be the clique containing $[x, a_1]$ and let $H_2, H_3, \ldots, H_m$ be the other cliques containing x.

By corollary 3.5.2, a_{11} is adjacent to no vertex that is adjacent to any vertex in H_1 other than a_1. By Lemma 5.3, there is a vertex a_{21} adjacent to both a_{11} and H_2. Then $[a_{11}, a_{21}]$ is contained in a clique H. If $v \in V(H)$ then $v \neq x$ since $d(x, a_{11}) = 2$. If $v \in A_1(x)$ then we may assume $v \neq a_1$. Then a_{11}, v, x, a_1 lie on a 4-circuit, thus generating a complete 4-graph with x adjacent to a_{11} which we noted is impossible. So $V(H) \subset A_2(x)$.

Assume $v_1, v_2 \in V(H)$ where v_1 and v_2 are adjacent to the same H_i for some $i \leqslant m$. Since we then have the edge $[v_1, v_2]$, Corollary 3.5.2 implies that v_1, v_2 are adjacent to the same vertex $v \in V(H_i)$. But $v \notin V(H)$ and so, by Theorem 3.5, $V(H) \cup \{v\}$ generates a larger complete graph than H, contradicting the fact that H is a clique.

Hence, H contains at most one vertex adjacent to H_i for each i, $1 \leqslant i \leqslant m$. Since $V(H) \subset A_2(x)$ and H contains k vertices we must have $m \geqslant k$.

Therefore, $\rho_0 = \rho(x) = m(k-1) \geqslant k(k-1)$.

THEOREM 5.11. *Let G have clique size $k \geqslant 3$ and assume G contains a clique H with the property that, for each vertex $a_i \in V(H)$, $i = 1, 2, \ldots, k$, there exists a clique H_i where*

$$V(H) \cap V(H_i) = \{a_i\} \qquad \textit{and} \qquad A_1(a_i) \subseteq V(H) \cup V(H_i).$$

Then $G_1 = G - H$ is geodetic of diameter two with clique size $k-1$. If G_1 contains cliques (i.e., $k \geqslant 4$) then each clique in G_1 is at distance two from every other clique.

Proof. Let $x, y \in V(G_1)$. If $\Gamma[x, y]$ in G contains two vertices a_i, $a_j \in V(H)$, then $\Gamma[x, y] = \Gamma[x, a_i] + [a_i, a_j] + \Gamma[a_j, y] \geqslant 3$. If $\Gamma[x, y]$ in G contains one vertex $a_i \in V(H)$, then since $d(x, y) \leqslant 2$, x and y would be adjacent to a_i. But then, $x, y \in V(H_i)$ which would make $\Gamma[x, y] = [x, y]$.

Thus, if $x, y \in V(G)$, then $\Gamma[x, y]$ in G does not contain a vertex of H and so $\Gamma[x, y] \subset G_1$. Hence, G_1 is geodetic of diameter two.

$H_i = U_k$ is a clique of G which means that $H_i - \{a_i\} = U_{k-1}$ is a clique of G_1 (or an edge if $k = 3$). Hence, $k - 1$ is the clique size of G_1.

To prove the last statement of the theorem, assume $k \geqslant 4$ and let $H_i' = H_i - \{a_i\}$, $i = 1, 2, \ldots, k$. The H_i' are the only cliques in G_1. If not, there would be a clique $H'' = U_{k-1}$ in G_1 which is also a clique in G. But this is impossible since the clique size in G was k. (Similarly, the H_i and H are the only cliques of G.)

Let $x \in V(H_i')$, $y \in V(H_j')$ where $i \neq j$. If $d(x, y) = 0$ then, in G, a_i would have been adjacent to two vertices of H_j, thus giving a complete $(k + 1)$-graph in G (by Theorem 3.5) which is impossible. If $d(x, y) = 1$ then x, y, a_j, a_i would be on a 4-circuit in G, generating a U_4, and hence giving a contradiction by Corollary 3.5.1. So $d(x, y) = 2$. Hence, $d(H_i', H_j') = 2$ for $i \neq j$.

6. Questions Remaining

While the preceding investigation of geodetic graphs of diameter two is quite detailed, it is still not complete. Some questions still to be answered are presented here, along with brief comments about them.

Question 1. *Are there geodetic lobe graphs of diameter two in which all vertices are clique vertices?*

If such graphs exist, theorem 5.10 gives a lower bound for ρ_0. It can be shown in a somewhat detailed but straightforward manner that for $k = 3$ and $\rho_0 = 6$ (the minimum possible value for ρ_0) no such graph exists.

We define the *clique incidence graph* of G, denoted G^I, by: $V(G^I)$ is the set of cliques of G; if $H_1, H_2 \in V(G^I)$ then $[H_1, H_2]$ exists if and only if H_1 and H_2 have a common vertex in G.

Question 2. *If G is a geodetic lobe graph of diameter two, is G^I geodetic? What can be said about the diameters of the connected components of G^I?*

The clique incidence graph was used by this author in trying to answer Question 1. It can easily be shown that if every vertex of G is a clique vertex, then G^I is connected and has diameter at most three; it is geodetic if and only if the diameter is two. In this case, if G^I is geodetic, then in G $\rho_0 = k(k - 1)$. It would then follow from the comments after Question 1 that the answer to that question would be no for $k = 3$.

Question 3. *For each set of values $\{n, k, \rho_0\}$ satisfying $n = \rho_0^2 + \rho_0(k - 2) + 1$ with $\rho_0 > 2(k - 1)$, does there exist a corresponding geodetic lobe graph of diameter two? If yes, is it unique?*

It can be shown that the graph G described in Theorem 5.11 exists and is unique for each $k \geqslant 3$. (In such graphs $\rho_0 = 2(k - 1)$.) Hence, for each k and the second smallest value for ρ_0, $\rho_0 = 2(k - 1) + 1$, a geodetic lobe graph of diameter two exists having $k + 1$ disjoint cliques and all non-clique vertices at distance one from each clique (this is the graph G_1 obtained in Theorem 5.11 if the clique size of G is $k + 1$). This graph is not uniquely determined by $\{n, k, \rho_0\}$ since for $n = 21$, $k = 3$, $\rho_0 = 5$ two geodetic lobe graphs can be constructed, one containing four cliques and the other containing ten.

References

1. A. J. Hoffman and R. R. Singleton, On moore graphs with diameters 2 and 3, *IBM J. Res. Develop.* 4 (1960), 497–504.
2. O. Ore, "Theory of Graphs," Amer. Math. Soc., Providence, RI, 1962.
3. J. G. Stemple, A further characterization of geodetic graphs, Dissertation, Yale University, 1966.
4. J. G. Stemple and M. E. Watkins, On planar geodetic graphs, *J. Combinatorial Theory* 4 (1968), 101–117.
5. M. E. Watkins, A characterization of the planar geodetic graph and some geodetic properties of nonplanar graphs, Dissertation, Yale University, 1964.