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Fusion rings for degenerate minimal models

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Abstract

We study fusion rings for degenerate minimal models ($p = q$ case) for $N = 0$ and $N = 1$ (super)conformal algebras. We consider a distinguished family of modules at the level $c = 1$ and $c = 3/2$ and show that the corresponding fusion rings are isomorphic to the representation rings for $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{osp}(1|2)$, respectively.

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1. Introduction

The Virasoro algebra and its minimal models are a good source of interesting vertex operator algebras. In [W] the rationality of the Virasoro vertex operator algebras $L(c_{p,q}, 0)$ was proved, where $c_{p,q} = 1 - 6(p - q)^2/(pq)$ and $(p, q) = 1, p, q \geq 2$. This result is used for the construction of the corresponding vertex tensor categories (cf. [H1]). A similar result is obtained for $N = 1$ case in [A, HM].

In this paper we study a non-rational vertex operator algebra $L(1, 0)$ ($p = q$ case) and the corresponding fusion ring for *degenerate* minimal models, i.e., the case $p = q$, with central charge $c = 1$. We also consider a $N = 1$ vertex operator superalgebra version based on $L(3/2, 0)$ (see below). These cases are substantially different for many reasons (let us focus on the case $L(1, 0)$ since

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the same problem persists for $L(3/2, 0)$. The vertex operator algebra $L(1, 0)$ is not *rational* (cf. [FZ]) but it has a distinguished family of irreducible modules (those that are not irreducible Verma modules) \mathcal{F}_1 , which consists of classes of irreducible modules isomorphic to $L(1, m^2/4)$ for some $m \in \mathbb{N}$. These modules have a quite simple embedding structure [KR,FF2].

We show that the fusion ring for the family \mathcal{F}_1 is isomorphic to the representation ring $\text{Rep}(\mathfrak{sl}(2, \mathbb{C}))$, i.e., we “formally” have

$$\begin{aligned} L\left(1, \frac{n^2}{4}\right) \times L\left(1, \frac{m^2}{4}\right) \\ = L\left(1, \frac{(n+m)^2}{4}\right) + L\left(1, \frac{(n+m-2)^2}{4}\right) + \cdots + L\left(1, \frac{(n-m)^2}{4}\right), \end{aligned}$$

where $m, n \in \mathbb{N}$ and $n \geq m$.

This result seems to be known—in some form—for a while by physicists (also in [FKRW] is stated as a part of more general conjecture concerning fusion rings for $W(\mathfrak{gl}_N)$ algebras—see also [FM]). The author of the current paper could not trace any proof in the language of vertex operator algebras. Some computations are done in [DG] but not complete. But instead of trying to patch missing proofs, there are two more important reasons for seeking such a proof.

- So far, not many computations of the fusion coefficients has been known for non-rational vertex operator algebras (here non-rational means *non-rational* in any reasonable category). In particular, we offer a proof that uses universal construction (induced modules), therefore it is very general.
- As noticed by H. Li in [L1,L2], Frenkel–Zhu’s formula [FZ] does not hold for non-rational vertex operator algebras. The right formula was provided in [L2] but it is a non-trivial matter to use it for computational purposes in non-rational setting.

We believe that our method can be used for more complicated models—like degenerate models associated to \mathcal{W} -algebras.

We have to stress that the fusion coefficients are simply derived from the space of intertwining operators among irreducible modules. In other words, it is *not* true that the only modules which “fuse” with $L(1, n^2/4)$, and $L(1, m^2/4)$ are completely reducible. This fact makes impossible to implement $P(z)$ -tensor product construction from [HL1,HL2]. The resolution might be to construct (a new) tensor product which takes only irreducible modules into account, but this approach will assume a good knowledge of matrix coefficients for product of intertwining operators. A different approach would be working in the larger family $\overline{\mathcal{F}}_1$, which consists of all quotients of Verma modules $M(1, m^2/4)$. The possible constructions will be discussed elsewhere.

We also provide a different proof of the fusion formulas by constructing all intertwining operators from the lattice vertex operator algebra V_L and its irreducible module $V_{L+1/2}$ (cf. [DG]).

A super $N = 1$ versions of the above result stems from the $N = 1$ Neveu–Schwarz Lie superalgebra at the level $3/2$. Again, there are essentially two approaches: one which uses the lattice construction (extended with a suitable fermionic Fock space) and the other which uses the singular vectors and projection formulas. For the future purposes we use the latter approach. We consider a set of equivalence classes of irreducible modules for the $N = 1$ superconformal algebra (see Section 3) with representatives $L(3/2, q^2/2)$ where $q \in \mathbb{N}$. We proved (see Theorem 10.1 and Corollary 10.1) that the corresponding fusion ring is isomorphic to the representation ring for $\mathfrak{osp}(1|2)$, i.e., we formally have:

$$\begin{aligned} L\left(\frac{3}{2}, \frac{r^2}{2}\right) \times L\left(\frac{3}{2}, \frac{q^2}{2}\right) \\ = L\left(\frac{3}{2}, \frac{(r+q)^2}{2}\right) + L\left(\frac{3}{2}, \frac{(r+q-1)^2}{2}\right) + \cdots + L\left(\frac{3}{2}, \frac{(r-q)^2}{2}\right), \end{aligned}$$

for every $r, q \in \mathbb{N}$, $r \geq q$, where \times stands for the fusion product (see the last section).

In particular, as in the Virasoro algebra case, these fusion coefficients are 0 or 1. However in [HM] we showed that for $N = 1$ case has some interesting features; for some vertex operator algebras $L(c, 0)$, fusion coefficients might be 2. In Proposition 11.1 we construct a non-trivial example with $c = 15/2 - 3\sqrt{5}$.

At the very end, we construct an example of a *logarithmic intertwining operator* (for the definition see [M]) for the $N = 1$ vertex operator superalgebra $L(27/2, 0)$.

These results can be extended for a more general class of vertex operator algebras $L(c, 0)$ where $c \neq c_{p,q}$; because of simplicity we treat only the case $c = 1$ and $c = 3/2$.

2. Representations of the Virasoro algebra at the level $c = 1$

The representation theory for the Virasoro algebra has been studied intensively in the last two decades [KR, FF1, FF2, FF3]. Kac's determinant formula is the most important tool in the highest (or lowest) weight theory. From the determinant formula it follows that the lowest weight Verma module with the central charge $c(t) = 13 - 6t - 6t^{-1}$ and the weight

$$h_{p,q}(t) = \frac{1-p^2}{4}t^{-1} - \frac{1-pq}{2} + \frac{1-q^2}{4}t,$$

has a singular vector of the weight $h_{p,q}(t) + pq$, $t \in \mathbb{C}$. We are interested in the case $t = 1$, i.e., $c = 1$. It is easy to see that $M(1, h)$ is irreducible if and only if $h \neq m^2/4$ for some $m \in \mathbb{N}$. In the case $h = m^2/4$ we have the following description.

Proposition 2.1. *The Verma module $M(1, m^2/4)$ has a unique singular vector of weight $m^2/4 + (m + 1)$. This vector generates the maximal submodule. In other words, we have the following exact sequence:*

$$0 \rightarrow M\left(1, \frac{(m+2)^2}{4}\right) \rightarrow M\left(1, \frac{m^2}{4}\right) \rightarrow L\left(1, \frac{m^2}{4}\right) \rightarrow 0. \tag{1}$$

Even though they do not exist in general, in the case $h_{1,q}(t)$, if $p = 1$ there are explicit formulas at each level $c(t)$ (in particular $t = 1$). When $c = 1$ Benoit and S. Aubin’s formula [BSA1] implies that

$$P_{\text{sing}}v_{1,q} = \sum_{\substack{I=(i_1, \dots, i_n) \\ |I|=q}} c_q(i_1, \dots, i_n)L(-i_1) \dots L(-i_n)v_{1,q} \tag{2}$$

is a singular vector for $M(1, h_{1,q}(1))$, where

$$c_r(i_1, \dots, i_n) = \prod_{\substack{1 < k < r \\ k \neq \sum_{j=1}^s v_{i_j}}} k(r - k).$$

Remark 2.1. Note that every singular vector (2) has form $L(-1)^{m+1} + \dots$, where dots represent lower degree terms (with respect to the universal enveloping algebra grading).

3. Vertex operator algebra $L(1, 0)$

3.1. Zhu’s algebra and intertwining operators

We will use the definition of vertex operator algebra and modules as stated in [FHL] or [FLM]. Let $L(1, 0) = M(1, 0)/\langle L(-1)\mathbf{1} \rangle$ be a simple vertex operator algebra associated to irreducible representation of the Virasoro algebra (cf. [FZ, W]).

It is known that to every vertex operator algebra V , we can associate Zhu’s associative algebra $A(V)$ (cf. [FZ,Z]). In the special case $V = L(1, 0)$, we know (see [FZ,W]) that $A(V) \cong \mathbb{C}[y]$, where $y = [L(-2) - L(-1)]$. We have chosen the multiplication in $A(V)$ as in [W] (which is slightly different then the one in [FZ]),

$$a * b = \text{Res}_x Y(a, x) \frac{(1-x)^{\text{deg}(a)}}{x} b,$$

where $a, b \in A(V)$.

By using standard techniques (see [FZ,W]) we have the following proposition.

Proposition 3.1. *Every irreducible module for the vertex operator algebra $L(1, 0)$ is isomorphic to $L(1, h)$, for some $h \in \mathbb{C}$.*

Proof. According to [Z], there is a one-to-one equivalence between equivalence classes of \mathbb{N} -gradable irreducible $L(1, 0)$ -modules and irreducible $\mathbb{C}[y]$ -modules. Every irreducible $L(1, 0)$ -module is a Vir-module. Any such module is \mathbb{N} -gradable and isomorphic to $L(1, h)$ for some $h \in \mathbb{C}$. On the other hand every finite-dimensional irreducible $\mathbb{C}[y]$ -module is one-dimensional so the proof follows. \square

Since the notion of intertwining operator is more subtle we include here the original definition [FHL].

Definition 3.1. Let W_1, W_2 , and W_3 be a triple of modules for vertex operator algebra V . A mapping

$$\mathcal{Y} \mapsto W_1 \otimes W_2 \rightarrow W_3\{x\},$$

is called an intertwining operator of type $\binom{W_3}{W_1 W_2}$, if it satisfies the following properties:

1. The *truncation property*: For any $w_i \in W_i, i = 1, 2, (w_1)_n w_2 = 0$, for n large enough.
2. The *$L(-1)$ -derivative property*: For any $v \in V$,

$$\mathcal{Y}(L(-1)w_1, x) = \frac{d}{dx} \mathcal{Y}(w_1, x).$$

3. The *Jacobi identity*: In $\text{Hom}(W_1 \otimes W_2, W_3)\{x_0, x_1, x_2\}$, we have

$$\begin{aligned} &x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) \mathcal{Y}(w_1, x_2) \\ &\quad - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_1, x_2) Y(u, x_1) \\ &= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y(u, x_0)w_1, x_2) \end{aligned} \tag{3}$$

for $u \in V$ and $w_1 \in W_1$.

We denote the space of all intertwining operators of the type $\binom{W_3}{W_1 W_2}$ by $I\left(\binom{W_3}{W_1 W_2}\right)$. The dimension of the space of intertwining operators (also known as “fusion rule”) of the type $\binom{W_3}{W_1 W_2}$ we denote by $\mathcal{N}_{W_1, W_2}^{W_3}$.

Our goal is to find the fusion rules for the degenerate minimal models, i.e.,

$$\dim I \left(\begin{matrix} L(1, r^2/4) \\ L(1, p^2/4) L(1, q^2/4) \end{matrix} \right).$$

Since our modules are irreducible we want to introduce Frenkel–Zhu’s formula which gives us (roughly) a prescription for calculating fusion rules. It is not hard to see, by using the Jacobi identity, that the space

$$I \left(\begin{matrix} L(1, r^2/4) \\ L(1, p^2/4) L(1, q^2/4) \end{matrix} \right)$$

is at most one-dimensional.

Now for every module M , we associate an $A(V)$ -bimodule $A(M) := M/O(M)$ (cf. [FZ]), where $O(M)$ is spanned by the elements of the form

$$\text{Res}_x Y(u, x) \frac{(1-x)^{\deg(a)}}{x^2} v,$$

$u \in V, v \in M$. In the case $M = M(c, h)$,

$$\begin{aligned} &O(M(c, h)) \\ &= \{(L(-n-3) - 2L(-n-2) + L(-1))v : n \geq 0, v \in M(c, h)\}. \end{aligned} \quad (4)$$

If we let

$$y = [L(-2) - L(-1)], \quad x = [L(-2) - 2L(-1) + L(0)],$$

then it follows from the formulas

$$[L(-n)v] = [(ny - x + wt(v))v] \quad \text{and} \quad [x, y]w = 0 \text{ mod } O(M(c, h))$$

$([x, y] = xy - yx)$ that

$$A(M(c, h)) \cong \mathbb{C}[x, y],$$

as a $\mathbb{C}[y]$ -bimodule (cf. [L2]) where the lowest weight vector is identified with $1 \in \mathbb{C}[x, y]$ and the actions of are

$$y * p(x, y) = xp(x, y), \quad p(x, y) * y = yp(x, y),$$

for every $p(x, y) \in \mathbb{C}[x, y]$.

The Frenkel–Zhu’s formula [FZ] states that it is possible to calculate the dimension of the space $\binom{M_3}{M_1 M_2}$ by knowing $A(V), A(M_1), M_2(0)$, and $M_3(0)$. Instead of giving the original statement from [FZ], we state the following refinement obtained in [L1,L2].

Theorem 3.1. *Let M_1, M_2 , and M_3 be lowest weight V -modules. Suppose that M_2 and M'_3 are generalized Verma V -modules (see Section 3.2). Then we have*

$$\mathcal{N}_{M_1 M_2}^{M_3} = \dim \text{Hom}_{A(V)}(A(M_1 \otimes_{A(V)} M_2(0), M_3(0)),$$

where $M_i(0), i = 1, 2, 3$, is the “top” level of M_i , respectively, equipped with the $A(V)$ -module structures.

This theorem is not so useful as it stands. On the other hand, its proof is important. Hence it will be necessary to understand a little bit deeper assumptions on M_2 and M_3 in our situation. For warm up let us start with the “easy-half” of the Frenkel–Zhu’s formula which says:

Lemma 3.1. *Let M_3 be an irreducible lowest weight V -module. Then*

$$\mathcal{N}_{M_1 M_2}^{M_3} \leq \dim \operatorname{Hom}_{A(V)}(A(M_1) \otimes_{A(V)} M_2(0), M_3(0)).$$

Define an infinite-dimensional Lie algebra \mathcal{L} spanned by

$$L(-n-2) - 2L(-n-1) + L(-n),$$

for $n \geq 1$. In the case of minimal models—which is the most interesting case—the homology groups $H_q(\mathcal{L}, L(c, h))$ where calculated in [FF2]. For the Verma modules the 0th homology, $H_0(\mathcal{L}, M(1, h))$ with the coefficients in the Verma modules is isomorphic to $\mathbb{C}[x, y]$ as an $A(L(1, 0))$ -bimodule (cf. [W]).

The following result is an application of a more general theory [FF1].

Theorem 3.2. *We have:*

- (a) $H_0\left(\mathcal{L}, L\left(1, \frac{m^2}{4}\right)\right)$, is infinite-dimensional.
- (b) $H_0(\mathcal{L}, L(1, m^2/4))$ is finitely generated as a (left) $A(L(1, 0))$ -module.
- (c) $\operatorname{Ext}_{\operatorname{Vir}, \mathbb{C}}^1\left(L\left(1, \frac{m^2}{4}\right), L\left(1, \frac{n^2}{4}\right)\right) = \begin{cases} \mathbb{C} & \text{if } |m - n| = 2, \\ 0 & \text{otherwise,} \end{cases}$

where $\operatorname{Ext}_{\operatorname{Vir}, \mathbb{C}}^1$ stands for the relative Ext with respect to the one-dimensional abelian subalgebra generated by $L(0)$.

Proof. (a) Since the maximal submodule of $M(1, m^2/4)$ is generated by one vector, in the projection (or homology) $A(L(1, m^2/4))$ is isomorphic to $\mathbb{C}[x, y]/I$, where I is a cyclic submodule (with respect to the left and right actions) generated by some polynomial $p(x, y)$ which is a projection of $v_{1,m}$ in $\mathbb{C}[x, y]$. It is clear that this space is infinite-dimensional.

(b) Note first that $[L(-1)v] = (y - x + \deg(v))[v]$. By using Remark 2.1 it follows that

$$[v_{\text{sing}}] = p(x, y) = \prod_{i=1}^{m+1} (x - y + i) + q(x, y),$$

where $\deg(q) < (m + 1)$. Thus, the pure monomials in $p(x, y)$ with the highest powers are x^{m+1} and y^{m+1} . Since, I is spanned by $p(x, y)\mathbb{C}[x]$, here we consider only the left action, it follows that $\mathbb{C}[x, y]/I$ finitely generated. The similar argument holds for the right action.

(c) The idea is the same as in [FF1]. The result is however different. It is known that

$$\text{Ext}_{\text{Vir}, \mathcal{O}}^*(M, N) \cong H^*(\text{Vir}, \mathcal{O}, \text{Hom}(M, N)).$$

Therefore

$$H^*(\text{Vir}, \mathcal{O}, \text{Hom}(M, N)) \cong \text{Tor}_*^{\text{Vir}, \mathcal{O}}(N^*, M),$$

where N^* is the dual module. Hence we can compute our cohomology by using the tensor product of complexes

$$\begin{aligned} M\left(1, \frac{(m+2)^2}{4}\right) &\longrightarrow M\left(1, \frac{m^2}{4}\right), \\ M\left(1, \frac{(n+2)^2}{4}\right)^{\text{opp}} &\longrightarrow M\left(1, \frac{n^2}{4}\right)^{\text{opp}}, \end{aligned}$$

where $M(c, h)^{\text{opp}}$ is the opposite Verma module (cf. [FF1, FF2]). The corresponding spectral sequence $E_2^{p,q}$ collapses at the second term. Therefore

$$\text{Tor}_1^{\text{Vir}, \mathcal{O}}\left(L\left(1, \frac{n^2}{4}\right)^*, L\left(1, \frac{m^2}{4}\right)\right) \cong E_2^{1,0} \cong \mathbb{C} \text{ or } 0,$$

where non-trivial homology occurs only if the Verma module $M(1, m^2/4)$ embeds inside $M(1, n^2/4)$ as the maximal submodule or vice-versa. This happens if and only if $|n - m| = 2$. Therefore we have the proof.² The corresponding short exact sequences are clearly

$$0 \rightarrow L\left(1, \frac{(m+2)^2}{4}\right) \rightarrow M\left(1, \frac{m^2}{4}\right) / M\left(1, \frac{(m+4)^2}{4}\right) \rightarrow L\left(1, \frac{m^2}{4}\right) \rightarrow 0, \tag{5}$$

and the one obtained from (5) by applying (exact) functor $(\)'$ taking modules to the corresponding contragredient modules. \square

For every $m, n \in \mathbb{N}$ (we exclude the case $mn = 0$), fix a multiset $J_{m,n} = \{m+n, m+n-2, \dots, m-n\}$. Let $\mathcal{F}_{\lambda, \mu}$ be a “density” module for the Virasoro algebra. $\mathcal{F}_{\lambda, \mu}$ is spanned by $w_r, r \in \mathbb{Z}$, and the action is given by

$$L_n.w_r = (\mu + r + \lambda(m+1))w_{r-n}.$$

In [FF1] the projection formula for the singular vectors (considered as an element of the enveloping algebra) on $\mathcal{F}_{\lambda, \mu}$ (more precisely w_0) was found. We want to relate the projection of the singular vectors on $\mathcal{F}_{\lambda, \mu}$ with the projection inside $A(M(1, m^2/4)) \otimes_{C[y]} L(1, n^2/4)$. It is easy to see that

² It is crucial to notice that our cohomology is relative one, otherwise our extension are not controllable inside category \mathcal{O} . Such (non-relative) extensions are studied in [M].

$$\begin{aligned}
 [L(-j_1) \dots L(-j_k)v_{m^2/4}] &= \prod_{r=1}^k \left(j_r \frac{n^2}{4} - y + \beta(r, k) \right) \cdot [v_{m^2/4}] \\
 &= \prod_{r=1}^k \left(j_r \frac{n^2}{4} - x + \beta(r, k) \right) \cdot [v_{m^2/4}]
 \end{aligned}
 \tag{6}$$

where $v_{m^2/4}$ is the lowest weight vector and

$$\beta(r, k) = j_{r+1} + \dots + j_k + \frac{m^2}{4}.$$

But the last factor in (6) is the same as the $P(j_1, \dots, j_k)$ where

$$L(-j_1) \dots L(-j_k).w_0 = P(j_1, \dots, j_k)w_{j_1+\dots+j_k},$$

and the projection is in $\mathcal{F}_{\lambda, \mu}$ for $\lambda = -n^2/4$ and $\mu = n^2/4 + m^2/4 - x$.

In the remarkable paper [FF2], projection formulas for all singular vectors on the density modules were found. In the slightly different notation, for the singular vectors we consider, these formulas appeared in [KA]. The result is

$$v_{1, m+1}.w_0 = \prod_{i \in J_{m, n}} \left(x - \frac{i^2}{4} \right) w_{m+1},
 \tag{7}$$

up to a multiplicative constant.

Now, by using (7) fact and the discussion above (cf. [W]) we obtain

Lemma 3.2. *As a $A(L(1, 0))$ -module $A(L(1, m^2/4)) \otimes_{A(L(1, 0))} L(1, n^2/4)(0)$ is isomorphic to $\mathbb{C}[x]/\langle \prod_{i \in J_{m, n}} (x - i^2/4) \rangle$.*

If $n \leq m$ notice that as an $A(L(1, 0))$ -module

$$A\left(L\left(1, \frac{m^2}{4}\right)\right) \otimes_{A(L(1, 0))} L\left(1, \frac{n^2}{4}\right)(0) \cong \bigoplus_{i \in J_{m, n}} \mathbb{C}v_i,
 \tag{8}$$

where v_i is an irreducible $A(L(1, 0))$ -module such that $y.v_i = i^2/4v_i$. But if $m < n$, then we have two-dimensional submodule in the above decomposition (and this module is *not* completely reducible). Thus, (8) is *not* symmetric if we switch m and n .

The similar failure was already noticed in [L1]. Anyhow, by using Lemmas 3.2 and 3.1 we obtain the following proposition.

Proposition 3.2. *Let $L(1, m^2/4)$, $L(1, n^2/4)$, and M be irreducible $L(1, 0)$ -modules. Then we have the following upper bounds:*

$$\dim I \left(L\left(1, \frac{m^2}{4}\right)^M L\left(1, \frac{n^2}{4}\right) \right) \leq \begin{cases} 1 & \text{if } M \cong L\left(1, \frac{r^2}{4}\right) \text{ for } r \in J_{m, n}, \\ 0 & \text{otherwise} \end{cases}
 \tag{9}$$

where $J_{m, n} = \{m + n, \dots, m - n\}$.

Now, we shall show that the equality holds in Eq. (9). We will provide two different proofs. One which uses the properties of Verma modules and the other which uses free field realization of the modules $L(1, m^2/4)$.

3.2. Lie algebra $g(V)$

Let V be a vertex operator algebra. Let $\widehat{V} = V \otimes \mathbb{C}[t, t^{-1}]$, $d = L(-1) \otimes 1 + 1 \otimes \frac{d}{dt}$, and $g(V) = \widehat{V}/d\widehat{V}$. It has been noticed by several authors that the space $g(V)$ has a Lie algebra structure if we let

$$[a(m), b(n)] = \sum_{i=0}^{\infty} \binom{m}{i} (a_i b)(m+n-i).$$

If we define the grading with $\deg a(m) = n - m - 1$, where $a \in V_{(n)}$, then we have the corresponding triangular decomposition $g(V) = g(V)_- \oplus g(V)_0 \oplus g(V)_+$. Let U be any $g(V)_0$ -module. We let (as in [L2])

$$F(U) = \text{Ind}_{U(g(V)_+ \oplus g(V)_0)}^{U(g)} U,$$

such that $g(V)_+$ acts as zero. We define also the quotient $\overline{F}(U) = F(U)/J(U)$ (the so-called generalized Verma module [L2]), where $J(U)$ is the intersection of all kernels of all $g(V)$ -homomorphisms from $F(U)$ to weak modules. Now, the assumption in Theorem 3.1 on M_2 and M'_3 means that $M_2 \cong \overline{F}(M_2(0))$ and $M_3 = \overline{F}(M_3^*(0))'$.

In [L1,L2] it was shown that every $A(V)$ homomorphism from $A(W_1) \otimes_{A(V)} W_2(0)$ to $W_3(0)$ does not necessary lead to an intertwining operator of the type $\binom{W_3}{W_1 W_2}$ but rather to $\binom{F(W_3(0)^*)'}{W_1 F(W_2(0))}$ (actually $F(W_2(0))$ might be replaced by $\overline{F}(W_2(0))$).

In the case when V is rational, $\overline{F}(W_2(0)) \cong W_2$ and $\overline{F}(W_3(0)^*)' \cong W_3$ [L2]. But if the vertex operator algebra V is not rational, the main difficulty is that the generalized Verma module $\overline{F}(W_2(0))$ may not be isomorphic to W_2 (let alone $F(W_2(0))!$) (cf. [L2]). Also, the spaces $F(U)$ and $\overline{F}(U)$ are extremely difficult to analyze explicitly. Still, because we are dealing with a particular example, Virasoro vertex operator algebra, we can make use of singular vectors and Verma modules to simplify the whole construction.

Let $V = L(1, 0)$. Pick $\omega = L(-2)\mathbf{1} \in L(1, 0)$. Then, inside $g(L(1, 0))$, we have

$$[\omega(m+1), \omega(n+1)] = (m-n)\omega(m+n+1) + \delta_{m+n,0} \frac{m^3-m}{12},$$

i.e., these operators close the Virasoro algebra. From the construction of $F(U)$ it is clear that $U(\text{Vir}_-) \otimes U \hookrightarrow U(g(V)_-) \otimes U \cong F(U)$. In particular, $M(1, h) \hookrightarrow F(M(1, h)(0))$.

3.3. The fusion rules computations

Assume first that

$$m \leq n. \tag{10}$$

First we replace the “big” space $F(M(1, h))$ with the smaller Verma module for the Virasoro algebra (we have seen already that the latter is a subspace inside $F(M(1, h))$).

Now, let us pick a non-trivial $A(L(1, 0))$ homomorphism from $A(L(1, m^2/4)) \otimes_{A(L(1,0))} L(1, n^2/4)(0)$ to $L(1, r^2/4)(0)$. Also let $T = L(1, m^2/4) \otimes \mathbb{C}[t, t^{-1}] \otimes M(1, n^2/4)$ be a $g(L(1, 0))$ -module as in [L2]. Then the construction in [L2] implies that there is a bilinear pairing between T and $M(1, r^2/4) \hookrightarrow F(M(1, r^2/4)(0)^*)$. This implies (again by applying Li’s construction in the proof of Theorem 2.11 in [L2]) that the corresponding intertwining operator *lands* in $M(1, r^2/4)'$, i.e., it is of the type

$$\begin{pmatrix} M(1, r^2/4)' \\ L(1, m^2/4) M(1, n^2/4) \end{pmatrix}.$$

Here $M(1, r^2/4)'$ is the contragredient Verma module (cf. [FF2]). The contragredient module $M(1, r^2/4)'$ is *not* of the lowest weight type (because $M(1, r^2/4)$ is reducible). In particular, if v' is the lowest weight vector

$$U(\text{Vir})v' \cong L\left(1, \frac{r^2}{4}\right),$$

i.e., we can “paste” the whole irreducible module by acting on the lowest weight subspace, but not the whole module $M(1, m^2/4)'$. Now, the question is

How to descend from $M(1, m^2/4)'$ to $L(1, m^2/4)$?

Here is the proof. We have either $n \leq r$ or $r < n$. For each of these two cases we consider

$$I\left(\begin{matrix} M(1, r^2/4)' \\ L(1, m^2/4) M(1, n^2/4) \end{matrix}\right), \tag{11}$$

or

$$I\left(\begin{matrix} M(1, n^2/4)' \\ L(1, m^2/4) M(1, r^2/4) \end{matrix}\right), \tag{12}$$

respectively. Notice that these two spaces are isomorphic because of

$$I\left(\begin{matrix} M_3 \\ M_1 M_2 \end{matrix}\right) \cong I\left(\begin{matrix} M'_2 \\ M_1 M'_3 \end{matrix}\right).$$

Suppose that $n \leq r$.

Now the aim is to construct intertwining operator of the type

$$I \left(\begin{matrix} M(1, r^2/4)' \\ L(1, m^2/4) M(1, n^2/4) \end{matrix} \right).$$

Therefore if we can check

$$\langle w'_3, \mathcal{Y}(w_1, x)w \rangle = 0, \tag{13}$$

for every $w \in M(1, (m + 2)^2/4) \hookrightarrow M(1, m^2/4)$, $w'_3 \in M(1, r^2/4)'' = M(1, r^2/4)$, and $w_1 \in L(1, n^2/4)$, then by defining $\bar{\mathcal{Y}}(w_1, x)[w_2] := \bar{\mathcal{Y}}(w_1, x)w_2$ where $[w_2] \in M(1, m^2/4)/M(1, (m + 2)^2/4)$, we obtain a (well-defined) non-trivial intertwining operator of the type

$$\left(\begin{matrix} M(1, r^2/4)' \\ L(1, m^2/4) L(1, n^2/4) \end{matrix} \right).$$

Let us check that (13) holds. First of all, because of the Jacobi identity and the fact that $M(1, r^2/4)$ is lowest weight module, it is enough to show that

$$\langle w'_3, \mathcal{Y}(w_1, x)v_{\text{sing}} \rangle = 0, \tag{14}$$

where $w'_3 \in M(1, r^2/4)''(0) = M(1, r^2/4)(0)$ is the lowest weight vector and v_{sing} is the singular vector that generates the maximal submodule of $M(1, m^2/4)$.

$$\begin{aligned} & \langle w'_3, \mathcal{Y}(w_1, x)L(-j_1) \dots L(-j_k)w \rangle \\ &= \prod_{i=1}^k - \left(x^{-j_i+1} \partial_x + (1 - j_i)x^{-j_i} \frac{n^2}{4} \right) \langle w'_3, \mathcal{Y}(w_1, x)w \rangle \\ &= \prod_{i=1}^k - \left(x^{-j_i+1} \partial_x + (1 - j_i)x^{-j_i} \frac{m^2}{4} \right) Cx^{\frac{r^2}{4} - \frac{m^2}{4} - \frac{n^2}{4}} \\ &= (-1)^{\sum_i j_i} \prod_{i=1}^k \left(\frac{r^2}{4} - \frac{m^2}{4} - \frac{n^2}{4} - \sum_{s=i+1}^k j_s + (1 - j_i) \frac{m^2}{4} \right) Cx^{\frac{r^2}{4} - \frac{m^2}{4} - \frac{n^2}{4}} \\ &= C \prod_{i=1}^k \left(j_i \frac{m^2}{4} - \frac{r^2}{4} + \sum_{s=i+1}^k j_s + \frac{n^2}{4} \right) Cx^{\frac{r^2}{4} - \frac{m^2}{4} - \frac{n^2}{4} - \sum_i j_i}, \tag{15} \end{aligned}$$

where C is a constant that depends on \mathcal{Y} (we may assume that C is equal to 1). If we compare (15) with (6) we see that products appearing in both expressions are the same if we interchange x with $r^2/4$ and $m^2/4$ with $n^2/4$. In other words, the expression $\langle w'_3, \mathcal{Y}(w_1, x)v_{\text{sing}} \rangle = 0$ if and only if the corresponding projection inside $A(L(1, n^2/4)) \otimes_{A(L(1,0))} A(L(1, m^2/4))$ is zero (notice that now $L(1, n^2/4)$ and $L(1, m^2/4)$ changed positions). We know that

$$A \left(L \left(1, \frac{n^2}{4} \right) \right) \otimes_{A(L(1,0))} A \left(L \left(1, \frac{m^2}{4} \right) \right) \cong \frac{\mathbb{C}[x]}{\prod_{i \in J_{n,m}} \langle x - i^2/4 \rangle}.$$

Because of (10), $J_{n,m} \subset J_{n,m}$ (as multisets). Therefore

$$\langle w'_3, \mathcal{Y}(w_1, x)v_{\text{sing}} \rangle = 0$$

holds. Thus we obtain a non-trivial intertwining operator $\overline{\mathcal{Y}}$ of the type

$$\begin{pmatrix} M(1, r^2/4)' \\ L(1, m^2/4) L(1, n^2/4) \end{pmatrix}.$$

Now,

$$I \begin{pmatrix} M(1, r^2/4)' \\ L(1, m^2/4) L(1, n^2/4) \end{pmatrix} \cong I \begin{pmatrix} L(1, n^2/4) \\ L(1, m^2/4) M(1, r^2/4) \end{pmatrix}.$$

Because of our initial assumption $n \leq r$, and $m - n \leq r \leq m + n$ it follows that $m - r \leq n \leq m + r$, therefore we can repeat the whole procedure for $M(1, r^2/4)$ so we end up with a non-trivial intertwining operator of the type

$$\begin{pmatrix} L(1, n^2/4) \\ L(1, m^2/4) L(1, r^2/4) \end{pmatrix}.$$

If $r < q$ then we pick the intertwining operator (12) and the same reasoning leads to a non-trivial intertwining operator of the type

$$\begin{pmatrix} L(1, r^2/4) \\ L(1, m^2/4) L(1, n^2/4) \end{pmatrix}.$$

This also follows from the duality property for the intertwining operators. If we summarized everything we obtain

Theorem 3.3.

$$\dim I \begin{pmatrix} L(1, r^2/4) \\ L(1, m^2/4) L(1, n^2/4) \end{pmatrix} = 1$$

if and only if $r \in \{m + n, \dots, |m - n|\}$.

Theorem 3.4. Let \mathcal{A} be a free Abelian group on the set $\{a(m) : m \in \mathbb{N}\}$ and

$$\times : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$$

a binary operation defined by the formula

$$a(m) \times a(n) = \sum_{r \in \mathbb{N} \cup 0} \mathcal{N}_{L(1, m^2/4) L(1, n^2/4)}^{L(1, r^2/4)} a(r).$$

Then \mathcal{A} is a commutative associative ring with the multiplication

$$a(m) \times a(n) = a(m + n) + a(m + n - 2) + \dots + a(|m - n|),$$

i.e., \mathcal{A} is isomorphic to the representation ring $\text{Rep}(\mathfrak{sl}(2, \mathbb{C}))$.

Remark 3.1. In general, if M is any $L(1, 0)$ -module and

$$\mathcal{Y} \in I \left(\begin{matrix} M \\ L(1, m^2/4) \ L(1, n^2/4) \end{matrix} \right),$$

then M is not necessary completely reducible. Also, note that we excluded the case $mn = 0$. If m or n are equal to zero then we deal with intertwining operators among two irreducible modules and vertex operator algebras, which are well known.

Another interesting fact is that in the case (10) the module

$$A(L(1, m^2/4)) \otimes_{A(L(1,0))} L(1, n^2/4)(0)$$

is not completely reducible. This fact was exploited in [M] where we study *logarithmic* intertwining operators.

Note that in our proof we actually analyzed more carefully the failure of Frenkel–Zhu’s formula. One should not expect to apply our procedure in the more general setting, because our Virasoro vertex operator algebra has a quite simple structure. Certainly it would be interesting to study a class of vertex operator algebra for which

$$A(W_1) \otimes_{A(V)} W_2(0) \cong A(W_2) \otimes_{A(V)} W_1(0), \tag{16}$$

for any choice of irreducible modules W_1 and W_2 . Then we hope that for this class of vertex algebras some version of Frenkel–Zhu’s formula indeed apply. Assumption (16) turns out to be very natural since

$$I \left(\begin{matrix} W_3 \\ W_1 \ W_2 \end{matrix} \right) \cong I \left(\begin{matrix} W_3 \\ W_2 \ W_1 \end{matrix} \right). \tag{17}$$

4. Construction of all intertwining operators for the family \mathcal{F}_1

4.1. V_L vertex operator algebra and its irreducible modules

Let L be a rank one even lattice with a generator β normalized such that $\langle \beta, \beta \rangle = 1$ and let $\alpha = \sqrt{2}\beta$. Thus $\langle \alpha, \alpha \rangle = 2$. As in [FLM,DL] we define V_L as a vector space

$$V_L = M(1) \otimes \mathbb{C}[L],$$

where $M(1)$ is the level one irreducible module for Heisenberg algebra \hat{h}_Z associated to one-dimensional abelian algebra $h = L \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbb{C}[L]$ is the group algebra of L with a generator e^α . Put $\omega = \frac{1}{2}\beta(-1)^2$. Then V_L is a vertex operator algebra (see [FLM]) with the Virasoro element ω . We have a decomposition

$$V_L = \bigoplus_{m \in \mathbb{Z}} M(1) \otimes e^{m\alpha}.$$

Let L° be a dual lattice, $L^\circ/L \cong \mathbb{Z}/2\mathbb{Z}$. Then (as in [DL]), for a nontrivial coset representative, we obtain an irreducible V_L -module $V_{L+1/2}$, which can be decomposed as

$$V_{L+1/2} = \bigoplus_{m \in \mathbb{Z}} M(1) \otimes e^{m\alpha+1/2\alpha}.$$

Moreover, $V_{L+1/2}, V_L$ is (up to equivalence) complete list of irreducible V_L -modules. Furthermore, one can equip the space $W = V_L \oplus V_{L+1/2}$ (as in [DL]) with the structure of the generalized vertex operator algebra. We will neglect this fact in our considerations.

For every module W for the Virasoro algebra on which $L(0)$ acts semisimple we define a formal character (or a q -graded dimension) by

$$\text{ch}_q(W) = \sum_{n \in \text{Spec}L(0)} \dim(W_n)q^n.$$

From the Proposition 2.1 it follows that

$$\text{ch}_q\left(L\left(1, \frac{m^2}{4}\right)\right) = \frac{q^{m^2/4} - q^{(m+2)^2/4}}{q^{-1/24}\eta(q)}.$$

Then it is not hard to obtain

$$\begin{aligned} \text{ch}_q(V_L) &= \sum_{n \geq 0} (2n + 1) \text{ch}_q(L(1, n^2)), \\ \text{ch}_q(V_{L+1/2}) &= \sum_{n \geq 0} (2n + 2) \text{ch}_q\left(L\left(1, \frac{(2n + 1)^2}{4}\right)\right). \end{aligned} \tag{18}$$

Consider the vectors

$$x = e^\alpha, \quad y = e^{-\alpha}, \quad h = \alpha(-1)t(0),$$

which span $(V_L)_1$. These vectors span a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. $x_0, y_0,$ and h_0 act as derivatives on W . The following result was obtained in [DG].

Proposition 4.1. *As $(L(1, 0), \mathfrak{sl}_2)$ -module*

$$V_L \cong \bigoplus_{m \geq 0} L(1, m^2) \otimes V(2m),$$

where $V(2m)$ is an irreducible $(2m + 1)$ -dimensional \mathfrak{sl}_2 -module.

The proof uses the result from [DLM,DM1,DM2] about the decomposition of the vertex operator algebra V with respect to a “dual” pair (V^G, G) where $G = \text{Aut}(G)$ is a compact (or finite) group and V^G is a G -stable subvertex operator algebra. This can be modified when instead of group G we work with the Lie algebra.

Since $V_{L+1/2}$ is a module for the pair $(V_L^{\mathfrak{sl}_2}, \mathfrak{sl}_2)$ then by using (18) we derive

$$V_{L+1/2} \cong \bigoplus_{m \geq 0} L\left(1, \frac{(2m+1)^2}{4}\right) \otimes V(2m+1), \tag{19}$$

where $V(2m+1)$ is a $(2m+2)$ -dimensional \mathfrak{sl}_2 -module. It is easy to see that $V(2m+1)$ is irreducible \mathfrak{sl}_2 -module.

Remark 4.1. Note that $V^{\mathfrak{sl}_2}$ (\mathfrak{sl}_2 —stable vertex operator algebra) is exactly V^G where $G \cong SO(3)$ is a (full) group of automorphisms of V_L . It is well known that every irreducible representation can be obtain as a representation of $SL(2, \mathbb{C})$, since $PSL(2, \mathbb{C}) \cong SO(3)$. In particular, every such finite-dimensional representation is odd-dimensional.

Since, $V_{L+1/2}$ is an irreducible V_L -module we have the Jacobi identity

$$\begin{aligned} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(v, x_2) w - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2) Y(u, x_1) w \\ = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2) w, \end{aligned} \tag{20}$$

for every $u \in V_L, v \in V_{L+1/2}$, and $w \in W$. Also, for

$$\mathcal{Y} \in I\left(\begin{matrix} V_L \\ V_{L+1/2} & V_{L+1/2} \end{matrix}\right),$$

we have

$$\begin{aligned} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) \mathcal{Y}(v, x_2) w - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(v, x_2) Y(u, x_1) w \\ = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y(u, x_0)v, x_2) w. \end{aligned} \tag{21}$$

Remark 4.2. Note that W can not be equipped with a vertex operator superalgebra structure. If $u, v \in V_{L+1/2}$ then we do not get Jacobi identity in the form (20) or (21), but rather generalized identity where the delta function is suitably multiplied with the terms of the type $((x_1 - x_0)/x_2)^{1/2}$. Studying this (generalized) Jacobi identity is useful for studying convergence and the extension properties for the intertwining operators (cf. [H1]).

4.2. Intertwining operators for the family \mathcal{F}_1

Let $V(i), i \in \mathbb{N}$, be an irreducible \mathfrak{sl}_2 -module considered as a subspace of W which corresponds to the decompositions in Proposition 4.1 and (19). Fix a positive integer j . We introduce a basis $u_j(m), m \in \{j, j-2, \dots, -j\}$ for $V(j)$, such that the following relations are satisfied:

$$\begin{aligned}
 h.u_j(m) &= mu_j(m), \\
 x.u_j(m) &= \frac{\sqrt{(j+m+2)(j-m)}}{2}u_j(m+2), \\
 y.u_j(m) &= \frac{\sqrt{(j+m)(j-m+2)}}{2}u_j(m-2),
 \end{aligned}
 \tag{22}$$

where $u_j(k) = 0$ for $k \notin \{j, \dots, -j\}$. Also, we choose a dual basis $u_j^*(m)$ for $V(j)^*$ such that $\langle u_j^*(m), u_j(n) \rangle = \delta_{m,n}$. Define $\langle g.u^*, v \rangle = -\langle u^*, g.v \rangle$. Then $V(j)^*$ became a \mathfrak{sl}_2 -module and an isomorphism from $V(j)$ to $V(j)^*$ is given by $\mu(u_j(m)) = (-1)^{j-m}u_j^*(-m)$. By using this identification, for $j_1, j_2, j_3 \in \mathbb{N}$ and $-j_i \leq m_i \leq j_i, i = 1, 2, 3$, we introduce real numbers (Clebsch–Gordan coefficients)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix},$$

such that

$$u_{j_1}(m_1) \otimes u_{j_2}(m_2) = \sum_{j_3=|j_1-j_2|}^{j_3=j_1+j_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} u_{j_3}(m_1+m_2).
 \tag{23}$$

First we need an auxiliary result which is slightly modified result from [DM1, DG].

Proposition 4.2. *Suppose that V is a vertex operator algebra and W_1, W_2 , and W_3 three irreducible V -modules. Let $v_i \in W_1, w_i \in W_2, i = 1, \dots, k$, be homogeneous elements such that $v_i \neq 0$ and w_i are linearly independent. Then*

$$\sum_{i=1}^k \mathcal{Y}(v_i, x)w_i \neq 0.$$

Now let us go back to our vertex operator algebra V_L . Let \mathcal{Y} be any intertwining operator of the type

$$\begin{pmatrix} V_L \\ V_{L+1/2} \ V_{L+1/2} \end{pmatrix}, \quad \begin{pmatrix} V_{L+1/2} \\ V_L \ V_{L+1/2} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} V_L \\ V_L \ V_L \end{pmatrix}.
 \tag{24}$$

By using the Proposition 4.2 the map

$$\mathcal{Y}(\cdot, x) : V(j_1) \otimes V(j_2) \rightarrow W\{x\}$$

is injective, and for every m_1, m_2 , and j_1, j_2 there is a $p \in \mathbb{C}$ such that

$$u_{j_1}(m_1)_p u_{j_2}(m_2) = \sum_{j_3=|j_1-j_2|}^{j_3=j_1+j_2} k(j_1, j_2, j_3, m_1, m_2, m_1+m_2)u_j(m_1+m_2),$$

where $k(j_1, j_2, j_3, m_1, m_2, m_1 + m_2)$ is a (non-zero) multiple of

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_1 + m_2 \end{pmatrix}$$

(in the special case $\mathcal{Y} = Y$ this fact was noticed in [DG]).

Now it is clear that if

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_1 + m_2 \end{pmatrix} \neq 0,$$

then the $L(1, 0)$ -module generated by $\mathcal{Y}(u_{j_1}(m_1), x)u_{j_2}(m_2)$ contains a copy of $L(1, j_3^2/4)$. Since $L(1, 0)$ is contained in V_L , and $L(1, m^2/4)$ is an $L(1, 0)$ -module then we obtain the following Jacobi identity:

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) \mathcal{Y}(v, x_2) w - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(v, x_2) Y(u, x_1) w \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y(u, x_0)v, x_2) w, \end{aligned} \tag{25}$$

for $u \in L(1, 0)$, $v \in L(1, j_1^2/4)$ and $w \in L(1, j_2^2/4)$ (here v and w lie in Vir-submodules generated by $u_{j_1}(m_1)$ and $u_{j_2}(m_2)$, respectively).

Now we can push down \mathcal{Y} to $L(1, j_3^2/4)$, which is generated by the vector $u_{j_3}(m_1 + m_2)$, since for every j_1, j_2 and $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$ we can choose a pair m_1, m_2 , and a \mathcal{Y} of the appropriate type (24) such that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_1 + m_2 \end{pmatrix} \neq 0.$$

We obtain an intertwining operator of the

$$\begin{pmatrix} L(1, j_3^2/4) \\ L(1, j_1^2/4) L(1, j_2^2/4) \end{pmatrix},$$

and this is the end of the construction.

5. Lie superalgebra $\mathfrak{osp}(1|2)$ and $\text{Rep}(\mathfrak{osp}(1|2))$

The Lie superalgebra $\mathfrak{osp}(1|2)$ is a graded extension of the finite-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. It has three even generators x, y and h , and two odd generators φ and χ that satisfy:

$$\begin{aligned} [h, x] &= 2x, & [h, y] &= -2y, & [x, y] &= h, \\ [x, \chi] &= \chi, & [x, \varphi] &= -\varphi, & [y, \chi] &= -\chi, & [y, \varphi] &= \varphi, \\ [h, \varphi] &= -\varphi, & [h, \chi] &= \chi, \\ [\chi, \varphi] &= 2h, & [\chi, \chi] &= 2x, & \{\varphi, \varphi\} &= 2y. \end{aligned}$$

Generators $\{x, y, h\}$ span a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, and this fact makes the representation theory of $\mathfrak{osp}(1|2)$ quite simple. All irreducible $\mathfrak{osp}(1|2)$ -modules can be constructed in the following way. Fix a positive half integer j ($2j \in \mathbb{N}$) and a $(4j + 1)$ -dimensional vector space $V(j)$ spanned by the vectors $\{v_j, v_{j-1/2}, \dots, v_{-j}\}$, with the following actions:

$$\begin{aligned} x.v_i &= \sqrt{[j-i][j+i+1]}v_{i+1}, \\ y.v_i &= \sqrt{[j+i][j-i+1]}v_{i-1}, \\ h.v_i &= 2i v_i. \end{aligned} \tag{26}$$

If $2(i - j) \in \mathbb{Z}$ then we define

$$\varphi.v_i = -\sqrt{j+i}v_{i-1/2}, \quad \chi.v_i = -\sqrt{j-i}v_{i+1/2}, \tag{27}$$

otherwise

$$\varphi.v_i = \sqrt{j-i+1/2}v_{i-1/2}, \quad \chi.v_i = -\sqrt{j+i+1/2}v_{i+1/2}. \tag{28}$$

In all these formulas $v_j = 0$ if $j \notin \{j, j - 1/2, \dots, -j\}$. It is easy to see that each $V(j)$ is an irreducible $\mathfrak{osp}(1|2)$ -module and that every finite-dimensional irreducible representation of $\mathfrak{osp}(1|2)$ is isomorphic to $V(j)$ for some $j \in \mathbb{N}/2$.

The representations with $j \in \mathbb{N}$ we call *even*, and the representations with $j \in \mathbb{N} + 1/2$ we call *odd*. We extend this definition for an arbitrary element of $V \in \text{Rep}(\mathfrak{osp}(1|2))$. The corresponding decomposition is $V = V_{\text{even}} + V_{\text{odd}}$.

It is a pleasant exercise to decompose the tensor product $V(i) \otimes V(j)$. The following result is well-known:

$$V(i) \otimes V(j) \cong \bigoplus_{k=|i-j|, k \in \mathbb{N}/2}^{i+j} V(k). \tag{29}$$

6. $N = 1$ Neveu–Schwarz superalgebra and its minimal models

The $N = 1$ Neveu–Schwarz superalgebra is given by

$$\mathfrak{ns} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C}G_{n+1/2} \oplus \mathbb{C}C,$$

together with the following $N = 1$ Neveu–Schwarz relations:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, G_{n+1/2}] &= \left(\frac{m}{2} - \left(n + \frac{1}{2}\right)\right)G_{m+n+1/2}, \\ [G_{m+1/2}, G_{n-1/2}] &= 2L_{m+n} + \frac{C}{3}(m^2 + m)\delta_{m+n,0}, \\ [C, L_m] &= 0, \quad [C, G_{m+1/2}] = 0 \end{aligned}$$

for $m, n \in \mathbb{Z}$. We have the standard triangular decomposition $\mathfrak{ns} = \mathfrak{ns}_+ \oplus \mathfrak{ns}_0 \oplus \mathfrak{ns}_-$ (cf. [KWa]). For every $(h, c) \in \mathbb{C}^2$, we denote by $M(c, h)$ Verma module for \mathfrak{ns} algebra. For each $(p, q) \in \mathbb{N}^2$, $p = q \pmod{2}$, let us introduce a family of complex ‘curves’ $(h_{p,q}(t), c(t))$:

$$h_{p,q}(t) = \frac{1-p^2}{8}t^{-1} + \frac{1-pq}{4} + \frac{1-q^2}{8}t,$$

$$c(t) = \frac{15}{2} + 3t^{-1} + 3t.$$

Then from the determinant formula (see [KWa]) it follows that $M(c, h)$ is reducible if and only if there is a $t \in \mathbb{C}$ and $p, q \in \mathbb{N}$, $p = q \pmod{2}$ such that $c = c(t)$ and $h = h_{p,q}(t)$. In this case $M(c, h)$ has a *singular vector* (i.e., a vector annihilated by \mathfrak{ns}_+) of the weight $h + pq/2$. Any such vector we denote by $v_{pq/2}$.

In this paper we are interested in the case $t = -1$. Then $c(-1) = 3/2$ and $h_{p,q}(-1) = (p - q)^2/8$. $h_{p,q}(-1) = h_{1,p-q+1}(-1)$, so we consider only the case $h_{1,q} := h_{1,q}(-1)$ (here q is odd and positive). Hence, each Verma module $M(3/2, h_{1,q})$ is reducible.

The following result easily follows from [D] (or [AA]) and [KWa].

Proposition 6.1. *For every odd q , $M(3/2, h_{1,q})$ has the following embedding structure:*

$$\dots \rightarrow M\left(\frac{3}{2}, h_{1,q+4}\right) \rightarrow M\left(\frac{3}{2}, h_{1,q+2}\right) \rightarrow M\left(\frac{3}{2}, h_{1,q}\right) \rightarrow 0. \tag{30}$$

Moreover, we have the following exact sequence:

$$0 \rightarrow M\left(\frac{3}{2}, h_{1,q+2}\right) \rightarrow M\left(\frac{3}{2}, h_{1,q}\right) \rightarrow L\left(\frac{3}{2}, h_{1,q}\right) \rightarrow 0, \tag{31}$$

where $L(3/2, h_{1,q})$ is the corresponding irreducible quotient.

Benoit and Saint-Aubin (cf. [BSA2]) found an explicit expression for the singular vectors $P_{\text{sing}}v_{1,q} \in M(3/2, h_{1,g})$ that generates the maximal submodule:

$$\sum_{N; k_1, \dots, k_N} \sum_{\sigma \in S_N} (-1)^{\frac{q-N}{2}} c(k_{\sigma(1)}, \dots, k_{\sigma(k)}) G(-k_1/2) \dots G(-k_N/2) v_{1,q}, \tag{32}$$

where S_N is a symmetric group on N letters and the first summation is over all the partitions of q into the odd integers k_1, \dots, k_N and

$$c(k_{\sigma(1)}, \dots, k_{\sigma(k)}) = \prod_{i=1}^N \binom{k_i - 1}{(k_i - 1)/2} \prod_{j=1}^{(N-1)/2} \frac{4}{\sigma_j \rho_{2j}},$$

where $\sigma_j = \sum_{l=1}^j k_l$ and $\rho_j = \sum_{l=j}^N k_l$.

In the special case: $q = 1, h_{1,1} = 0, M(3/2, 0)$ has a singular vector $G(-1/2)v$ which generate the maximal submodule. By quotienting we obtain a vacuum module $L(3/2, 0) = M(3/2, 0)/\langle G(-1/2)v_{3/2,0} \rangle$.

7. $N = 1$ superconformal vertex operator superalgebra and intertwining operators

We use the definition of $N = 1$ superconformal vertex operator superalgebra (with and without odd variables) as in [B] (cf. [KV]) and [HM] (see also [KW]).

Let φ be a Grassman (odd) variable such that $\varphi^2 = 0$. Every $N = 1$ superconformal vertex operator superalgebra $(V, Y, \mathbf{1}, \tau)$ can be equipped with a structure of $N = 1$ superconformal vertex operator algebra with an odd variable via

$$Y(\cdot, (x, \varphi)) : V \otimes V \rightarrow V((x))[\varphi],$$

$$u \otimes v \mapsto Y(u, (x, \varphi))v,$$

where

$$Y(u, (x, \varphi))v = Y(u, x)v + \varphi Y(G(-1/2)u, x)v$$

for $u, v \in V$.

The same formula can be used in the case of modules for the superconformal vertex operator superalgebra $(V, Y, \mathbf{1}, \tau)$ (see [HM]).

It is known (see [KW]) that $V(c, 0) := M(c, 0)/\langle G(-1/2)v_{c,0} \rangle^3$ is a $N = 1$ superconformal vertex operator superalgebra. Also, every lowest weight ns -module with the central charge c is a $V(c, 0)$ -module. If $c = 3/2$ then $V(3/2, 0) = L(3/2, 0)$.

Proposition 7.1. *Every irreducible $L(3/2, 0)$ -module is isomorphic to $L(3/2, h)$, for some $h \in \mathbb{C}$.*

Proof. The proof is essentially the same as the one of Proposition 3.1. \square

Among all irreducible $L(3/2, 0)$ -modules we distinguish modules isomorphic to $L(3/2, h_{1,q})$, $q \in 2\mathbb{N} - 1$. These representations we call *degenerate minimal models*.

7.1. Intertwining operators and its matrix coefficients

The notation of an intertwining operators for $N = 1$ superconformal vertex operator algebras is introduced in [KW, HM].

³ We write $L(c, 0)$ if $V(c, 0)$ is irreducible.

Let $W_1, W_2,$ and W_3 be a triple of V -modules and \mathcal{Y} an intertwining operator of type (W_1, W_2, W_3) . Then we consider the corresponding intertwining operator with an odd variable (cf. [HM]):

$$\begin{aligned} \mathcal{Y}(\cdot, (x, \varphi)) : W_1 \otimes W_2 &\rightarrow W_3\{x\}[\varphi], \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, (x, \varphi))w_{(2)}, \end{aligned}$$

such that

$$\mathcal{Y}(w_{(1)}, (x, \varphi))w_{(2)} = \mathcal{Y}(w_{(1)}, x)w_{(2)} + \varphi\mathcal{Y}(G(-1/2)w_{(1)}, x)w_{(2)}.$$

Let w_1 be a lowest weight vector for the Neveu–Schwarz algebra of the weight h . From the Jacobi identity we derive the following formulas:

$$\begin{aligned} [L(-n), \mathcal{Y}(w_1, x_2)] &= \left(x_2^{-n+1} \frac{\partial}{\partial x_2} + (1-n)h\right) \mathcal{Y}(w_1, x_2), \\ [G(-n-1/2), \mathcal{Y}(w_1, x_2)] &= x_2^{-n} \mathcal{Y}(G(-1/2)w_1, x_2), \\ [L(-n), \mathcal{Y}(G(-1/2)w_1, x_2)] \\ &= \left(x_2^{-n+1} \frac{\partial}{\partial x_2} + (1-n)\left(h + \frac{1}{2}\right)\right) \mathcal{Y}(G(-1/2)w_1, x_2), \\ [G(-n-1/2), \mathcal{Y}(G(-1/2)w_1, x_2)] \\ &= \left(x_2^{-n} \frac{\partial}{\partial x_2} - 2nhx_2^{-n-1}\right) \mathcal{Y}(w_1, x_2). \end{aligned} \tag{33}$$

In the odd formulation we obtain

$$\begin{aligned} [L(-n), \mathcal{Y}(w_1, (x_2, \varphi))] \\ &= (x_2^{-n+1} \partial_{x_2} + (1-n)x_2^{-n}(h + 1/2\varphi\partial_\varphi)) \mathcal{Y}(w_1, (x_2, \varphi)), \\ [G(-n-1/2), \mathcal{Y}(w_1, (x_2, \varphi))] \\ &= (x_2^{-n}(\partial_\varphi - \varphi\partial_{x_2}) - 2nx_2^{-n-1}(h\varphi)) \mathcal{Y}(w_1, (x_2, \varphi)), \end{aligned} \tag{34}$$

where ∂_φ is the odd (Grassmann) derivative.

7.2. Even and odd intertwining operators

In [HM] we proved that every intertwining operator

$$\mathcal{Y} \in I \left(\begin{matrix} L(c, h_3) \\ L(c, h_1) \ L(c, h_2) \end{matrix} \right)$$

is uniquely determined by the operators $\mathcal{Y}(w_1, x)$ and $\mathcal{Y}(G(-1/2)w_1, x)$, where w_1 is the lowest weight vector of $L(c, h_1)$. This fact will be used later in connection with the following definition.

Definition 7.1. Let $||$ denote the $(\mathbb{Z}/2\mathbb{Z}$ -valued) parity operator from the union of odd and even subspaces for V -modules W_i , $i = 1, 2, 3$. An intertwining operator $\mathcal{Y} \in I\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ is:

- even, if $|\text{Coeff}_{x^s} \mathcal{Y}(w_1, x)w_2| = |w_1| + |w_2|$,
- odd, if $|\text{Coeff}_{x^s} \mathcal{Y}(w_1, x)w_2| = |w_1| + |w_2| + 1$,

for every $s \in \mathbb{C}$ and every $\mathbb{Z}/2\mathbb{Z}$ -homogeneous vectors w_1 and w_2 .

The space of even (odd) intertwining operators of the type $\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ we denote by $I\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)_{\text{even}}$ ($I\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)_{\text{odd}}$). In general, we do not have a decomposition of $I\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ into the even and odd subspaces.

7.3. Frenkel–Zhu’s theorem for vertex operator superalgebras

According to [KW] (after [Z]), to every vertex operator superalgebra we can associate the Zhu’s associative algebra $A(V)$. If $V = L(c, 0)$, $A(L(c, 0)) \cong \mathbb{C}[y]$, where $y = [(L(-2) - L(-1))\mathbf{1}] = [L(-2)\mathbf{1}]$ (because of the calculations that follow it is convenient to use $y = [(L(-2) - L(-1))\mathbf{1}]$). Also to every V -module W we associate a $A(V)$ -bimodule $A(W)$ (cf. [KW]). In a special case $W = M(c, h)$, we have

$$A(M_{n\bar{s}}(c, h)) = M_{n\bar{s}}(c, h) / O(M_{n\bar{s}}(c, h)),$$

where

$$O(M_{n\bar{s}}(c, h)) = \{L(-n - 3) - 2L(-n - 2) + L(-1)v, \\ G(-n - 1/2) - G(-n - 3/2)v: n \geq 0, v \in M(c, h)\}. \tag{35}$$

It is not hard to see that, as $\mathbb{C}[y]$ -bimodule,

$$A(M(c, h)) \cong \mathbb{C}[x, y] \oplus \mathbb{C}[x, y]v,$$

where $v = [G(-1/2)v_h]$ and

$$y = [L(-2) - L(-1)], \quad x = [L(-2) - 2L(-1) + L(0)].$$

Let W_1 , W_2 , and W_3 be three $\mathbb{N}/2$ -gradable irreducible V -modules such that $\text{Spec}L(0)|_{W_i} \in h_i + \mathbb{N}$, $i = 1, 2, 3$, and $\mathcal{Y} \in I\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$. We define $o(w_1) := \text{Coeff}_{x^{h_3-h_1-h_2}} \mathcal{Y}(w_1, x)$. Because the fusion rules formula in [FZ] needs some modifications (cf. [L1]) the same modification is necessary for the main theorem in [KW] (this can be done with a minor super-modifications along the lines of [L1]). Nevertheless (cf. [KW]) we have the following theorem.

Theorem 7.1. *The mapping*

$$\pi : I \begin{pmatrix} W_3 \\ W_1 \ W_2 \end{pmatrix} \rightarrow \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0)),$$

such that

$$\pi(\mathcal{Y})(w_1 \otimes w_2) = o(w_1)w_2, \tag{36}$$

is injective.

8. Some Lie superalgebra homology

In this section we recall some basic definition from the homology theory of infinite-dimensional Lie superalgebras which is in the scope of the monograph [F] (in the cohomology setting though).

Let \mathcal{L} be an any (possibly infinite-dimensional) $\mathbb{Z}/2\mathbb{Z}$ -graded Lie superalgebra with the $\mathbb{Z}/2\mathbb{Z}$ -decomposition $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ and let $M = M_0 \oplus M_1$ be any \mathbb{Z}_2 -graded \mathcal{L} -module, such that the gradings are compatible. Then, we form a chain complex (C, d, M) (for details see [F]),

$$0 \xleftarrow{d_0} C_0(\mathcal{L}, M) \xleftarrow{d_1} C_1(\mathcal{L}, M) \xleftarrow{d} \dots,$$

where

$$C_q(\mathcal{L}, M) = \bigoplus_{q_0+q_1=q} M \otimes \Lambda^{q_0} \mathcal{L}_0 \otimes S^{q_1} \mathcal{L}_1,$$

$$C_q^p(\mathcal{L}, M) = \bigoplus_{\substack{q_0+q_1=q \\ q_1+r=p \pmod 2}} M_r \otimes \Lambda^{q_0} \mathcal{L}_0 \otimes S^{q_1} \mathcal{L}_1,$$

for $p = 0, 1$. The mappings d are super-differentials. For $q \in \mathbb{N}$ and $p = 0, 1$, we define q th homology with coefficients in M as

$$H_q^p(\mathcal{L}, M) = \text{Ker}(d_q(C_q^p(\mathcal{L}, M)))_p / (d_{q+1}(C_{q+1}^p(\mathcal{L}, M)))_p. \tag{37}$$

In a special case $q = 0$, we have

$$H_0^0(\mathcal{L}, M) = M_0 / (\mathcal{L}_0 M_0 + \mathcal{L}_1 M_1)$$

and

$$H_0^1(\mathcal{L}, M) = M_1 / (\mathcal{L}_1 M_0 + \mathcal{L}_0 M_1).$$

We want to calculate $H_q(\mathcal{L}_s, L(3/2, h_{1,q}))$ for the Lie superalgebra

$$\mathcal{L}_s = \bigoplus_{n \geq 0} \mathcal{L}_s(n),$$

where $\mathcal{L}_s(n)$ is spanned by the vectors $L(-n - 3) - 2L(-n - 2) + L(-n - 1)$ and $G(-n - 1/2) - G(-n - 3/2)$, $n \in \mathbb{N}$. From (35) we see (cf. [HM]) that $H_0(\mathcal{L}_s, M(c, h))$ is a $\mathbb{C}[y]$ -bimodule such that

$$H_0(\mathcal{L}_s, M(c, h)) \cong A(M(c, h)) \cong \mathbb{C}[x, y] \oplus \mathbb{C}[x, y]v. \tag{38}$$

Remark 8.1. It is more involved to calculate $H_0(\mathcal{L}_s, L(c, h))$, so we consider only the special case $c = 3/2$, $h = h_{1,q}$, q odd. As in the Virasoro case, it is easy to show that the space $H_p(\mathcal{L}_s, L(3/2, h_{1,q}))$ is infinite-dimensional for very $p, q, s \in \mathbb{N}$, and finitely generated as a $A(L(3/2, 0))$ -module. Moreover, it is not hard to see (by using the same method as in the Virasoro case) that

$$\text{Ext}_{\text{ns}, \mathcal{O}}^1 \left(L\left(\frac{3}{2}, h_{1,q}\right), L\left(\frac{3}{2}, h_{1,r}\right) \right)$$

is non-trivial (and one-dimensional) if and only if $|r - q| = 2$.

In the case of minimal models we expect a substantially different result (cf. [FF1]).

Conjecture 8.1. Let

$$c_{p,q} = \frac{3}{2} \left(1 - 2 \frac{(p-q)^2}{pq} \right) \quad \text{and} \quad h_{p,q}^{m,n} = \frac{(np - mq)^2 - (p - q)^2}{8pq}.$$

Then

$$\dim H_q(\mathcal{L}_s, L(c_{p,q}, h_{p,q}^{m,n})) < \infty,$$

for every $q \in \mathbb{N}$.

There is strong evidence that Conjecture 8.1 holds based on [A] and an example $c = -\frac{11}{14}$ treated in Appendix of [HM].

The main difference between the minimal models and the degenerate models is the fact that the maximal submodule for a minimal model is generated by two singular vectors, compared to $M(3/2, h_{1,q})$ where the maximal submodule is generated by a single singular vector.

9. Benoit–Saint-Aubin’s formula projection formulas

9.1. Odd variable formulation

We have seen before how to derive the commutation relation between generators of ns superalgebra and $\mathcal{Y}(w_1, x)$ where w_1 is a lowest weight vector for ns. We fix

$$\mathcal{Y} \in I \left(\begin{matrix} L(3/2, h) \\ L(3/2, h_{1,r}) \quad L(3/2, h_{1,q}) \end{matrix} \right)$$

and consider the following matrix coefficient:

$$\langle w'_3, \mathcal{Y}(w_1, x, \varphi) P_{\text{sing}} w_2 \rangle, \tag{39}$$

where $P_{\text{sing}} w_2 = v_{1,q}$ (cf. (7), $\deg(P_{\text{sing}}) = q/2$) and $w_i, i = 1, 2, 3$, are the lowest weight vectors.

Since all modules are irreducible, by using a result from [HM, Proposition 2.2], we get

$$\langle w'_3, \mathcal{Y}(w_1, x, \varphi) w_2 \rangle = c_1 x^{h-h_{1,q}-h_{1,r}} + c_2 \varphi x^{h-h_{1,q}-h_{1,r}-1/2},$$

where c_1 and c_2 are constants with the property

$$c_1 = c_2 = 0 \quad \text{implies} \quad \mathcal{Y} = 0. \tag{40}$$

From the formula (34),

$$\langle w'_3, \mathcal{Y}(w_1, x, \varphi) P_{\text{sing}} w_2 \rangle = P(\partial_{x_2}, \varphi) \langle w'_3, \mathcal{Y}(w_1, x, \varphi) w_2 \rangle,$$

where $P(\partial_{x_2}, \varphi)$ is a certain super-differential operator such that

$$\deg(P_{\text{sing}}) = \deg P(\partial_{x_2}, \varphi) = q/2.$$

Therefore

$$P(\partial_{x_2}, \varphi) c_1 x^{h-h_{1,q}-h_{1,r}} = \varphi C_1(h_{1,q}, h_{1,r}, h) x^{h-h_{1,q}-h_{1,r}-q/2}$$

and

$$P(\partial_{x_2}, \varphi) \varphi c_2 x^{h-h_{1,q}-h_{1,r}-q/2} = C_2(h_{1,q}, h_{1,r}, h) x^{h-h_{1,q}-h_{1,r}-q/2}.$$

Constants $C_1(h_{1,q}, h_{1,r}, h)$ and $C_2(h_{1,q}, h_{1,r}, h)$ (in slightly different form, but in more general setting) were derived in [BSA2]. Considering these coefficients was motivated by deriving formulas for singular vectors from already known singular vectors. By slightly modifying result from [BSA2] we obtain the following proposition.

Proposition 9.1. *Suppose that*

$$\mathcal{Y} \in I \left(\begin{matrix} L(3/2, h) \\ L(3/2, h_{1,r}) \ L(3/2, h_{1,q}) \end{matrix} \right)$$

and $P(\partial_x, \varphi)$ are as the above. Then, up to a multiplicative constant,

$$C_1(h_{1,q}, h_{1,r}, h) = \prod_{-j \leq k \leq j} (h - h_{1,q+4k})$$

and

$$C_2(h_{1,q}, h_{1,r}, h) = \prod_{-j+1/2 \leq k \leq j-1/2} \left(h + \frac{1}{2} - h_{1,q+4k} \right),$$

for $j = (r - 1)/4, j > 0$ (when $j = 0, C_2(h_{1,1}, h_{1,r}, h) = 1$).

Proof. The superdifferential operator $P(\partial_x, \varphi)$ is obtained by replacing generators $L(-m)$ and $G(-n - 1/2)$ by the superdifferential operators

$$L(-m) \mapsto -(x_2^{-m+1} \partial_{x_2} + (1 - m)x_2^{-m}(h_1 + 1/2\varphi\partial_\varphi)) \tag{41}$$

and

$$G(-n - 1/2) \mapsto (x_2^{-n}(\partial_\varphi - \varphi\partial_{x_2}) - 2nx_2^{-n-1}(h_1\varphi)), \tag{42}$$

acting on $\langle w'_3, \mathcal{Y}(w_1, x, \varphi)w_2 \rangle$. This action was calculated in [BSA2]. Their results [BSA2, Formula 3.10] implies the statement.⁴ \square

9.2. BSA formula without odd variables

Since Frenkel–Zhu’s formula does not involve odd variables we need a version of Proposition 9.1 without odd variables (which is of course equivalent). Again

$$\mathcal{Y} \in I \left(\begin{matrix} L(3/2, h) \\ L(3/2, h_{1,r}) \ L(3/2, h_{1,q}) \end{matrix} \right)$$

is the same as the above. Then

$$\langle w'_3, \mathcal{Y}(w_1, x)P_{\text{sing}}w_2 \rangle = P_2(\partial_x)\langle w'_3, \mathcal{Y}(G(-1/2)w_1, x)w_2 \rangle$$

and

$$\langle w'_3, \mathcal{Y}(G(-1/2)w_1, x)P_{\text{sing}}w_2 \rangle = P_1(\partial_x)\langle w'_3, \mathcal{Y}(w_1, x)w_2 \rangle,$$

where P_1 and P_2 are certain differential operators. If

$$P_2(\partial_x)c_2x^{h-h_{1,q}-h_{1,r}-1/2} = c_2K_2(h_{1,q}, h_{1,r}, h)x^{h-h_{1,q}-h_{1,r}-q/2}$$

and

$$P_1(\partial_x)c_1x^{h-h_{1,q}-h_{1,r}} = c_1K_1(h_{1,q}, h_{1,r}, h)x^{h-h_{1,q}-h_{1,r}-q/2},$$

then, by comparing corresponding coefficients, we obtain

$$\begin{aligned} K_1(h_{1,q}, h_{1,r}, h) &= C_1(h_{1,q}, h_{1,r}, h), \\ K_2(h_{1,q}, h_{1,r}, h) &= C_2(h_{1,q}, h_{1,r}, h). \end{aligned} \tag{43}$$

Let us mention that the projection formulas from Proposition 9.1 have a simple explanation terms of *super density modules* for the Neveu–Schwarz superalgebra.

⁴ In [BSA2] a different sign was used in Eq. (41). Still, we obtain the same result if we consider an isomorphic algebra with the generators $\tilde{L}(n) := -L(n)$. The same generators were used in [FF2].

10. Fusion ring for the degenerate minimal models

In order to obtain an upper bound for the fusion coefficients (cf. Theorem 7.1), we first compute

$$A\left(L\left(\frac{3}{2}, h_{1,q}\right)\right) \otimes_{A(L(3/2,0))} L\left(\frac{3}{2}, h_{1,r}\right)(0).$$

$\mathbb{Z}/2\mathbb{Z}$ -grading of the 0th homology group (37) enables us (see Theorem 10.1) to study odd and even intertwining operators (see Definition 7.1). For that purpose we introduce the following splitting:

$$\begin{aligned} A^0\left(L\left(\frac{3}{2}, h_{1,q}\right)\right) &:= H_0^0\left(\mathcal{L}_s, L\left(\frac{3}{2}, h_{1,q}\right)\right) \cong \frac{\mathbb{C}[x, y]}{I_1}, \\ A^1\left(L\left(\frac{3}{2}, h_{1,q}\right)\right) &:= H_0^1\left(\mathcal{L}_s, L\left(\frac{3}{2}, h_{1,q}\right)\right) \cong \frac{\mathbb{C}[x, y]v}{I_2}, \end{aligned} \tag{44}$$

where I_1 and I_2 are cyclic submodules (the maximal submodule for $M(3/2, h_{1,q})$ is cyclic!). It seems hard to obtain explicitly these polynomials. First we obtain some useful formulas. Inside $A(M(c, h))$ (cf. [W]),

$$\begin{aligned} [L(-n)v] &= [(n-1)(L(-2) - L(-1)) + L(-1)]v \\ &= [n(L(-2) - L(-1)) - (L(-2) - 2L(-1) + L(0)) \\ &\quad + L(0)]v \\ &= (ny - x + wt(v))[v] \end{aligned} \tag{45}$$

for every $n \in \mathbb{N}$ and every homogeneous $v \in M(c, h)$. Therefore in

$$A\left(M\left(\frac{3}{2}, h_{1,q}\right)\right) \otimes_{A(L(3/2,0))} L\left(\frac{3}{2}, h_{1,r}\right)(0)$$

we have

$$\begin{aligned} [L(-n)v] &= (nh_{1,q} - x + L(0))[v], \\ [G(-n - 1/2)v] &= [G(-1/2)v]. \end{aligned} \tag{46}$$

Also, we have:

$$\begin{aligned} [G(-n - 1/2)G(-m - 1/2)v] &= [G(-1/2)G(-m - 1/2)v] \\ &= [(2L(-m - 1) - G(-m - 1/2)G(-1/2))v] \\ &= [(2L(-m - 1) - L(-1))v] = ((2m + 1)y - x + wt(v))[v]. \end{aligned} \tag{47}$$

By using (45) and (47) we obtain

$$\begin{aligned}
 & [G(-m-1-1/2) \dots G(-m_{2r}-1/2)L(-n_1) \dots L(-n_s)v_{1,q}] \\
 &= \prod_{i=1}^r \left((2m_{2i}+1)h_{1,r} - x + \sum_{p=2i+1}^{2r} (m_p+1/2) + h_{1,q} \right) \\
 & \cdot \prod_{j=1}^s \left(n_j h_{1,r} - x + \sum_{p=j+1}^s n_p + h_{1,q} \right) [v] \tag{48}
 \end{aligned}$$

inside

$$A \left(M \left(\frac{3}{2}, h_{1,q} \right) \right) \otimes_{A(L(3/2,0))} L \left(\frac{3}{2}, h_{1,r} \right) (0).$$

It is easy to obtain a similar formula for the vector

$$[G(-m_1-1/2) \dots G(-m_{2r+1}-1/2)L(-n_1) \dots L(-n_s)v_{1,q}].$$

Lemma 10.1. *Let $[P_{\text{sing}}v_{1,q}] = Q_1(x)[G(-1/2)v_{1,q}]$ and $[G(-1/2)P_{\text{sing}}v_{1,q}] = Q_2(x)[v_{1,q}]$ be the projections inside*

$$A \left(M \left(\frac{3}{2}, h_{1,q} \right) \right) \otimes_{A(L(3/2,0))} L \left(\frac{3}{2}, h_{1,r} \right) (0).$$

Then

$$Q_1(h) = K_2(h_{1,q}, h_{1,r}, h), \quad Q_2(h) = K_1(h_{1,q}, h_{1,r}, h), \tag{49}$$

for every $h \in \mathbb{C}$.

Proof. We use the notation from the Section 6.2, where

$$\mathcal{Y} \in I \left(\begin{array}{c} L(3/2, h) \\ L(3/2, h_{1,r}) \quad L(3/2, h_{1,q}) \end{array} \right).$$

By using (33), we obtain

$$\begin{aligned}
 & \langle w'_3, \mathcal{Y}(w_1, x)G(-m_1-1/2) \dots G(-m_{2r}-1/2)L(-n_1) \dots L(-n_s)w_2 \rangle \\
 &= \prod_{i=1}^r - \left(x^{-m_{2i-1}-m_{2i}} \frac{\partial}{\partial x} - 2m_{2i}h_{1,r}x^{-m_{2i-1}-m_{2i}-1} \right) \\
 & \cdot \prod_{j=1}^s - \left(x^{-n_j+1} \frac{\partial}{\partial x} + (1-n_j)h_{1,r}x^{-n_j} \right) \langle w'_3, \mathcal{Y}(w_1, x)w_2 \rangle \\
 &= c_1 \prod_{i=1}^r \left((2m_{2i}+1)h_{1,r} - h + h_{1,q} + \sum_{p=2i+1}^{2r} (m_p+1/2) \right) \\
 & \cdot \prod_{j=1}^s \left(n_j h_{1,r} - h + \sum_{p=j+1}^s n_p + h_{1,q} \right) x^{h-h_{1,q}-h_{1,r}-r-\sum m_i-\sum_j n_j}, \tag{50}
 \end{aligned}$$

for the constant c_1 (see Sections 6.1 and 6.2) that depends only on \mathcal{Y} . There is a similar expression for

$$\langle w'_3, \mathcal{Y}(w_1, x)G(-m_1 - 1/2) \dots G(-m_{2r+1} - 1/2)L(-n_1) \dots L(-n_s)w_2 \rangle. \tag{51}$$

If we compare (48) with (50) (and corresponding formulas for (51)) it follows that $Q_1(h)$ is, up to a non-zero multiplicative constant, equal to $K_2(h_{1,r}, h_{1,q}, h)$ (singular vector is odd!) and $Q_2(h)$ is, up to a multiplicative constant, equal to $K_1(h_{1,r}, h_{1,q}, h)$. \square

Thus, Proposition 9.1 and Theorem 10.1 gives us the following theorem.

Theorem 10.1. (a) As a $A(L(3/2, 0))$ -module

$$\begin{aligned} &A(L(3/2, h_{1,q})) \otimes_{A(L(3/2,0))} L(3/2, h_{1,r})(0) \\ &\cong \frac{\mathbb{C}[x]}{\langle \prod_{-j \leq k \leq j} (x - h_{1,q+4k}) \rangle} \\ &\oplus \frac{\mathbb{C}[x]}{\langle \prod_{-j+1/2 \leq k \leq j+1/2} (h + 1/2 - h_{1,q+4k}) \rangle}. \end{aligned} \tag{52}$$

(b) The space

$$I \left(\begin{array}{c} M(3/2, h)' \\ L(3/2, h_{1,q}) \ L(3/2, h_{1,r}) \end{array} \right)$$

is non-trivial if and only if $h = h_{1,s}$ for some $s \in \{q + r - 1, q + r - 3, \dots, q - r + 1\}$.

(c) The space

$$I \left(\begin{array}{c} L(3/2, h) \\ L(3/2, h_{1,q}) \ L(3/2, h_{1,r}) \end{array} \right)$$

is one-dimensional if and only if $h = h_{1,s}$, $s \in \{q + r - 1, q + r - 3, \dots, |q - r| + 1\}$.

Proof. (a) From Lemma 10.1 it follows that

$$A(L(3/2, h_{1,r})) \otimes_{A(L(3/2,0))} L(3/2, h_{1,q}) \cong \frac{\mathbb{C}[x]}{\langle Q_1(x) \rangle} \oplus \frac{\mathbb{C}[x]}{\langle Q_2(x) \rangle}. \tag{53}$$

Now we apply (43) and Proposition 9.1.

(b) As in the Virasoro case, by examining carefully the main construction of intertwining operators in [L1] with a minor super-modifications, for every $A(L(3/2, 0))$ -morphism from $A(L(3/2, h_{1,q})) \otimes_{A(L(3/2,0))} L(3/2, h_{1,r})$ to $L(3/2, h)(0)$ we can construct a non-trivial intertwining operator of the form

$$I \left(\begin{array}{c} M(3/2, h)' \\ L(3/2, h_{1,q}) \ L(3/2, h_{1,r}) \end{array} \right).$$

(c) The proof and all the arguments involved are the same as in Section 3, so we omit the details. We obtain a non-trivial intertwining operator of the type

$$\begin{pmatrix} L(3/2, h) \\ L(3/2, h_{1,q}) L(3/2, h_{1,r}) \end{pmatrix}$$

if $h = h_{1,s}$ for

$$s \in \{q + r - 1, q + r - 3, \dots, q - r + 1\} \cap \{r + q - 1, r + q - 3, \dots, r - q + 1\},$$

i.e., $s \in \{q + r - 1, r + q - 3, \dots, |q - r| + 1\}$. \square

Theorem 10.2. *Suppose that $q \geq r$.⁵*

$$\dim I \begin{pmatrix} L(3/2, h_{1,s}) \\ L(3/2, h_{1,q}) L(3/2, h_{1,r}) \end{pmatrix}_{\text{even}} = 1 \tag{54}$$

if and only if $s \in \{q + r - 1, q + r - 5, \dots, q - r + 1\}$;

$$\dim I \begin{pmatrix} L(3/2, h_{1,s}) \\ L(3/2, h_{1,q}) L(3/2, h_{1,r}) \end{pmatrix}_{\text{odd}} = 1 \tag{55}$$

if and only if

$$s \in \{q + r - 3, q + r - 7, \dots, q - r + 3\}.$$

Proof. By using (52) we obtain the following decomposition:

$$\begin{aligned} & A^0 \left(L \left(\frac{3}{2}, h_{1,q} \right) \right) \otimes_{A(L(3/2,0))} L \left(\frac{3}{2}, h_{1,r} \right) (0) \\ & \cong \mathbb{C}v_{q+r-1} \oplus \mathbb{C}v_{q+r-5} \oplus \dots \oplus \mathbb{C}v_{q-r+1}, \\ & A^1 \left(L \left(\frac{3}{2}, h_{1,q} \right) \right) \otimes_{A(L(3/2,0))} L \left(\frac{3}{2}, h_{1,r} \right) (0) \\ & \cong \mathbb{C}v_{q+r-3} \oplus \mathbb{C}v_{q+r-7} \oplus \dots \oplus \mathbb{C}v_{q-r+3}, \end{aligned} \tag{56}$$

where $\mathbb{C}v_i$ is a $\mathbb{C}[y]$ -module such that

$$y.v_i = \frac{(i-1)^2}{8} v_i.$$

⁵ $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix} \cong \begin{pmatrix} W_3 \\ W_2 W_1 \end{pmatrix}.$

Claim. *Let*

$$\psi \in \text{Hom}_{A(L(c,0))} \left(A^0 \left(L \left(\frac{3}{2}, h_{1,q} \right) \right) \otimes_{A(L(3/2,0))} L \left(\frac{3}{2}, h_{1,r} \right) (0), \right. \\ \left. L \left(\frac{3}{2}, h_{1,s} \right) (0) \right),$$

then the corresponding intertwining operator is even. Similarly if we start from

$$\psi \in \text{Hom}_{A(L(c,0))} \left(A^1 \left(L \left(\frac{3}{2}, h_{1,q} \right) \right) \otimes_{A(L(3/2,0))} L \left(\frac{3}{2}, h_{1,r} \right) (0), \right. \\ \left. L \left(\frac{3}{2}, h_{1,s} \right) (0) \right),$$

the corresponding intertwining operator is odd.

Proof (of the Claim). Let us elaborate the proof when ψ is “even.” From the construction in [FZ,L2] \mathcal{Y} is obtained by lifting ψ to a mapping from $L(3/2, h_{1,q}) \otimes L(3/2, h_{1,r})(0)$ to $L(3/2, h_{1,s})(0)$, such that

$$L(3/2, h_{1,q})_{\text{odd}} \otimes L(3/2, h_{1,r})(0) \mapsto 0.$$

To extend this map to a mapping $L(3/2, h_{1,q}) \otimes M(3/2, h_{1,r}) \rightarrow M(3/2, h_{1,s})'$ one uses generators and PBW so the sign is preserved. Because the isomorphism

$$I \begin{pmatrix} W_3 \\ W_1 \ W_2 \end{pmatrix} \cong I \begin{pmatrix} W'_2 \\ W_1 \ W'_3 \end{pmatrix}$$

preserves the sign, i.e., odd intertwining operators are mapped into odd and even into even, the result follows from the construction of intertwining operators. When ψ is odd a similar argument works. \square

Let us summarize everything.

Corollary 10.1. *Let \mathcal{A}_s be a free abelian group with generators $b(m)$, $m \in 2\mathbb{N} + 1$. Define a binary operation $\times : \mathcal{A}_s \times \mathcal{A}_s \rightarrow \mathcal{A}_s$,*

$$b(q) \times b(r) = \sum_{j \in \mathbb{N}} \dim I \begin{pmatrix} L(3/2, h_{1,j}) \\ L(3/2, h_{1,q}) \ L(3/2, h_{1,r}) \end{pmatrix} b(j).$$

Then \mathcal{A}_s is a commutative associative ring, and the mapping $b(m) \mapsto V((m - 1)/4)$ gives an isomorphism to the representation ring $\text{Rep}(\mathfrak{osp}(1|2))$.

Proof. The proof follows from Theorem 10.1(c) and (29). \square

11. Multiplicity 2 fusion rules and super logarithmic intertwiners

11.1. A multiplicity 2 case

We have seen that in the $c = 3/2$ case all fusion coefficients are 0 or 1. Still, we expect (according to [HM]) that for some vertex operator superalgebras $L(c, 0)$, fusion coefficients are 2.

Here is one example. If $c = 0$, as in the case of the Virasoro algebra, the vertex operator superalgebra

$$L(0, 0) = \frac{M(0, 0)}{\langle G(-1/2)v_0, G(-3/2)v_0 \rangle}$$

is trivial. Still we can consider a vertex operator superalgebra

$$V(0, 0) := \frac{M(0, 0)}{\langle G(-1/2)v \rangle}.$$

Clearly, for every $h \in \mathbb{C}$, we have (all modules are considered to be $V(0, 0)$ -modules):

$$\dim I \left(\begin{matrix} L(0, 0) \\ L(0, h) \ L(0, h) \end{matrix} \right) = 2. \tag{57}$$

The previous example is little bit awkward. Here is a nice example with irrational central charge.

Proposition 11.1.

$$\dim I \left(\begin{matrix} L\left(\frac{15}{2} - 3\sqrt{5}, \frac{\sqrt{5}}{2} - 1\right) \\ L\left(\frac{15}{2} - 3\sqrt{5}, \frac{3}{4}\left(\frac{\sqrt{5}}{2} - 1\right)\right) \ L\left(\frac{15}{2} - 3\sqrt{5}, \frac{3}{4}\left(\frac{\sqrt{5}}{2} - 1\right)\right) \end{matrix} \right) = 2. \tag{58}$$

Proof. It is not hard to see (by using a result from [AA] or [D]) that

$$M\left(\frac{15}{2} - 3\sqrt{5}, \frac{3}{4}\left(\frac{\sqrt{5}}{2} - 1\right)\right)$$

has the unique submodule that is irreducible (the case Π_+ in [AA]). If we analyze the determinant formula [KWa], singular vectors, and then use Theorem 9.1, we obtain (58). \square

11.2. A logarithmic intertwiner

In [M] we studied several examples of logarithmic intertwining operators. Roughly, logarithmic intertwiners exist if matrix coefficients yield some logarithmic solutions. Our analysis can be extended for vertex operator superalgebras:

$$\dim I \left(\begin{array}{c} W_2\left(\frac{27}{2}, \frac{-3}{2}\right) \\ L\left(\frac{27}{2}, \frac{-3}{2}\right) L\left(\frac{27}{2}, \frac{-3}{2}\right) \end{array} \right) = 2, \quad (59)$$

where $W_2(27/2, -3/2)$ is certain logarithmic module (cf. [M]). The proof of this result and the discussion will appear in a separate publication.

12. Future work and open problems

- We know that it is possible to obtain *intertwining operator algebras* (see [H2]) from the rational vertex operator algebras (satisfying some natural convergence and extension condition and an additional condition involving generalized modules). Since the notation of intertwining operator algebra can be (obviously) generalized such that fusion algebra is an infinite-dimensional associative, commutative algebra, one hopes that it is possible to construct tensor categories for degenerate minimal models. In the language of conformal field theory this involves explicit calculations of correlation functions for both products and iterates of intertwining operators (cf. Remark 4.2).
- Open problem: For rational vertex operator algebras, construct a *canonical* isomorphism

$$A(M_1) \otimes_{A(V)} M_2(0) \cong A(M_2) \otimes_{A(V)} M_1(0).$$

- ($N = 1$ case) For which triples $L(c, h_1)$, $L(c, h_2)$, and $L(c, h_3)$ do we have

$$\dim I \left(\begin{array}{c} L(c, h_3) \\ L(c, h_1) L(c, h_2) \end{array} \right) = 2?$$

- Determine the fusion ring for degenerate minimal models for $N = 2$ superconformal algebra by using our method (it should be related to $\text{Rep}(\mathfrak{osp}(2|2))$).
- Construct an analogue of the vertex tensor categories constructed in [HM] (by using the main result in [A]), for the models studied in this paper.

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