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Groups of Cohomological Dimension One

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Stallings [29] has recently shown that all finitely generated groups of cohomological dimension one are free. His methods are a remarkable combination of topological and algebraic ideas. Unfortunately, they do not seem to generalize to infinitely generated groups. However, by using Stallings' results for the finitely generated case, I have been able to solve the general problem.

If R is a ring with unit which is nontrivial (i.e., $R \neq 0$), we write $\operatorname{cd}_R G \leqslant n$ if $H^i(G, M) = 0$ for all $i > n$ and all RG-modules M.

THEOREM A. *Let R be a nontrivial ring with unit and let G be a torsion-free group. If* $\operatorname{cd}_R G \leqslant 1$ *then G is free.*

We do not even need R to be commutative here. However, by Proposition 3.3 below, we can immediately reduce to the case where R is a prime field. Throughout this paper, R will be any nontrivial ring with unit.

If $R = \underline{Z}$, we write cd for cd_R. Now, $\operatorname{cd} G < \infty$ implies that G is torsion free ([29], 6.6).

COROLLARY. *If* $\operatorname{cd} G \leqslant 1$, *then G is free.*

Stallings [29] also gives an affirmative answer to a question of Serre [24] for the case of finitely generated groups. Using Theorem A, we can now settle the general case.

THEOREM B. *Let G be a torsion-free group. If G has a free subgroup of finite index, then G is free.*

By Shapiro's Lemma ([3], Prop. 7.4), we know that if $H \subset G$ and $\operatorname{cd}_R G \leqslant n$, then $\operatorname{cd}_R H \leqslant n$. This will often be used without further comment.

I will use the categorical notation $G \coprod_A H$ for the free product of G and H with amalgamated subgroup A, but will stick to the old notation $G * H$ and

$* G_i$ for ordinary free products to avoid confusion if the groups happen to be abelian.

1. Free Differential Calculus

In [*18*] Lyndon gives a method of constructing the first three terms of a free resolution of $\underline{Z}$ over the group ring $\underline{Z}G$ of a group G given by generators and relations. In this section I will give a brief and less computational account of these results. The construction of the free derivatives can be generalized to a result on nonabelian cohomology. If a group G acts by automorphism on a (not necessarily abelian) group A, we recall ([*22*], Ch. VII) that $Z^1(G, A)$ is defined to be the set of functions $f : G \to A$ satisfying the cocycle condition $f(\sigma\tau) = f(\sigma)\,\sigma(f(\tau))$. The set $H^1(G, A)$ is the quotient of $Z^1(G, A)$ by the equivalence relation $f \sim g$ if and only if there is an element $a \in A$ such that $g(\sigma) = a^{-1}f(\sigma)\,\sigma(a)$ for all $\sigma \in G$.

If S is any set, let $F(S, A)$ be the set of all functions from S to A. If $S \subset G$ is a set of generators for G, it is easy to see that the restriction map $Z^1(G, A) \to F(S, A)$ is injective. In fact the cocycle relation implies that $f(1) = 1$ and $f(\sigma^{-1}) = (\sigma^{-1}f(\sigma))^{-1}$. Thus $f \mid S$ determines f on the generators and their inverses and hence, by induction on the length of the words, on all products of these. In the case where G is free, we have a stronger result.

Lemma 1.1. *If G is a free group with base S, then $Z^1(G, A) \to F(S, A)$ is bijective.*

Proof. We define an inverse map. Let $f : S \to A$. Set $f(1) = 1$ and $f(\sigma^{-1}) = (\sigma^{-1}f(\sigma))^{-1}$ for $\sigma \in S$. For each sequence $\sigma_1, \ldots, \sigma_n$ where each $\sigma_i \in S \cup \{1\} \cup S^{-1}$, define $f(\sigma_1, \ldots, \sigma_n) = f(\sigma_1, \ldots, \sigma_{n-1})\,\sigma_1 \cdots \sigma_{n-1}(f(\sigma_n))$. Then f satisfies

$$f(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n) = f(\sigma_1, \ldots, \sigma_m)\,\sigma_1 \cdots \sigma_m(f(\tau_1, \ldots, \tau_n)). \tag{1}$$

We claim that $f(\sigma_1, \ldots, \sigma_n)$ depends only on the product $\sigma_1 \ldots \sigma_n$. Since G is free on S, it will suffice to show that $f(\sigma_1, \ldots, \sigma_n)$ is unchanged by the insertion or deletion of terms 1 or σ, σ^{-1} or $\sigma^{-1}\sigma$. Using (1) to isolate the offending terms, e.g.

$$\begin{aligned} &f(\sigma_1, \ldots, \sigma_r, \sigma, \sigma^{-1}, \sigma_{r+1}, \ldots, \sigma_n) \\ &\quad = f(\sigma_1, \ldots, \sigma_r) \cdot \sigma_1 \cdots \sigma_r f(\sigma, \sigma^{-1}) \cdot \sigma_1 \cdots \sigma_r \sigma\sigma^{-1} f(\sigma_{r+1}, \ldots, \sigma_n), \end{aligned}$$

this reduces to showing $f(1) = f(\sigma, \sigma^{-1}) = f(\sigma^{-1}, \sigma) = 1$ which is clear from the definition. Set $f(\sigma_1, \ldots, \sigma_n) = g(\sigma)$. It follows from (1) that g is a cocycle. The map $F(S, A) \to Z^1(G, A)$ by $f \to g$ is clearly an inverse for the restriction map.

Remark. If G is defined by a set of generators S with relations $R_\alpha = 1$, it is easy to see that $Z^1(G, M)$ maps onto the subset of $F(S, M)$ consisting of those f such that the extended f just defined satisfies $f(R_\alpha) = 1$ for all α.

COROLLARY 1.2. *If G is free and $h : A \to B$ is a surjective G-homomorphism, $h_* : H^1(G, A) \to H^1(G, B)$ is also surjective.*

In fact, $F(S, A) \to F(S, B)$ is clearly surjective and therefore so is $Z^1(G, A) \to Z^1(G, B)$.

If A and B are abelian, the assertion of Corollary 1.2 is easily seen to be equivalent to the assertion cd $G \leqslant 1$. The more general assertion of Corollary 1.2 is considerably stronger than this (but is, of course, equivalent to it by Theorem A). In fact, this assertion trivially implies that G is free. We merely map a free group F onto G. Let G act trivially on F and G. Then $H^1(G, F)$ is the set of homomorphisms of G into F modulo inner automorphisms of F and similarly for $H^1(G, G)$. Thus we can lift the identity map $G \to G$ to a map $G \to F$ which identifies G with a subgroup of F.

Now let F be free on a set of generators $\{x_\alpha\}$. Let G act on RG by left multiplication, R being a commutative ring with unit. By Lemma 1.1, there is a unique cocycle $D_\alpha : G \to RG$ such that $D_\alpha x_\beta = 0$ for $\beta \neq \alpha$ and $D_\alpha x_\alpha = 1$. This D_α is denoted by $\partial/\partial x_\alpha$. It is trivial to check that $D(x) = x - 1$ for $x \in G$ is also a cocycle. Lemma 1.1 shows that

$$x - 1 = \sum \frac{\partial x}{\partial x_\alpha}(x_\alpha - 1) \tag{2}$$

since both sides are cocycles and they agree on the base $\{x_\alpha\}$.

For any group G, we let $I_G = I(G)$ denote the kernel of the augmentation $\epsilon : RG \to R$.

PROPOSITION 1.3. *If F is free on $\{x_\alpha\}$ then I_F as a left ideal of RF is a free module with base $x_\alpha - 1$.*

Proof. Let M be RF-free on a base e_α. Map $M \to I_F$ by $e_\alpha \mapsto x_\alpha - 1$ and $I_F \to M$ by $x - 1 \mapsto \sum (\partial x/\partial x_\alpha)\, e_\alpha$. Using (2) these maps are easily seen to be inverse isomorphisms.

The application of the free differential calculus to the construction of a resolution is given by the following result.

THEOREM 1.4. *Let F be free on $\{x_\alpha\}$. Let $G = F/N$ where $N \lhd F$. Then there is an exact sequence of $\underline{Z}G$-modules*

$$0 \to N/[N, N] \xrightarrow{\theta} X \xrightarrow{\partial} \underline{Z}G \xrightarrow{\epsilon} \underline{Z} \to 0 \tag{3}$$

where X is $\underline{Z}G$-free on a base e_α, $\partial e_\alpha = x_\alpha - 1$, and θ is given by

$$\theta(\bar{n}) = \sum \eta\left(\frac{\partial n}{\partial x_\alpha}\right) e_\alpha \tag{4}$$

where $\bar{n}$ denotes n $\mathrm{mod}[N, N]$ and η is the canonical map $\underline{Z}F \to \underline{Z}G$ induced by the quotient map $F \to G$.

The action of G on $N/[N, N]$ is, of course, given by $\sigma \cdot n = \overline{\tau n \tau^{-1}}$ where $\tau \in F$, $\tau \mapsto \sigma$.

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_N & \longrightarrow & \underline{Z}N & \longrightarrow & \underline{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & I_F & \longrightarrow & \underline{Z}F & \longrightarrow & \underline{Z} & \longrightarrow & 0, \end{array}$$

where the vertical arrows are inclusions. Taking homology with respect to N and using $H_1(N, \underline{Z}N) = 0 = H_1(N, \underline{Z}F)$, we get

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_1(N, Z) & \longrightarrow & I_N/I_N{}^2 & \xrightarrow{0} & \underline{Z} & \xrightarrow{\approx} & \underline{Z} & \longrightarrow & 0 \\ & & = \downarrow & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & H_1(N, Z) & \longrightarrow & I_F/I_N I_F & \longrightarrow & \underline{Z}G & \longrightarrow & \underline{Z} & \longrightarrow & 0. \end{array} \tag{5}$$

By Proposition 1.3, I_F is free over $\underline{Z}F$ on $\{x_\alpha - 1\}$. Therefore $X = I_F/I_N I_F$ is $\underline{Z}G$ free on the images e_α of the $x_\alpha - 1$. Now (5) gives the exact sequence

$$0 \to I_N/I_N{}^2 \to X \to \underline{Z}G \to \underline{Z} \to 0.$$

Now ([*3*], p. 190 (7)), $N/[N, N] \approx I_N/I_N{}^2$ by $n \leftrightarrow n - 1 \bmod I_N{}^2$. This gives the required sequence and (4) follows immediately from (2).

COROLLARY 1.5. *If R is any commutative ring, we have an exact sequence*

$$0 \to R \otimes_{\underline{Z}} N/[N, N] \xrightarrow{\theta} R \otimes_{\underline{Z}} X \xrightarrow{\psi} RG \xrightarrow{\epsilon} R \to 0 \tag{6}$$

and $R \otimes_{\underline{Z}} X$ is RG free with base $\{e_\alpha\}$.

This follows from the fact that $\underline{Z}$, $\underline{Z}G$, and X are $\underline{Z}$-free so (3) splits over $\underline{Z}$. As a consequence of Theorem 1.4 we get the resolution given by Lyndon ([*18*], §5). Let $\{R_\beta \in F\}$ be a set of defining relations for G. Then N is the normal closure in F of the subgroup generated by the R_β. Therefore $N/[N, N]$ is generated as a $\underline{Z}G$-module by the $\bar{R}_\beta$. Therefore we have an exact sequence

$$Y \xrightarrow{\partial} X \xrightarrow{\partial} \underline{Z}G \to \underline{Z} \to 0, \tag{7}$$

where Y is a free $\underline{Z}G$-module on a base e'_β and $\partial e'_\beta = \sum \eta(\partial R_\beta/\partial x_\alpha)\, e_\alpha$. The kernel of $\partial : Y \to X$ is generated by the "relations between the relations," i.e., the relations which must be imposed on the e'_β to give a presentation of $N/[N, N]$ in terms of the generators $\bar{R}_\beta$. By tensoring (7) with R, we get a similar resolution for R over RG.

We conclude this section by recalling a converse to Proposition 1.3 given in [*6*].

PROPOSITION 1.6. (Bass–Nakayama): *Let R be a nontrivial commutative ring. Let G be a group and S a subset of G such that I_G is RG-free on the base $\{\sigma - 1\}$ for $\sigma \in S$. Then G is a free group with base S.*

Since [*6*] is now out of print, I will give a proof here. We begin with some general remarks.

Let H be a subgroup of G and let G/H be the set of left cosets of H in G. Let $R[G/H]$ be the free R-module on G/H and let $\epsilon : RG \to R[G/H]$ by $\epsilon(\sigma) = \sigma \bmod H$ for $\sigma \in G$. We obtain $R[G/H]$ from RG by identifying σh with σ for all $\sigma \in G$, $h \in H$. Since I_H is R-free on the elements $h - 1$ for $h \in H$, $h \neq 1$, we see that $\ker \epsilon = RG \cdot I_H$, i.e.,

$$0 \to RG \cdot I_H \to RG \to R[G/H] \to 0. \tag{8}$$

Note also that $\epsilon : RG \to R$ induces $\epsilon : R[G/H] \to R$ by sending all elements of G/H to 1.

We now turn to the proof of Proposition 1.6. Let H be the subgroup of G generated by S. The hypothesis implies $\underline{Z}G \cdot I_H = I_G$. Thus $\epsilon : R[G/H] \to R$ is an isomorphism. Therefore $H = G$, i.e., S generates G. Now let F be free on e_σ, $\sigma \in S$. The map $F \to G$ by $e_\sigma \to \sigma$ identifies G with a quotient F/N. In (5), the map $\partial : X \to I_G \subset RG$ sends the generators e_σ of X onto the free generators $\sigma - 1$ of I_G. Thus $\partial : X \approx I_G$ so $R \otimes_{\underline{Z}} N/[N, N] = 0$ by (5). But N is free so $N/[N, N]$ is free abelian. Since R is nontrivial, $N/[N, N] = 0$ and so $N = \{1\}$.

Remark 1.7. This argument also shows that if S is a subset of G and I_G is generated as a left ideal by the elements $\sigma - 1$, $\sigma \in S$, then G is generated by S.

2. MAYER-VIETORIS SEQUENCE

In ([*18*], §6), Lyndon gives without proof some results which imply the existence of a Mayer–Vietoris sequence for a free product with amalgamation.

Since this sequence plays a vital role in the proof of Theorem A, I will supply proofs for these results here. A Mayer–Vietoris sequence was obtained topologically by Stallings [27] but only for the case where the groups act trivially on the coefficient module.

Let A be a subgroup of a group G. Let

$$\cdots \to X_2 \to X_1 \to RA \xrightarrow{\epsilon} R \to 0 \tag{9}$$

be a projective resolution of R over RA. Tensoring (9) with RG over RA gives a projective resolution

$$\cdots \to X_2' \to X_1' \to RG \to R[G/A] \to 0 \tag{10}$$

for $R[G/A]$ over RG. I will denote the resolutions (9) and (10) by X and X' respectively. If M is an RG-module, $\mathrm{Hom}_{RG}(X', M) = \mathrm{Hom}_{RA}(X, M)$ so $H^i(\mathrm{Hom}_{RG}(X', M)) = H^i(A, M)$.

Since $\epsilon : RG \to R$ factors as $RG \to R[G/A] \to R$, we can obtain R from RG by factoring out an ideal containing the image of X_1'. Choosing generators, we can find a free RG-module Y_1' and a map $Y_1' \to RG$ so that

$$Y_1' \oplus X_1' \to RG \to R \to 0 \tag{11}$$

is exact. We can now find a free Y_2' so that $Y_2' \oplus X_2'$ maps onto the kernel in (11). Continuing in this way, we arrive at a resolution for R over RG of the shape

$$\begin{array}{ccccccccc} \cdots \longrightarrow & Y_2' & \longrightarrow & Y_1' & & & & & \\ \searrow & \oplus & \searrow & \oplus & \searrow & & & & \\ \cdots \longrightarrow & X_2' & \longrightarrow & X_1' & \longrightarrow & RG & \longrightarrow & R & \longrightarrow 0\,. \end{array} \tag{12}$$

I will denote this resolution by Y. If M is any RG-module, restriction to X' gives a natural map $\mathrm{Hom}_{RG}(Y, M) \to \mathrm{Hom}_{RG}(X', M) = \mathrm{Hom}_{RA}(X, M)$.This leads to a map of cohomological δ functors $H^*(G, M) \to H^*(A, M)$. In dimension 0 this is easily verified to be the inclusion $M^A \to M^G$. By the universal property ([9], 2.2, 2.3), we see that this map must be the usual restriction map, ([22], Ch. VII, §5).

Remark. We can obtain a standard resolution of the form (12) by taking the standard resolution Y ([3], Ch. X, §4) for G, letting X' be the subcomplex spanned by all $[\sigma_1, \ldots, \sigma_n]$ with all $\sigma_i \in A$, and letting Y' be spanned by all $[\sigma_1, \ldots, \sigma_n]$ with at least one $\sigma_i \notin A$.

Suppose now that A is also a subgroup of a group H. Repeating the above construction gives a resolution Z for R over RH of the form

$$\begin{array}{ccccccccc} \longrightarrow & Z_2'' & \longrightarrow & Z_1'' & & & & & \\ \searrow & \oplus & \searrow & \oplus & \searrow & & & & \\ \cdots\longrightarrow & X_2'' & \longrightarrow & X_1'' & \longrightarrow & RH & \longrightarrow & R & \longrightarrow 0 . \end{array} \tag{13}$$

Here $X'' = RH \otimes_{RA} X$.

Now let $K = G \coprod_A H$ be the free product of G and H with A amalgamated. Tensoring (12) and (13) with RK over RG and RH respectively gives us resolutions

$$\begin{array}{ccccccccc} \longrightarrow & \tilde{Y}_2' & \longrightarrow & \tilde{Y}_1' & & & & & \\ \searrow & \oplus & \searrow & \oplus & \searrow & & & & \\ \cdots\longrightarrow & \tilde{X}_2 & \longrightarrow & \tilde{X}_1 & \longrightarrow & RK & \longrightarrow & R[K/G] & \longrightarrow 0 \end{array} \tag{14}$$

and

$$\begin{array}{ccccccccc} \longrightarrow & \tilde{Z}_2' & \longrightarrow & \tilde{Z}_1' & & & & & \\ \searrow & \oplus & \searrow & \oplus & \searrow & & & & \\ \cdots\longrightarrow & \tilde{X}_2 & \longrightarrow & \tilde{X}_1 & \longrightarrow & RK & \longrightarrow & R[K/H] & \longrightarrow 0 \end{array} \tag{15}$$

which I will denote by $\tilde{Y}$ and $\tilde{Z}$ respectively. Note that in each case $\tilde{X} = RK \otimes_{RA} X$ is the same. As above, if M is an RK-module, we have $\mathrm{Hom}_{RK}(\tilde{Y}, M) = \mathrm{Hom}_{RG}(Y, M)$, $\mathrm{Hom}_{RK}(\tilde{Z}, M) = \mathrm{Hom}_{RH}(Z, M)$, and $\mathrm{Hom}_{RK}(\tilde{X}, M) = \mathrm{Hom}_{RA}(X, M)$.

Now map $\tilde{X}$ into $\tilde{Y} \oplus \tilde{Z}$ by sending x to $(x, -x)$ and define W to be the cokernel

$$0 \to \tilde{X} \to \tilde{Y} \oplus \tilde{Z} \to W \to 0. \tag{16}$$

Then W is obtained by identifying the two copies of $\tilde{X}$ in $\tilde{Y} \oplus \tilde{Z}$ and so is of the shape

$$\begin{array}{ccccccccc} \longrightarrow & \tilde{Y}_2 & \longrightarrow & \tilde{Y}_1 & & & & & \\ \searrow & \oplus & \searrow & \oplus & \searrow & & & & \\ \cdots\longrightarrow & \tilde{X}_2 & \longrightarrow & \tilde{X}_1 & \longrightarrow & RK & \longrightarrow & R & \longrightarrow 0 \\ \nearrow & \oplus & \nearrow & \oplus & \nearrow & & & & \\ \longrightarrow & \tilde{Z}_2 & \longrightarrow & \tilde{Z}_1 & & & & & \end{array} \tag{17}$$

I claim that W is a free resolution of R over RK. The sequence (16) leads to a Mayer–Vietoris sequence

$$\cdots \to H_1(\tilde{Y}) \oplus H_1(\tilde{Z}) \to H_1(W) \to H_0(\tilde{X}) \to H_0(\tilde{Y}) \oplus H_0(\tilde{Z}) \to H_0(W) \to 0.$$

This shows that $H_n(W) = 0$ for $n \geqslant 2$ and

$$0 \to H_1(W) \to R[K/A] \to R[K/G] \oplus R[K/H] \to H_0(W) \to 0. \tag{18}$$

To show that $H_1(W) = 0$ and $H_0(W) = R$, we need the following lemma

LEMMA 2.1. *The sequence*

$$0 \to R[K/A] \to R[K/G] \oplus R[K/H] \to R \to 0$$

is exact.

Proof. Choose a presentation for A with generators a_α and relations $R_{\alpha'} = 1$. Enlarge this to a presentation for G with generators a_α and g_β with relations $R_{\alpha'} = 1$ and $S_{\beta'} = 1$. Similarly, we get a presentation for H with generators a_α and h_γ with relations $R_{\alpha'} = 1$ and $T_{\gamma'} = 1$.

Apply the free differential calculus to these presentations as in Section 1. This gives part of a resolution for G of exactly the form (12) and one for H of the form (13). Now K is presented with generators a_α, g_β, h_γ and relations $R_{\alpha'} = 1$, $S_{\beta'} = 1$, $T_{\gamma'} = 1$. The free differential calculus gives part of a resolution W of exactly the form (17) and the maps are induced by those in (12) and (13). Since the sequence resulting from the application of the free differential calculus is automatically exact, this particular W has $H_1(W) = 0$ and $H_0(W) = R$. Substitution in (18) now yields the required result.

COROLLARY 2.2. *The complex W is a free resolution for R over RK.*

This resolution is exactly the one obtained by Lyndon ([*18*], §6) in matrix form.

Now let M be an RK-module. Apply $\text{Hom}_{RK}(-, M)$ to (16). This gives an exact sequence

$$\begin{aligned} 0 \to \text{Hom}_{RK}(W, M) &\to \text{Hom}_{RG}(Y, M) \oplus \text{Hom}_{RH}(Z, M) \\ &\to \text{Hom}_{RA}(X, M) \to 0 \end{aligned} \tag{19}$$

using the identifications observed above. Taking the exact cohomology sequence now yields the main result of this section. A similar result for homology is obtained by applying $- \otimes_{RG} M$ to (16).

THEOREM 2.3. *Let $K = G \coprod_A H$ be a free product with amalgamated* *subgroup A. Let M be any RK-module. Then there are exact Mayer–Vietoris sequences*

$$\cdots \to H^n(K, M) \xrightarrow{(\mathrm{res},\mathrm{res})} H^n(G, M) \oplus H^n(H, M) \xrightarrow{(\mathrm{res},-\mathrm{res})} H^n(A, M) \xrightarrow{\delta} H^{n+1}(K, M) \to \cdots$$

and

$$\cdots \to H_n(A, M) \xrightarrow{(\mathrm{cor},-\mathrm{cor})} H_n(G, M) \oplus H_n(H, M) \xrightarrow{(\mathrm{cor},\mathrm{cor})} H_n(K, M) \xrightarrow{\partial} H_{n-1}(A, M) \to \cdots .$$

These are, of course, natural in all reasonable senses.

3. A Remark on Stallings' Result

The purpose of this section is to point out that Stallings' proof [29] of his Theorem 0.3 actually yields a somewhat stronger result.

We begin by pointing out some consequences of the condition $\mathrm{cd}_R\, G \leqslant n$. The proofs of 3.1, 3.2, 3.3 are arranged so that they apply even to noncommutative rings.

Lemma 3.1. *If $R \to R'$ is a ring homomorphism and $\mathrm{cd}_R\, G \leqslant n$, then $\mathrm{cd}_{R'}\, G \leqslant n$.*

This is immediate since any $R'G$-module can be regarded as an RG-module.

Lemma 3.2. *If $R \neq 0$ is a ring with unit, then at least one of the rings $\underset{\sim}{Q} \otimes_Z R$ and $\underset{\sim}{Z}_p \otimes_{\underset{\sim}{Z}} R = R/pR$ (for prime p) is nontrivial.*

Proof. If $\underset{\sim}{Q} \otimes_{\underset{\sim}{Z}} R = 0$, the additive group of R is a torsion group. Therefore $n \cdot 1 = 0$ for some integer $n \neq 0$. If $R/pR = 0$ for all p, the additive group of R is divisible. If $x \in R$, we can write $x = ny$ for some y. Thus $x = ny = (n \cdot 1)y = 0y = 0$.

Proposition 3.3. *If $\mathrm{cd}_R\, G \leqslant n$ for some nontrivial ring R with unit, then $\mathrm{cd}_K\, G \leqslant n$ for some prime field K.*

Proof. By Lemma 3.2, there is a prime field K with $R' = K \otimes_{\underset{\sim}{Z}} R \neq 0$. By Lemma 3.1, $\mathrm{cd}_{R'}\, G \leqslant n$. Since R' has a unit $1 \neq 0$, we can identify K with $K \cdot 1 \subset R'$ and write $R' = K \oplus V$ as a K-module. If M is a KG-module, $R' \otimes_K M$ is an $R'G$-module so $H^i(G, R' \otimes_K M) = 0$ for $i > n$. But $R' \otimes_K M = M \oplus V \otimes_K M$ as a KG-module so $H^i(G, M) = 0$ for $i > n$.

We now give a slight generalization of Stallings' theorem.

Theorem 3.4. *Let R be a nontrivial ring with unit. Let G be a finitely generated torsion-free group. If $\mathrm{cd}_R\, G \leqslant 1$, then G is free.*

Proof. By Proposition 3.3, there is a field K with $\mathrm{cd}_K G \leqslant 1$. As in ([*29*], 6.4), we see that the kernel of $\epsilon : KG \to K$ is finitely generated and projective. Then, just as in ([*29*], 6.7) we show that $H^1(G, KG) \neq 0$. If we had the same result with K replaced by $\underline{Z}_2$, the argument of ([*29*], 6.8) could be applied to show that G is free. There are two other points to be noted. First, it is unnecessary to show that G is almost finitely presented because of Bergman's result [*2*]. Second, we need G to be torsion free. This does not follow from $\mathrm{cd}_K G \leqslant 1$ and so must be assumed. Therefore, all that remains is to verify the following lemma.

LEMMA 3.5. *If G is any finitely generated group, then* $\dim_K H^1(G, KG)$ *is the same for all fields K.*

Proof. This follows from the topological interpretation of this dimension as $e - 1$ where e is the number of ends of G [*26*]. Here is an algebraic version of this proof. Choose a finite set of generators $\sigma_1, \ldots, \sigma_k$ for G. If A is a subset of G, a path in A from x to y will mean a sequence $x_1, \ldots, x_r \in A$ with $x_1 = x$, $x_r = y$ such that $x_{i+1} = x_i \sigma_{\nu_i}^{\epsilon_i}$, $\epsilon_i = \pm 1$ for each $i = 1, \ldots, r - 1$. Write $x \sim_A y$ if such a path exists. This is an equivalence relation. Its equivalence classes are called the components of A. If S is a finite subset of G, and $x \in G - S$, we can join x to 1 by a path $x_1, \ldots, x_r$ in G. If it meets S, let i be minimal with $x_{i+1} \in S$. Then $x_1, \ldots, x_i$ is a path in $G - S$. It follows that $G - S$ has a finite number of components determined by 1 and some of the elements $s\sigma_\nu^{\pm 1}$, $s \in S$. Let $G = \{g_1, g_2, \ldots\}$. Let $S_0 = \varnothing$. If S_n is defined let $S'_{n+1} = S_n \cup \{g_{n+1}\}$ and let S_{n+1} be the union of S'_{n+1} with all finite components of $G - S'_{n+1}$. The S_n are finite, $S_n \subset S_{n+1}$, and $G = \cup S_n$. Let C_n be the set of components of $G - S_n$. Each component of $G - S_{n+1}$ lies in a unique component of $G - S_n$. This defines a map $C_{n+1} \to C_n$ which is onto since all components of $G - S_n$ are infinite. We can now define the space of ends of G to be $\varprojlim C_n$, a compact totally disconnected space. However, we will instead proceed algebraically.

As in ([*29*], §3), let F be the set of functions from G to R with G acting by $\sigma f \cdot (x) = f(x\sigma)$. Let $\Phi \subset F$ consist of all functions with $f(x) = 0$ for almost all x (i.e., all but a finite number). Since $F = \mathrm{Hom}_{\underline{Z}}(\underline{Z}G, R)$, we have $H^1(G, F) = H^1(\{1\}, R) = 0$ by Shapiro's lemma ([3], Ch. X, Prop. 7.4) cf ([*29*], §3). Therefore, the cohomology sequence of $0 \to \Phi \to F \to F/\Phi \to 0$ gives

$$0 \to \Phi^G \to R \to E(G, R) \to H^1(G, RG) \to 0 \tag{20}$$

where $E(G, R) = (F/\Phi)^G$. Note $\Phi^G = R$ if G is finite and otherwise $\Phi^G = 0$. By (20), it will suffice to show $\dim_K E(G, K)$ is independent of K. This follows immediately from the next lemma (cf [*26*]).

LEMMA 3.6. *$E(G, R) = R \otimes_{\underline{Z}} E(G, \underline{Z})$ and $E(G, \underline{Z})$ is free abelian of rank $\leqslant \aleph_0$.*

Proof. Let $E_n(G, R)$ be the set of functions from C_n to R. The surjection $C_{n+1} \to C_n$ defines a map $E_n(G, R) \to E_{n+1}(G, R)$ which is a monomorphism onto a direct summand. Now $E(G, R) = H/\Phi$ where H is the set of functions f from G to R with $f(x\sigma) - f(x) = 0$ for almost all x for each $\sigma \in G$. It is sufficient to require this only for $\sigma = \sigma_1, \ldots, \sigma_k$ since these generate G. Define $\varphi_n : E_n(G, R) \to E(G, R)$ by sending f to f' where $f'(x) = f(c)$ if x lies in the component c of $G - S_n$ and $f'(x) = 0$ if $x \in S_n$. This is well-defined and gives a map $\varphi : \varinjlim E_n(G, R) \to E(G, R)$. If $f \in E(G, R)$, then $f(x\sigma_\nu) = f(x)$ for all ν and all $x \in G - S$ with S finite. For large n, $S \subset S_n$. Thus f is constant on the components of $G - S_n$. Therefore φ is onto. If $f \in E_n(G, R)$ and $\varphi_n(f) = 0$, then $f'(x) = 0$ for $x \in G - S$ with S finite. For large m, $S \subset S_m$. Thus $f' \to 0$ in $E_m(G, R)$ so φ is injective. Now $E_n(G, R) = R \otimes E_n(G, \underline{Z})$ trivially. Taking limits gives $E(G, R) = R \otimes E(G, \underline{Z})$. Also $E_{n+1}(G, \underline{Z}) = E_n(G, \underline{Z}) \oplus X_n$ where X_n is finitely generated free abelian. Clearly $E(G, \underline{Z}) = \coprod X_n$.

COROLLARY 3.7. *$H^1(G, RG) = R \otimes_{\underline{Z}} H^1(G, \underline{Z}G)$ and $H^1(G, \underline{Z}G)$ is free abelian of rank $\leqslant \aleph_0$.*

Proof. This is trivial if G is finite since then $H^1(G, RG) = 0$. Assume G is infinite. Then $C_n \neq \varnothing$ for all n. Let $c \in \varprojlim C_n$ so $c = (c_n)$, $c_n \in C_n$, $c_{n+1} \to c_n$. Define $E_n(G, R) \to R$ by evaluation at c_n. These maps define a map $E(G, R) = \varinjlim E_n(G, R) \to R$ which is clearly a left inverse for $R \to E(G, R)$. Therefore, by (20), we have an isomorphism $E(G, R) \approx R \oplus H^1(G, RG)$ which is natural in R. Therefore it follows from Lemma 3.6 that $H^1(G, \underline{Z}G)$ is free abelian of rank $\leqslant \aleph_0$. The first statement follows from the diagram

$$\begin{array}{ccccccc} R \otimes_{\underline{Z}} \underline{Z} & \longrightarrow & R \otimes_{\underline{Z}} E(G, \underline{Z}) & \longrightarrow & R \otimes_{\underline{Z}} H^1(G, \underline{Z}G) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ R & \longrightarrow & E(G, R) & \longrightarrow & H^1(G, RG) & \longrightarrow & 0. \end{array}$$

4. SOME ELEMENTARY LEMMAS

In this section we collect a few simple results which will be useful in Sections 5 and 7. As usual R will be any nontrivial commutative ring with unit.

LEMMA 4.1. *Let H be a subgroup of a group G. Then there are natural isomorphisms $H^n(G, M) \approx \mathrm{Ext}^n_{RG}(R, M)$, $H^n(H, M) \approx \mathrm{Ext}^n_{RG}(R[G/H], M)$ where G/H is the space of left cosets. Under these isomorphisms, the restriction map* res: $H^n(G, M) \to H^n(H, M)$ *corresponds to the map*

$$\mathrm{Ext}^n_{RG}(R, M) \to \mathrm{Ext}^n_{RG}(R[G/H], M)$$

induced by the map $\epsilon : R[G/H] \to R$ sending all elements of G/H to 1.

Proof. This follows immediately from the universal property ([9], 2.2, 2.3) [20]. In fact $H^n(H, M)$ and $\mathrm{Ext}^n_{RG}(R[G/H], M)$ are cohomological δ-functors on the category of RG-modules. Clearly

$$H^0(H, M) = M^G \approx \mathrm{Hom}_{RG}(R[G/H], M).$$

Therefore the δ-functors are isomorphic. The map induced by ϵ is a map of δ functors which in dimension 0 is given by $M^G \subset M^H$. Therefore, it must coincide with the restriction map ([22], Ch. VII, §5).

As usual, we denote I_H the augmentation ideal $\ker[\epsilon : RH \to R]$.

LEMMA 4.2. *Let H be an infinite subgroup of a group G. Let $J = RGI_H \subset RG$. The following conditions are equivalent.*

(a) res : $H^1(G, RG) \to H^1(H, RG)$ *is a monomorphism.*

(b) *The map* $\mathrm{Hom}_{RG}(I_G, RG) \to \mathrm{Hom}_{RG}(J, RG)$ *induced by the inclusion $J \subset I_R$, is a monomorphism.*

(c) $\mathrm{Hom}_{RG}(I_G/J, RG) = 0$.

Proof. It is clear that b and c are equivalent by the left exactness of Hom and the sequence $0 \to J \to I_G \to I_G/J \to 0$. Consider now the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & RG & \longrightarrow & R[G/H] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \epsilon} & & \\ 0 & \longrightarrow & I_G & \longrightarrow & RG & \longrightarrow & R & \longrightarrow & 0. \end{array} \qquad (21)$$

Since H is infinite, $\mathrm{Hom}_{RG}(R[G/H], RG) = (RG)^H = 0$ and

$$\mathrm{Hom}_{RG}(R, RG) = (RG)^G = 0.$$

Applying $\mathrm{Ext}^*_{RG}(-, RG)$ to (21) thus gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & RG & \longrightarrow & \mathrm{Hom}_{RG}(J, RG) & \longrightarrow & \mathrm{Ext}^1_{RG}(R[G/H], RG) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle \epsilon^*} & & \\ 0 & \longrightarrow & RG & \longrightarrow & \mathrm{Hom}_{RG}(I_G, RG) & \longrightarrow & \mathrm{Ext}^1_{RG}(R, RG) & \longrightarrow & 0. \end{array} \qquad (22)$$

Thus i^* is a monomorphism if and only if ϵ^* is. Use Lemma 4.1 to identify ϵ^* with res.

LEMMA 4.3. *If H is a subgroup of G and I is any left ideal of RH, then $RG \otimes_{RH} I \to RGI$ is an isomorphism.*

Proof. Since RG is free and therefore flat as a right RH-module, applying $RG \otimes_{RH}$ to $0 \to I \to RH$ gives $0 \to RG \otimes_{RH} I \to RG$ with image RGI.

LEMMA 4.4. *Let G be a finitely generated free group and H a finitely generated subgroup of G. Let $J = RGI_H$. Then the map*

$$\mathrm{Hom}_{RG}(I_G, RG) \to \mathrm{Hom}_{RG}(J, RG),$$

induced by the inclusion $J \subset I_G$, is not an isomorphism unless $H = G$.

Proof. By Proposition 1.3, I_H is free and finitely generated as an RH-module. Therefore, by Lemma 4.3, the same is true of J as an RG-module. Also I_G is free and finitely generated as an RG-module. If

$$\mathrm{Hom}_{RG}(I_G, RG) \to \mathrm{Hom}_{RG}(J, RG)$$

is an isomorphism, we apply $\mathrm{Hom}_{RG}(-, RG)$ to it. As in ([*29*], 6.1, 6.2) this produces the original map $J \to I_G$ which is therefore an isomorphism, i.e., $J = I_G$. Now $RG/J = R[G/H]$ so $\epsilon : R[G/H] \to R = RG/I_G$ is an isomorphism. This shows that G/H can only have one element.

LEMMA 4.5. *Let $H \subset K$ be subgroups of a group G. If*

$$\mathrm{res} : H^1(G, M) \to H^1(H, M)$$

is surjective for all RG-modules M, then $\mathrm{res} : H^1(K, N) \to H^1(H, N)$ *is surjective for all RK-modules N.*

Proof. Let $M = \mathrm{Hom}_{RK}(RG, N)$ with G acting by $\sigma f \cdot (x) = f(x\sigma)$. Define RK-homomorphisms $\nu : N \to M$, $\mu : M \to N$ by $\nu(n) \cdot (x) = xn$ and $\mu(f) = f(1)$. Then $\mu\nu = 1_N$ so $M \approx N \oplus N'$ as an RK-module. Since M is an RG-module, the composition $H^1(G, M) \to H^1(K, M) \to H^1(H, M)$ is surjective and therefore so is $H^1(K, M) \to H^1(H, M)$ but this is the direct sum of $H^1(K, N) \to H^1(H, N)$ and $H^1(K, N') \to H^1(H, N')$ so these maps must also be surjective.

LEMMA 4.6. *If H is a subgroup of a group G, the following conditions are equivalent.*

(a) $\mathrm{res} : H^1(G, M) \to H^1(H, M)$ *is surjective for all RG-modules M.*

(b) $J = RGI_H$ *is a direct summand of I_G.*

Proof. Consider the diagram (21). Applying $\mathrm{Ext}^*_{RG}(-, M)$ to it gives

$$\begin{array}{ccccccccc} 0 \longrightarrow & M^H & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_{RG}(J, M) & \longrightarrow & H^1(H, M) & \longrightarrow 0 \\ & \uparrow & & \uparrow = & & \uparrow i^* & & \uparrow \mathrm{res} & \\ 0 \longrightarrow & M^G & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_{RG}(I_G, M) & \longrightarrow & H^1(G, M) & \longrightarrow 0, \end{array} \qquad (23)$$

where we have used Lemma 4.1 to identify ϵ^* with res. If J is a direct summand of I_G, then i^* is clearly onto and hence so is res. If res is onto, then so is i^*. Applying this to $M = J$ shows that $1_J = i^*(f)$ so $f : I_G \to J$ is a retraction of I_G on J.

5. A Relative Version of Stallings' Theorem

As usual R will be a nontrivial commutative ring with unit. If G is a finitely generated torsion free group and $H^1(G, RG) \neq 0$, it follows from Corollary 3.7 that $H^1(G, \underline{Z}_2G) \neq 0$. Therefore, by Stallings' main theorem ([*29*], 0.1) with Bergman's improvement [*2*], G is either $\underline{Z}$ or a nontrivial free product. Using this, I will prove the following relative version of this theorem.

Theorem 5.1. *Let G be a finitely generated torsion free group and let H be a subgroup of G. If* res : $H^1(G, RG) \to H^1(H, RG)$ *is not a monomorphism, then H is contained in a proper free factor of G.*

In other words, there is a free product decomposition $G = G_1 * G_2$ with $H \subset G_1$ and $G_1 \neq G$.

Proof. Since is countable and torsion free, it can be embedded in a finitely generated torsion free group H_1 [*12*]. There is a very simple embedding in a 3 generator group obtained by taking a free product of H with a free group and then forming a free product, amalgamating a subgroup, with another free group ([*21*], §20). Let $K = H_1 \times \underline{Z} \times \underline{Z}$ and let $L = G \coprod_H K$ be the free product of G and K with the amalgamated subgroup H. Then L is torsion free since G and K are ([*17*], p. 32) [*21*] and L is finitely generated.

Since RL is free as an RG-module, we can write $RL = RG \oplus X$. The map $H^1(G, RL) \to H^1(H, RL)$ is thus the direct sum of $H^1(G, RG) \to H^1(H,RG)$ and $H^1(G, X) \to H^1(H, X)$. Since the first of these is not a monomorphism by hypothesis, neither is $H^1(G, RL) \to H^1(H, RL)$. By Theorem 2.3, wehave an exact sequence

$$H^1(L, RL) \to H^1(G, RL) \oplus H^1(K, RL) \to H^1(H, RL). \qquad (24)$$

Let $a \in H^1(G, RL)$ be a nonzero element with zero image in $H^1(H,RL)$.

The element $(a, 0)$ in the middle term of (24) is nonzero but has zero image in $H^1(H, RL)$, so the exactness of (24) shows that $H^1(L, RL) \neq 0$. By Stallings' theorem we see that either $L = \underset{\sim}{Z}$ or a nontrivial free product. Since $\underset{\sim}{Z} \times \underset{\sim}{Z} \subset K \subset L$, the case $L = \underset{\sim}{Z}$ is impossible. Therefore $L = L_1 * L_2$ with $L_1, L_2 \neq 1$. By the Kuroš subgroup theorem [*1*][*17*][*19*], $K \subset L$ is the free product of a free group F and of subgroups of the form $K \cap \sigma L_\nu \sigma^{-1}$, $\nu = 1$ or 2. But $K = H_1 \times \underset{\sim}{Z} \times \underset{\sim}{Z}$ is a nontrivial direct product and hence is freely indecomposable ([*1*], Folg. 4). Therefore either $K = F$ or $K = K \cap \sigma L_\nu \sigma^{-1}$ for some $\sigma \in G$, $\nu = 1$ or 2. But $\underset{\sim}{Z} \times \underset{\sim}{Z} \subset K$ so $K = F$ is impossible. Therefore, $K \subset \sigma L_\nu \sigma^{-1}$ for some σ, ν, say for $\nu = 1$. Applying the inner automorphism $x \to \sigma x \sigma^{-1}$ to L shows that $L = \sigma L_1 \sigma^{-1} * \sigma L_2 \sigma^{-1}$. Therefore, if we replace L_ν by $\sigma L_\nu \sigma^{-1}$ we have $L = L_1 * L_2$ and $K \subset L_1$. Now $G \subset L$ so by ([*1*], p. 395(b)] ([*25*], §2.2.2, Satz 8*), $G_1 = G \cap L_1$ is a free factor of G. Clearly, $H \subset G_1$. If G_1 is not a proper free factor of G, then $G_1 = G$ so $G, K \subset L_1$ but G and K generate L. Thus we would have $L_1 = L$, contradicting the choice of $L = L_1 * L_2$ as a nontrivial free factorization.

Using Lemma 4.2 we can reformulate this theorem as follows.

COROLLARY 5.2. *Let G be a finitely generated torsion free group and let H be a subgroup of G which is not contained in any proper free factor of G. Let $J = RGI_H$. Then the map* $\mathrm{Hom}_{RG}(I_G, RG) \to \mathrm{Hom}_{RG}(J, RG)$, *induced by the inclusion $J \subset I_G$, is a monomorphism.*

Note that H must be infinite otherwise $H = \{1\}$ which is certainly a proper free factor if $G \neq \{1\}$.

Using Lemma 4.4 we get the following consequence.

COROLLARY 5.3. *Let G be a finitely generated free group. Let H be a finitely generated proper subgroup of G which is not contained in any proper free factor of G. Let $J = RGI_H$. Then the map* $\mathrm{Hom}_{RG}(I_G, RG) \to \mathrm{Hom}_{RG}(J, RG)$, *induced by the inclusion $J \subset I_G$, is a monomorphism but not an epimorphism.*

6. THE COUNTABLE CASE

We now prove Theorem A under the additional hypothesis that G is countable. By Theorem 3.4, G is locally free. If G is countable but not free, it follows from a theorem of G. Higman ([*10*], Th. 1) that there is an ascending sequence of subgroups $\{1\} \neq G_1 \subset G_2 \subset \cdots$ such that each inclusion is proper, each G_n is free and finitely generated, and no G_n lies in any proper free factor of G_{n+1}. By replacing G by $\cup G_n$, it will suffice to obtain a contradiction under the additional assumption that $G = \cup G_n$. Let $J_n = RGI_{G_n}$. As in the proof of Lemma 4.4, it follows from Proposition 1.3 and Lemma 4.3 that

J_n is a free, finitely generated RG-module. Since $G = \cup G_n$, $I_G = \cup J_n$. We can now define a resolution for R over RG. Let $J = \coprod J_n$ be the direct sum of the J_n. Then J is free and we can define a resolution by

$$0 \to J \xrightarrow{\psi} J \xrightarrow{\theta} RG \to R \to 0, \tag{25}$$

where $\theta \mid J_n$ is the inclusion $J_n \subset RG$ and ψ is defined by $\psi(a_n) = (b_n)$ where $b_n = a_n - i(a_{n-1})$ where a_n is the J_n-component of (a_n), $a_0 = 0$, and i is the inclusion $J_n \subset J_{n+1}$. It is trivial to verify the exactness of (25).

Remark. Since we have not yet used the condition that G_n is not contained in a free factor of G_{n+1}, the argument just given shows that if G is any countable locally free group, then cd $G \leqslant 2$.

If M is any RG-module, we can calculate $H^2(G, M)$ using (25). If $u \in \mathrm{Hom}_{RG}(J, M) = \prod \mathrm{Hom}_{RG}(J_n, M)$, let u_n be its component in $\mathrm{Hom}_{RG}(J_n, M)$. The map $\mathrm{Hom}_{RG}(\psi, M)$ sends v into u where $u_n = v_n - \varphi(v_{n+1})$ and $\varphi : \mathrm{Hom}_{RG}(J_{n+1}, M) \to \mathrm{Hom}_{RG}(J_n, M)$ is induced by the inclusion $J_n \subset J_{n+1}$. Since $\mathrm{cd}_R\, G \leqslant 1$, $H^2(G, M) = 0$. Therefore, any sequence $u_n \in \mathrm{Hom}_{RG}(J_n, M)$ can be expressed in the form $u_n = v_n - \varphi(v_{n+1})$ where $v_n \in \mathrm{Hom}_{RG}(J_n, M)$.

We now apply this to the case $M = J$. Since J_n is finitely generated the functor $\mathrm{Hom}_{RG}(J_n, -)$ preserves direct sums. Therefore, $\mathrm{Hom}_{RG}(J_n, J) = \coprod \mathrm{Hom}_{RG}(J_n, J_i)$. If $v_n \in \mathrm{Hom}_{RG}(J_n, J)$, let $v_{n,i}$ be the component of v_n in $\mathrm{Hom}_{RG}(J_n, J_i)$. Note that for each n, $v_{n,i} = 0$ for almost all i. If $u_n = v_n - \varphi(v_{n+1})$, projecting on the ith summand shows that $u_{n,i} = v_{n,i} - \varphi(v_{n+1,i})$.

Choose u to be the identity map of J. Then u_n is the inclusion $J_n \subset J$ so $u_{n,i} = 0$ for $i \neq n$ and $u_{n,n}$ is the identity map 1_n of J_n. Let $u_n = v_n - \varphi(v_{n+1})$. Choose i so large that $v_{1,i} = 0$. Let $w_n = v_{n,i} \in \mathrm{Hom}_{RG}(J_n, J_i)$. Then $w_1 = 0$, $w_n - \varphi(w_{n+1}) = 0$ for $n \neq i$ and 1_i for $n = i$.

The fact that G_n is not contained in a proper free factor of G_{n+1} now permits us to apply Corollary 5.3. This shows that

$$\mathrm{Hom}_{RG_{n+1}}(I_{G_{n+1}}, RG_{n+1}) \to \mathrm{Hom}_{RG_{n+1}}(RG_{n+1}I_{G_n}, RG_{n+1})$$

is a monomorphism but not an epimorphism. Since $I_{G_{n+1}}$ and $RG_{n+1}I_{G_n}$ are finitely generated, $\mathrm{Hom}_{RG_{n+1}}(I_{G_{n+1}}, -)$ and $\mathrm{Hom}_{RG_{n+1}}(RG_{n+1}I_{G_n}, -)$ preserve direct sums. Since RG is a free RG_{n+1}-module, we see that $\mathrm{Hom}_{RG_{n+1}}(I_{G_{n+1}}, RG) \to \mathrm{Hom}_{RG_{n+1}}(RG_{n+1}I_{G_n}, RG)$ is a monomorphism but not an epimorphism. For any RG_{n+1}-module M, we have

$$\mathrm{Hom}_{RG_{n+1}}(M, RG) = \mathrm{Hom}_{RG}(RG \otimes_{RG_{n+1}} M, RG).$$

By Lemma 4.3, $RG \otimes_{RG_{n+1}} I_{G_{n+1}} = J_{n+1}$ and

$$RG \otimes_{RG_{n+1}} (RG_{n+1} I_{G_n}) = RG \otimes_{RG_{n+1}} (RG_{n+1} \otimes_{RG_n} I_{G_n})$$
$$= RG \otimes_{RG_n} I_{G_n} = J_n .$$

We conclude that $\mathrm{Hom}_{RG}(J_{n+1}, RG) \to \mathrm{Hom}_{RG}(J_n, RG)$ is a monomorphism but not an epimorphism. Therefore, the same is true of $\varphi : \mathrm{Hom}_{RG}(J_{n+1}, J_i) \to \mathrm{Hom}_{RG}(J_n, J_i)$ since J_i is free, finitely generated and nonzero.

Now, for $n < i$, we have $w_n - \varphi(w_{n+1}) = 0$. If $w_n \neq 0$ for some $n \leqslant i$, let n be the least such value. Then $n > 1$ since $w_1 = 0$. But $w_{n-1} - \varphi(w_n) = 0$ since $n - 1 < i$, and $w_{n-1} = 0$. Since φ is a monomorphism we get $w_n = 0$, a contradiction. Thus $w_n = 0$ for $n \leqslant i$. Since $w_i - \varphi(w_{i+1}) = 1_i$ and $w_i = 0$, we see that $1_i = \varphi(f)$ where $f \in \mathrm{Hom}_{RG}(J_{i+1}, J_i)$. In other words, $f : J_{i+1} \to J_i$ and $f \mid J_i = 1_i$. This shows that J_i is a direct summand of J_{i+1} but this implies that $\varphi : \mathrm{Hom}_{RG}(J_{i+1}, J_i) \to \mathrm{Hom}_{RG}(J_i, J_i)$ is onto. This is the required contradiction.

7. Splitting Criteria

In this section we give some sufficient conditions for a subgroup of a free group to be a free factor. It would be interesting to know whether the countability hypothesis in Theorem 7.3 can be dropped.

Lemma 7.1. *Let G be a free group and H a finitely generated subgroup of G. If* res : $H^1(G, M) \to H^1(H, M)$ *is surjective for all RG-modules M, then H is a free factor of G.*

Proof. By expressing the generators of H in terms of a base B for G, we find a finite subset B_1 of B such that H is contained in the subgroup G_1 generated by B_1. Clearly $G = G_1 * G_2$ where G_2 is generated by $B - B_1$. This argument is used by Higman in [*10*]. It will clearly suffice to show that H is a free factor of G_1. By Lemma 4.5, the hypothesis on G and H is inherited by $H \subset G_1$. Therefore, it will suffice to prove the theorem under the additional hypothesis that G is finitely generated. We now use induction on the number of generators of G. If H lies in a proper free factor G_1 of G, then $G = G_1 * G_2$, $G_2 \neq 1$. By abelianizing, we see that G_1 has fewer generators than G. As above, the hypothesis is inherited by $H \subset G_1$. Therefore H will be a free factor of G_1 and hence of G. Thus we can also assume that H is not contained in any proper free factor of G.

Let $J = RGI_H$. By Lemma 4.6, J is a direct summand of I_G. Therefore

$\mathrm{Hom}_{RG}(I_G, RG) \to \mathrm{Hom}_{RG}(J, RG)$ is onto. This will contradict Corollary 5.3 if H is a proper subgroup of G. Thus $H = G$ and so is trivially a free factor.

We now improve this by showing H need only be countable.

PROPOSITION 7.2. *Let G be a free group and H a countable subgroup of G. If* res : $H^1(G, M) \to H^1(H, M)$ *is surjective for all RG-modules M, then H is a free factor of G.*

Proof. As in Lemma 7.1, H is contained in a countable free factor of G and we can reduce to the case that G is countable. Let F be a free group on 2 generators. Since $[F, F]$ is free on $\aleph_0$ generators, we can identify H with a subgroup of F. Let $K = G \coprod_H F$ be the free product of G and F with H amalgamated. Since G, F, and H are free, they have cd $\leqslant 1$. Let M be any RK-module. By Theorem 2.3 we have an exact sequence

$$H^{n-1}(H, M) \to H^n(K, M) \to H^n(G, M) \oplus H^n(F, M).$$

If $n \geqslant 3$ both ends are zero so $H^n(K, M) = 0$. For $n = 2$, we have the exact sequence

$$H^1(G, M) \oplus H^1(F, M) \to H^1(H, M) \to H^2(K, M) \to H^2(G, M) \oplus H^2(F, M).$$

The right hand term is 0. Since $H^1(G, M) \to H^1(H, M)$ is onto by hypothesis, it follows that $H^2(K, M) = 0$. Therefore $\mathrm{cd}_R\, K \leqslant 1$. Since K is torsion free [*17*] [*21*] and countable, it follows from the countable case of Theorem A, which we proved in §6, that K is free.

By Theorem 2.3, if M is any RG-module, we have the exact sequence

$$H^1(K, M) \xrightarrow{(\mathrm{res},\mathrm{res})} H^1(G, M) \oplus H^1(F, M) \xrightarrow{(\mathrm{res},-\mathrm{res})} H^1(H, M). \quad (26)$$

Let u be any element of $H^1(F, M)$. Since res : $H^1(G, M) \to H^1(H, M)$ is onto by hypothesis, we can find $v \in H^1(G, M)$ such that res $v =$ res u. Thus (v, u) has zero image in $H^1(H, M)$, so by the exactness of (26), there is an element $w \in H^1(K, M)$ with image (v, u). Therefore, res : $H^1(K, M) \to H^1(F, M)$ is onto. Since F is finitely generated and K is free, Lemma 7.1 shows that F is a free factor of K. By ([*1*], p. 395(b)) ([*25*], §2.2.2, Satz 8*) it follows that $F \cap G = H$ is a free factor of G.

I will now show that a different sort of countability condition will suffice.

THEOREM 7.3. *Let G be a free group and H a subgroup of G. Suppose G is generated by $H \cup S$ where S is a countable subset of G. If* res : $H^1(G, M) \to H^1(H, M)$ *is surjective for all RG-modules M, then H is a free factor of G.*

Proof. By Lemma 4.6, $J = RGI_H$ is a direct summand of $I_G = J \oplus C$. Since $H \cup S$ generates G, I_G is generated as a left ideal by the elements $h - 1$, $h \in H$ and $s - 1$, $s \in S$. Now all $h - 1 \in J$. Therefore I_G/J is generated by the images of the $s - 1$, $s \in S$, so $C \approx I_G/J$ is countably generated, say by $c_1, c_2, \ldots$. Each c_i can be expressed in terms of a finite number of elements of G. This gives us a countable subset X_0 of G. Let $K_0 = \langle X_0 \rangle$ be the subgroup of G generated by X_0. Then $c_1, c_2, \ldots \in RK_0$. Let C_0 be the left ideal of RK_0 generated by $c_1, c_2, \ldots$. Then $C = RG \cdot C_0$. We now apply a saturation procedure as in [*16*]. We define a sequence of countable subgroups $K_0 \subset K_1 \subset \ldots$ of G and a sequence of countable subgroups $H_0 \subset H_1 \subset \ldots$ of H. We have just defined K_0. Set $H_0 = \{1\}$. Suppose K_i and H_i have been defined for $i \leqslant n$. Now I_{K_n} is generated as an R-module by the elements $\sigma - 1$, $\sigma \in K_n$. Since $I_{K_n} \subset I_G = J \oplus C = RG \cdot I_H \oplus RG \cdot C_0$, we can express each $\sigma - 1$ as an R-linear combination of elements of the form $g(h - 1)$ with $g \in G$, $h \in H$ and $g'c$ with $g' \in G$, $c \in C_0$. To express each $\sigma - 1$, $\sigma \in K_n$ in this way requires a countable number of $g, g' \in G$ and $h \in H$. Let H_{n+1} be generated by H_n and all elements $h \in H$ which we have just obtained. Let K_{n+1} be generated by the $g, g' \in G$ which we have obtained together with all elements of H_{n+1}. The sequences so obtained have the property that $H_n \subset K_n$ and $I_{K_n} \subset RK_{n+1}I_{H_{n+1}} + RK_{n+1}C_0$.

Now let $K = \bigcup K_n$ and $L = \bigcup H_n$. Then $L \subset K$. Since $I_K = \bigcup I_{K_n}$, we have $I_K \subset RKI_L + RKC_0$. This sum is direct since $L \subset H$ implies $RKI_L \subset J$, $RKC_0 \subset C$, and $I_G = J \oplus C$. Also $L \subset K$ so $RKI_L \subset I_K$ and $C_0 \subset I_{K_0} \subset I_K$. Therefore $I_K = RKI_L \oplus RKC_0$. By Lemma 4.6, $H^1(K, M) \to H^1(L, M)$ is onto for all RK-modules M. Since K is countable we can apply Proposition 7.2. Therefore L is a free factor of K. Let $K = L * P$. Choose bases U, V for L, P. Then $U \cup V$ is a base for K. Now RKI_L is free on the elements $u - 1$, $u \in U$, RKI_P is free on the $v - 1$, $v \in V$ and I_K is free on all $u - 1$, $v - 1$. Therefore $I_K = RKI_L \oplus RKI_P$. But $I_K = RKI_L \oplus D$ where $D = RKC_0$. Consider the projection $p : I_G \to C$ with kernel RGI_H. The restriction of p to I_K is the projection of I_K on the summand D. Therefore $p : RKI_P \approx D$. Consider the diagram

$$\begin{array}{ccc} RG \otimes_{RK} RKI_P & \xrightarrow{1 \otimes p} & RG \otimes_{RK} D \\ \downarrow & & \downarrow \\ RGI_P & \xrightarrow[p]{} & C. \end{array}$$

This commutes since p is an RG-homomorphism. We have just shown that the top map $1 \otimes p$ is an isomorphism. By Lemma 4.3 the vertical maps are isomorphisms (note that $C = RGD$). Therefore $p : RGI_P \to C$ is an isomorphism and so $I_G = RGI_H \oplus RGI_P$. Let V be a base for P and let W

be a base for H. Then RGI_P is free on the $v - 1$, $v \in V$ and RGI_H is free on the $w - 1$, $w \in W$. Therefore I_G is free on the elements $x - 1$, $x \in V \cup W$ and $V \cap W = \varnothing$. By Proposition 1.6, G is free on $V \cup W$. Since H is free on W and P is free on V, this shows that $G = H * P$ so H is a free factor of G.

8. Proof of Theorem A

Let G be torsion-free with $\mathrm{cd}_R\, G \leqslant 1$. By Proposition 3.3, we have $\mathrm{cd}_k\, G \leqslant 1$ for some prime field k. The use of k instead of R makes it a bit easier to count elements although it is not really necessary. By section 6, all countable subgroups of G are free. I will show that G is free using a very useful method introduced by Kaplansky [*16*].

This method has recently been formalized and applied with great success to abelian groups by Hill [*13*], Hill and Megibben [*14*], and Griffith [*7*].

We will also use the actual theorem proved by Kaplansky in [*16*]. Let I_G be the augmentation ideal of kG. Since $\mathrm{cd}_k\, G \leqslant 1$, I_G is projective as a left kG-module ([*29*], 6.4). By Kaplansky's theorem [*16*], I_G is a direct sum of countably generated modules.

$$I_G = \coprod_{\alpha \in S} P_\alpha , \qquad P_\alpha \neq 0. \tag{27}$$

We may clearly assume that G is uncountable.

By transfinite induction on $|G|$, we can assume that Theorem A holds for groups of cardinality less than $|G|$.

Lemma 8.1. *If G is uncountable, $|G| = |S|$, i.e., G and S have the same cardinal number.*

Proof. Since k is countable, $|I_G| = |G|$. Therefore $|S| \leqslant |G|$. Conversely, since each P_α is countably generated, I_G is generated by $\aleph_0 |S|$ elements. Each of these elements is a finite linear combination of elements $\sigma - 1$, $\sigma \in G$. Therefore I_G is generated by $\aleph_0 |S|$ elements $\sigma - 1$. By Remark 1.7, the corresponding elements σ generate G. Therefore $|G| \leqslant \aleph_0 |S|$. Since G is uncountable so is $|S|$ and thus $\aleph_0 |S| = |S|$.

Now let Ω be the least ordinal number with $|\Omega| = |G|$. We may assume that S consists of all ordinals less than Ω. We also index the elements of G by these ordinals $G = \{g_\alpha\}_{\alpha < \Omega}$.

Lemma 8.2. *There are subsets $S_\alpha \subset S$, $\alpha < \Omega$ and subgroups $G_\alpha \subset G$, $\alpha < \Omega$ satisfying the following conditions*

(a) *If $\alpha < \beta$, then $S_\alpha \subset S_\beta$, $G_\alpha \subset G_\beta$*

(b) *If λ is a limit ordinal,* $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$, *and* $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$

(c) $\alpha \in S_{\alpha+1}$ *and* $g_\alpha \in G_{\alpha+1}$

(d) $S_{\alpha+1} - S_\alpha$ *is countable and* $G_{\alpha+1}$ *is generated by the union of* G_α *and a countable set.*

(e) $kG \cdot I_{G_\alpha} = \coprod_{S_\alpha} P_\beta$.

Proof. Let $S_0 = \varnothing$ and $G_0 = \{1\}$. Note (c) is vacuous in this case. Suppose S_β and G_β have been defined for all $\beta < \alpha$. If α is a limit ordinal, define S_α and G_α by (b). Since $I_{G_\alpha} = \bigcup_{\beta<\alpha} I_{G_\beta}$, it is clear that (e) holds. The other conditions are clear.

Suppose now that $\alpha = \beta + 1$. Let $T_1 = S_\beta \cup \{\beta\}$ and $H_1 = \langle G_\beta, g_\beta \rangle$. We will define sequences $T_1 \subset T_2 \subset \ldots$ and $H_1 \subset H_2 \subset \ldots$ and set $S_\alpha = \bigcup T_n$, $G_\alpha = \bigcup H_n$. Clearly (a), (b), and (c) will be trivially satisfied. We must choose the sequences to satisfy (d) and (e). Suppose T_n and H_n are defined such that $T_n - S_\beta$ is countable and H_n is generated by G_β and a countable set X_n. Then I_{H_n} is generated by I_{G_β} and the elements $x - 1$ for $x \in X_n$. Write each $x - 1$ as $\sum p_\gamma(x)$ with $p_\gamma(x) \in P_\gamma$. For each $x \in X_n$, only a finite number of $p_\gamma(x)$ will be nonzero. Let T_{n+1} be the union of T_n and the set of all γ such that $p_\gamma(x) \neq 0$ for some $x \in X_n$. Therefore $T_{n+1} - T_n$ is countable and

$$kGI_{H_n} \subset \coprod_{T_{n+1}} P_\gamma . \tag{28}$$

Now $\coprod_{T_n - S_\beta} P_\gamma$ is countably generated. Each generator is a finite kG-linear combination of elements $h - 1$, $h \in G$. In this way we get a countable set of elements $h \in G$. Let H_{n+1} be generated by H_n and these elements. Then

$$\coprod_{T_n} P_\gamma \subset kGI_{H_{n+1}} \tag{29}$$

since we already have $\coprod_{S_\beta} P_\gamma \subset kGI_{G_\beta}$.

Set $S_\alpha = \bigcup T_n$, $G_\alpha = \bigcup H_n$. Then (d) is clear and (e) follows from (28) and (29).

We can now prove Theorem A. It follows from (d) that G_α is generated by at most $\aleph_0 \mid \alpha \mid$ elements. Therefore $\mid G_\alpha \mid < \mid G \mid$ so G_α is free by our transfinite induction hypothesis. By (e) and (27), $kG \cdot I_{G_\alpha}$ is a direct summand of I_G. By Lemma 4.6, res : $H^1(G, M) \to H^1(G_\alpha, M)$ is onto for all kG-modules M. By Lemma 4.5, res : $H^1(G_{\alpha+1}, N) \to H^1(G_\alpha, N)$ is onto for all $kG_{\alpha+1}$-modules N. Because of (d) we can apply Theorem 7.3 and conclude that G_α is a free factor of $G_{\alpha+1}$. Let $G_{\alpha+1} = G_\alpha * K_\alpha$. Then K_α is free since $G_{\alpha+1}$ is.

LEMMA 8.3. *The map* $\theta : \ast K_\alpha \to G$, *defined by the inclusions* $K_\alpha \subset G$, *is an isomorphism.*

This clearly implies Theorem A.

Proof. Let α be least such that $G_\alpha \not\subset \operatorname{im} \theta$. Let $x \in G_\alpha$, $x \notin \operatorname{im} \theta$. If α is a limit ordinal, $x \in G_\gamma$ for some $\gamma < \alpha$ and so $x \in \operatorname{im} \theta$. Therefore $\alpha = \beta + 1$ so $G_\alpha = G_\beta * K_\beta$ but G_β, $K_\beta \subset \operatorname{im} \theta$. Thus θ is onto.

If $\ker \theta \neq 1$, choose $x \in \ker \theta$, $x \neq 1$ with $x \in \ast_{\gamma<\alpha} G_\gamma$ with the least possible α. If α is a limit ordinal, then $x \in \ast_{\gamma<\delta} G_\gamma$ for some $\delta < \alpha$. Therefore $\alpha = \beta + 1$. By the choice of α, θ is injective on $L = \ast_{\gamma<\beta} G_\gamma$. Therefore $\theta : \ast_{\gamma<\alpha} K_\gamma = L * K_\beta \to G_\beta * K_\beta \subset G$ is injective.

9. PROOF OF THEOREM B

We begin with a general result on subgroups of finite index. This is the discrete analogue of a theorem of Tate [*24*] ([*23*], I-20). The proof is the same as his.

THEOREM 9.1. *Let H be a subgroup of finite index in a group G. Then either* $\mathrm{cd}_R\, G = \mathrm{cd}_R\, H$ *or* $\mathrm{cd}_R\, G = \infty$.

Proof. Clearly $\mathrm{cd}_R\, H \leqslant \mathrm{cd}_R\, G$. Suppose $\mathrm{cd}_R\, G = n < \infty$. The exact cohomology sequence shows that the functor $H^n(G, -)$ is right exact. Let M be an RG-module with $H^n(G, M) \neq 0$. By Shapiro's Lemma ([*3*], Ch. X, Prop. 7.4), $H^n(H, M) = H^n(G, \mathrm{Hom}_{RH}(RG, M))$. If we had an RG-epimorphism $\mathrm{Hom}_{RH}(RG, M) \to M$, the right exactness of $H^n(G, -)$ would give us an epimorphism $H^n(H, M) \to H^n(G, M)$. Therefore we would have $H^n(H, M) \neq 0$ so $\mathrm{cd}_R\, H \geqslant n = \mathrm{cd}_R\, G$. Now there is an epimorphism $RG \otimes_{RH} M \to M$ by $g \otimes m \mapsto gm$. Therefore our result follows from the following well-known lemma.

LEMMA 9.2. *If H has finite index in G, then for any RH-module M there is an RG-isomorphism* $RG \otimes_{RH} M \approx \mathrm{Hom}_{RH}(RG, M)$.

Proof. Let $G = \bigcup \sigma_i H$ be a coset decomposition. Define a map $\theta : \mathrm{Hom}_{RH}(RG, M) \to RG \otimes_{RH} M$ by sending f to $\sum \sigma_i \otimes f(\sigma_i^{-1})$. This is bijective. In fact, any $f \in \mathrm{Hom}_{RH}(RG, M)$ is determined by the values $f(\sigma_i^{-1}) = m_i$ since $G = \bigcup H\sigma_i^{-1}$, and any set of m_i can occur. But $RG \otimes_{RH} M = \coprod \sigma_i \otimes M$. Now θ is independent of the choice of coset representatives because $\sigma_i h \otimes f((\sigma_i h)^{-1}) = \sigma_i h \otimes h^{-1} f(\sigma_i) = \sigma_i \otimes f(\sigma_i^{-1})$. If $\sigma \in G$, then $G = \bigcup \sigma^{-1}\sigma_i H$ so $\theta(\sigma f) = \sum \sigma_i \otimes \sigma f(\sigma_i^{-1}) = \sum \sigma_i \otimes f(\sigma_i^{-1}\sigma) = \sigma \sum \sigma^{-1}\sigma_i \otimes f((\sigma^{-1}\sigma_i)^{-1}) = \sigma\theta(f)$. Thus θ is an RG-homomorphism.

It is natural to conjecture that $\operatorname{cd} H = \operatorname{cd} G$ in Theorem 9.1 provided G is torsion free. This would imply that the group G in Theorem B has $\operatorname{cd} G \leqslant 1$ and so Theorem B would follow from Theorem A. My original proof of Theorem B proceeded by proving this conjecture for the case $\operatorname{cd} H = 1$. It followed closely the proof of Serre [*24*] except at one point where Stallings' results [*29*] were used. However, Serre has recently shown me a very simple and elegant proof of the general conjecture. We say that a group G has no R-torsion if for every finite subgroup K of G, the order n of K is a unit in R (more precisely, $n \cdot 1$ is a unit in R). With this definition, we now state Serre's result.

THEOREM 9.2. (Serre). *Let R be a commutative ring and let G be a group having no R-torsion. If H is a subgroup of finite index in G, then* $\operatorname{cd}_R G = \operatorname{cd}_R H$.

Clearly Theorem B is a consequence of this and Theorem A.

Since Serre's proof is unpublished, I will give a brief account of it here. At the same time, I would like to express my profound thanks to Serre for showing me this proof.

By Theorem 9.1, it is sufficient to show that $\operatorname{cd}_R G < \infty$ if $\operatorname{cd}_R H < \infty$. Let P be a finite dimensional projective resolution for R over RH. Since P and R are projective over R, P splits over R. Therefore $\underset{\sim}{Q} = \otimes^n P = P \otimes_R \cdots \otimes_R P$ is an acyclic resolution of R over R. Let $n = [G : H]$ and define an action of G on $\underset{\sim}{Q}$ as follows. Choose coset representatives x_i so that $G = \bigcup x_i H$. If $g \in G$, let $g^{-1}x_i = x_{\nu_i} h_{\nu_i}^{-1}$ with $h_{\nu_i} \in H$ and define $g(p_1 \otimes \cdots \otimes p_n) = h_{\nu_i} p_{\nu_i} \otimes \cdots \otimes h_{\nu_n} p_{\nu_n}$. This extends uniquely to an R-automorphism of $\underset{\sim}{Q}$ and it is easy to see that we have defined an action of G on $\underset{\sim}{Q}$. This construction is an analogue of Frobenius' construction of induced representations.

It will now suffice to show that $\underset{\sim}{Q}$ is RG-projective since $\underset{\sim}{Q}$ will then be a finite-dimensional projective resolution of R over RG. To do this we can forget the boundary operator on P and regard it as projective RH-module. Let $P \oplus P' = F$ be a free RH-module. Then $\otimes^n F = \otimes^n P \oplus X$ where X is the sum of all $P^{(i_1)} \otimes \cdots \otimes P^{(i_n)}$ with $i_\nu = 0, 1$, $P^{(0)} = P$, $P^{(1)} = P'$ and not all $i_\nu = 0$. This decomposition is stable under G so $\otimes^n P$ is a direct summand of $\otimes^n F$ as an RG-module. Therefore it will suffice to show that $\otimes^n F$ is RG-projective. Let $(b_\alpha)_{\alpha \in J}$ be an RG-base for F. Then $(hb_\alpha)_{\alpha \in J, h \in H}$ is an R-base for F so $\otimes^n F$ has an R-base consisting of all elements $w = h_1 b_{\alpha_1} \otimes \cdots \otimes h_n b_{\alpha_n}$. This base is clearly permuted by G. If K_w is the set of $g \in G$ with $gw = w$, the G-orbit of w spans a submodule isomorphic to $R[G/K_w]$ and $\otimes^n F$ is the direct sum of these modules, choosing one w in each orbit. Therefore we need only show each $R[G/K_w]$ is RG-projective.

We first show that K_w is finite. Let N be the kernel of the permutation representation of G on the finite set G/H. Then $[G : N] < \infty$. If $g \in N$, we have $g^{-1}x_i = x_i h_i^{-1}$ where $h_i = x_i^{-1}gx_i$ so

$$g(w) = g(h_1 b_{\alpha_1} \otimes \cdots \otimes h_n b_{\alpha_n}) = x_1^{-1}gx_1h_1b_{\alpha_1} \otimes \cdots \otimes x_n^{-1}gx_nh_nb_{\alpha_n} \neq w$$

unless $g = 1$. Therefore $K_w \cap N = 1$ so K_w is isomorphic to a subgroup of the finite group G/N.

Finally, if K is any finite subgroup of G and $m = |K|$, we know that m is a unit in R. Define a map $R[G/K] \to RG$ by sending the coset gK to the element $m^{-1}\sum_{k \in K} gk$. This is an RG-homomorphism which splits the canonical epimorphism $RG \to R[G/K]$. Therefore $R[G/K]$ is a direct summand of RG and so is projective.

10. A Topological Remark

The starting point for this investigation was the result of Curtis and Fort [5] which states that if X is a 1-dimensional separable metric space, then X is aspherical and $\pi_1(X)$ is locally free so $H_i(X) = 0$ for $i > 1$. This suggested that we might have cd $\pi_1(X) \leqslant 1$ for such a space X. The simplest example is the union S of the circles $(x - 1/n)^2 + y^2 = 1/n^2$ in the plane. The group $\pi_1(S)$ was computed by Griffiths [*8*] and shown to be isomorphic to a group studied by Higman [*11*]. Higman constructs a subgroup P of $\pi_1(S)$ which is countable but not free. My proof that cd $P = 2$ was expanded into Section 6 of the present paper. By ([*4*], Th. 2.2), if X is a locally connected one-dimensional continuum, then either X is locally 1-connected in which case $\pi_1(X)$ is free and finitely generated, or else $\pi_1(X)$ contains a subgroup isomorphic to $\pi_1(S)$ so cd $\pi_1(X) \geqslant 2$. I do not know the exact value of cd $\pi_1(S)$.

Note added in proof. Serre has pointed out to me that I have inadvertantly omitted the signs in the proof of Theorem 9.2. The definition of $g(p_1 \otimes \cdots \otimes p_n)$ should include a sign $(-1)^s$ where $s = \sum \deg(p_i)\deg(p_j)$, the sum being taken over all (i, j) with $i < j$ and $\nu_i > \nu_j$. The base of $\otimes^n F$ is only permuted up to sign by G and we must define K_w as the subgroup of all $g \in G$ with $gw = \pm w$. The G-orbit of w spans a submodule isomorphic to $RG \otimes_{RK_w} E_w$ where $E_w = Rw$ as an RK_w-module. Therefore we need only show that E_w is RK_w-projective. The epimorphism $RK_w \to Rw$ is split by $h : Rw \to RK_w$ with $h(w) = |K_w|^{-1} \sum \epsilon(k)\, k$, the sum being over all $k \in K_w$ with $\epsilon(k) = \pm 1$ such that $kw = \epsilon(k)\, w$.

The construction of the "induced representation" can be generalized to any functor of two variables which is coherently commutative and associative (S. MacLane, Categorical Algebra. *Bull. Am. Math. Soc.* **71** (1965), 40–106).

Using this approach, we can avoid having to compute the exact value of the sign involved.

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