Factorization of Nonnegative Matrices—II

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ABSTRACT

Suppose A is an $n\times n$ nonnegative matrix. Necessary and sufficient conditions are given for A to be factored as LU, where L is a lower triangular nonnegative matrix, and U is an upper triangular nonnegative matrix with $u_{ii}=1$.

I. INTRODUCTION

Suppose A is a matrix of order n over the complex field. Necessary and sufficient conditions are given for A to have an LU-factorization in [3], where other related results may be found. In this paper, we consider the problem of factoring a nonnegative A $(A\geqslant 0)$ as LU, where L is a lower triangular nonnegative matrix, and U is an upper triangular nonnegative matrix with main diagonal consisting entirely of ones.

In [4], factorizations of this type were considered with the restriction that all principal minors of A are nonzero. In Theorem 1 of this paper, these restrictions are removed. Throughout the paper, the Schur complement of a nonsingular principal submatrix of A [2] plays an important role in these factorizations, and some of the later results elucidate this role.

II. NONNEGATIVE FACTORIZATIONS

We introduce first the notation that we shall use. Let α and β be increasing sequences on $\{1,\ldots,n\}$. Then $A(\alpha|\beta)$ is the minor of A with rows indexed by α and columns indexed by β, whereas $A[\alpha|\beta]$ denotes the

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submatrix of A in rows α and columns β. A principal submatrix of A is written $A[\alpha]$. For $1 \leqslant k \leqslant n$, $A_k = A[1,\dots,k]$. Further, $\hat{\alpha}$ denotes the complement of α.

If A is an $n\times n$ matrix, and A_k is nonsingular, the Schur complement of A_k in

$$A=\begin{pmatrix} A_k & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is defined as $(A|A_k) = A_{22} - A_{21}A_k^{-1}A_{12}$ [2].

THEOREM 1. *An $n\times n$ nonnegative matrix A has an LU-factorization with $L \geqslant 0$, $U \geqslant 0$ and $u_{ii}=1$ iff A_{n-1} has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$, $\tilde{u}_{ii}=1$, and there exist nonnegative $1\times(n-1)$ vectors v and w such that $\tilde{L}w^T = A[\hat{n}|n]$, $v\tilde{U} = A[n|\hat{n}]$ and $a_{nn} - vw^T \geqslant 0$.*

Proof. Assume A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. Partition A, L and U conformally so that

$$A=\begin{pmatrix} A_{n-1} & z^T \\ y & a_{nn} \end{pmatrix} = \begin{pmatrix} L_{n-1} & 0 \\ r & l \end{pmatrix}\begin{pmatrix} U_{n-1} & s^T \\ 0 & 1 \end{pmatrix} = LU.$$

Let $\tilde{L} = L_{n-1}$, $\tilde{U} = U_{n-1}$, $v = r$ and $w = s$. Then $\tilde{L}, \tilde{U}, v, w$ are all nonnegative, $\tilde{L}\tilde{U} = L_{n-1}U_{n-1} = A_{n-1}$, $\tilde{L}w^T = L_{n-1}s^T = z^T = A[\hat{n}|n]$, $v\tilde{U} = rU_{n-1} = y = A[n|\hat{n}]$, and $a_{nn} - vw^T = l \geqslant 0$.

Conversely, assume A_{n-1} has an $\tilde{L}\tilde{U}$-factorization and there exist v and w satisfying conditions on the right-hand side of the statement. Partition A as above. Let

$$L=\begin{pmatrix} \tilde{L} & 0 \\ v & a_{nn} - vw^T \end{pmatrix} \quad \text{and} \quad U=\begin{pmatrix} \tilde{U} & w^T \\ 0 & 1 \end{pmatrix}.$$

Then $L\geqslant 0$, $U\geqslant 0$, and

$$LU=\begin{pmatrix} \tilde{L} & 0 \\ v & a_{nn} - vw^T \end{pmatrix}\begin{pmatrix} \tilde{U} & w^T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{L}\tilde{U} & \tilde{L}w^T \\ v\tilde{U} & a_{nn} \end{pmatrix} = A. \qquad \blacksquare$$

COROLLARY 1. *Let A be an $n\times n$ nonnegative matrix with* $\det(A_{n-1})\neq 0$. *Then A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$, $u_{ii}=1$ iff A_{n-1} has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$, $\tilde{u}_{ii}=1$; and for $y=A[n|\hat{n}]$ and $z^T=A[\hat{n}|n]$, we have $y\tilde{U}^{-1}\geqslant 0$, $\tilde{L}^{-1}z^T\geqslant 0$, and $(A|A_{n-1})=a_{nn}-yA_{n-1}^{-1}z^T\geqslant 0$.*

Proof. Note that since $\tilde{U}$ is always invertible, $\tilde{L}$ is invertible if and only if A_{n-1} is invertible. Thus for $v\tilde{U}=y$ and $\tilde{L}w^T=z^T$, we have $v=y\tilde{U}^{-1}$, $w^T=\tilde{L}^{-1}z^T$ and $a_{nn}-vw^T=a_{nn}-y\tilde{U}^{-1}\tilde{L}^{-1}z^T=a_{nn}-yA_{n-1}^{-1}z^T=(A_n|A_{n-1})$. It is now easy to see that the corollary is a direct consequence of Theorem 1. ■

COROLLARY 2. *Let A be an $n\times n$ nonnegative matrix with nonzero leading principal minors. If A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$, then $(A_{k+1}|A_k)>0$ for $k=1,2,\dots,n-1$. Consequently* $\det(A_k)>0$ *for $k=1,2,\dots,n$.*

Proof. Since A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$, it follows from Corollary 1 that A_{n-1} has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$ and $(A|A_{n-1})\geqslant 0$. Now apply the same argument to A_{n-1}. The process can be continued, yielding $(A_{k+1}|A_k)\geqslant 0$ for $k=1,2,\dots,n-1$. Suppose $(A_k|A_{k-1})=0$ for some k. Then it follows from the proofs of Theorem 1 and Corollary 1 that $l_{kk}=0$, contradicting the fact that A is nonsingular. Finally, note that $a_{11}>0$ and $\det(A_{k+1})=\det(A_k)(A_{k+1}|A_k)$. The fact that $\det(A_k)>0$ for $k=1,2,\dots,n$ follows iteratively. ■

COROLLARY 3. *Let A be an $n\times n$ nonnegative matrix with the property that for $k=1,2,\dots,n-1$, there exist nonnegative vectors v_k and w_k such that $A_kw_k^T=A[1,2,\dots,k|k+1]$, $v_kA_k=A[k+1|1,2,\dots,k]$ and $a_{k+1,k+1}-v_kA_kw_k^T\geqslant 0$. Then A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$.*

Proof. We shall use induction on n. For $n=1$, the statement is trivially true. Assume it is true for $n=k$, and consider $n=k+1$. Partition A into $\begin{pmatrix} A_k & z^T \\ y & a_{nn}\end{pmatrix}$, where y,z are $1\times k$ nonnegative vectors. A_k satisfies conditions of the hypothesis and hence has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$. Also by hypothesis, there exist nonnegative vectors v_k and w_k such that $A_kw_k^T=A[\hat{n}|n]$, $v_kA_k=A[n|\hat{n}]$ and $a_{nn}-v_kA_kw_k^T\geqslant 0$. Let $v=v_k\tilde{L}$ and $w^T=\tilde{U}w_k^T$. Then $v\geqslant 0$, $w\geqslant 0$, $\tilde{L}w^T=A_kw_k^T=A[\hat{n}|n]$, $v\tilde{U}=v_kA_k=A[n|\hat{n}]$, and $a_{nn}-vw^T=a_{nn}-v_k\tilde{L}\tilde{U}w_k^T=a_{nn}-v_kA_kw_k^T\geqslant 0$. Thus by Theorem 1, A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. ■

EXAMPLE 1. The converse of Corollary 1 is not true, as can be seen in

$$A=\begin{bmatrix}2&2&0\\2&3&1\\0&2&3\end{bmatrix}=\begin{bmatrix}2&0&0\\2&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix},$$

where

$$A[1,2|3]=\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2&2\\2&3\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=A_2w_2^T.$$

w_2^T is unique but not nonnegative, and yet A has a nonnegative factorization.

COROLLARY 4. *Let A be an $n\times n$ nonnegative matrix with nonzero proper leading principal minors such that for $k=1,2,\ldots,n-1$, we have $A_k^{-1}s_k^T\geqslant 0$, $r_kA_k^{-1}\geqslant 0$, and $(A_{k+1}|A_k)\geqslant 0$, where $r_k=A[k+1|1,2,\ldots,k]$ and $s_k^T=A[1,2,\ldots,k|k+1]$. Then A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$.*

Proof. Let $w_k^T=A_k^{-1}s_k^T$ and $v_k=r_kA_k^{-1}$. Then $A_kw_k^T=s_k^T$, $v_kA_k=r_k$, and $a_{k+1,k+1}-v_kA_kw_k^T=a_{k+1,k+1}-r_kA_k^{-1}s_k^T=(A_{k+1}|A_k)\geqslant 0$. It follows directly from Corollary 3 that A has an LU-factorization of the specified type. ■

THEOREM 2. *Let A be an $n\times n$ nonnegative matrix. If A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$, then every almost principal submatrix of the type $A[1,2,\ldots,k,i|1,2,\ldots,k,j]$ also has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$.*

Proof. Let $A=LU$ with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. Note that $a_{ij}=L[i|\hat{\phi}]U[\hat{\phi}|j]$. Let $v=L[i|1,2,\ldots,k]$ and $w^T=U[1,2,\ldots,k|j]$. Then we have $a_{ij}-vw^T\geqslant 0$, $L_kw^T=A[1,2,\ldots,k|j]$, and $vU_k=A[i|1,2,\ldots,k]$. Finally, letting

$$\tilde{L}=\begin{pmatrix}L_k&0\\v&a_{ij}-vw^T\end{pmatrix}\quad\text{and}\quad\tilde{U}=\begin{pmatrix}U_k&w^T\\0&1\end{pmatrix},$$

we obtain

$$\tilde{L}\tilde{U}=\begin{pmatrix}L_kU_k&L_kw^T\\vU_k&a_{ij}\end{pmatrix}=A[1,2,\ldots,k,i|1,2,\ldots,k,j].$$ ■

COROLLARY 5. *Let A be an $n\times n$ nonnegative matrix with* $\det(A_k)\neq 0$ *for some k. If A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$, then* $(A|A_k)\geqslant 0$.

Proof. Theorem 2 implies that $A[1,2,\dots,k,i|1,2,\dots,k,j]$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$. Thus $(A[1,2,\dots,k,i|1,2,\dots,k,j]|A_k)\geqslant 0$ by Corollary 1. Partition A into $\begin{pmatrix} A_k & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$. Then $(A|A_k)=A_{22}-A_{21}A_k^{-1}A_{12}=(h_{ij})_{i,j=k+1}^n$. Note that $h_{ij}=(A[1,2,\dots,k,i|1,2,\dots,k,j]|A_k)$. Hence $(A|A_k)\geqslant 0$. ∎

THEOREM 3. *Let A be an $n\times n$ nonnegative matrix with $a_{11}>0$. Then A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$ iff $(A|a_{11})$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$.*

Proof. Assume A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. Partition A, L and U conformally so that

$$A=\begin{pmatrix} a_{11} & z \\ y^T & A_{22}\end{pmatrix}=\begin{pmatrix} a_{11} & 0 \\ r^T & L_{22}\end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & U_{22}\end{pmatrix}=LU,$$

where y, z, r and s are $1\times(n-1)$ nonnegative vectors. Necessarily $r=y$, $s=a_{11}^{-1}z$, and $A_{22}=r^Ts+L_{22}U_{22}$. Thus $(A|a_{11})=A_{22}-y^Ta_{11}^{-1}z=A_{22}-r^Ts=L_{22}U_{22}$, which is a factorization of the type specified.

Conversely assume $(A|a_{11})$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$. Partition A as above. Note that $(A|a_{11})=A_{22}-y^Ta_{11}^{-1}z=\tilde{L}\tilde{U}$. Let

$$L=\begin{pmatrix} a_{11} & 0 \\ y^T & \tilde{L}\end{pmatrix}\quad\text{and}\quad U=\begin{pmatrix} 1 & a_{11}^{-1}z \\ 0 & \tilde{U}\end{pmatrix}.$$

Then L is nonnegative lower triangular, U is nonnegative upper triangular, $u_{ii}=1$, and

$$LU=\begin{pmatrix} a_{11} & 0 \\ y^T & \tilde{L}\end{pmatrix}\begin{pmatrix} 1 & a_{11}^{-1}z \\ 0 & \tilde{U}\end{pmatrix}=\begin{pmatrix} a_{11} & z \\ y^T & A_{22}\end{pmatrix}=A.$$ ∎

COROLLARY 6. *Let A be an $n\times n$ nonnegative matrix with nonzero proper leading principal minors. If $(A|A_k)\geqslant 0$ for $k=1,2,\dots,n-1$, then A*

has an LU-factorization with $L \geqslant 0$, $U \geqslant 0$ *and* $u_{ii}=1$. *Necessarily* $\det(A_k)>0$ *for* $k=1,2,\dots,n-1$.

Proof. Since $(A|A_{n-1})\geqslant 0$ and consists of a single element, trivially it has a factorization of the specified type. Now using the quotient formula (see [1]) of Schur complements, we have $(A|A_{n-1})=\big((A|A_{n-2})|(A_{n-1}|A_{n-2})\big)$. Note that $(A_{n-1}|A_{n-2})$ is the element in the first row and first column of $(A|A_{n-2})$ and hence is nonnegative. In fact $(A_{n-1}|A_{n-2})>0$, since $(A_{n-1}|A_{n-2})=0$ would imply that $\det(A_{n-1})=\det(A_{n-2})(A_{n-1}|A_{n-2})=0$, a contradiction. Now it follows from Theorem 3 that $(A|A_{n-2})$ has a factorization of the specified type. Next apply the same argument to $(A|A_{n-2})=\big((A|A_{n-3})|(A_{n-2}|A_{n-3})\big)$ to obtain a nonnegative factorization for $(A|A_{n-3})$. The process can be continued until eventually we obtain $A=LU$ with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. Also, since $a_{11}>0$ and $\det(A_{k+1})=\det(A_k)(A_{k+1}|A_k)$, we obtain iteratively $\det(A_k)>0$ for $k=1,2,\dots,n-1$. ∎

THEOREM 4. *Let A be an* $n\times n$ *nonnegative matrix with nonzero proper leading principal minors. Then A has an LU-factorization with* $L\geqslant 0$, $U\geqslant 0$ *and* $u_{ii}=1$ *iff* $(A|A_k)\geqslant 0$ *for* $k=1,2,\dots,n-1$.

Proof. This is a direct consequence of Corollaries 5 and 6. ∎

The statement in Theorem 4 is a known result (cf. [4]).

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