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TOPOLOGY AND ITS APPLICATIONS

On weak shape equivalences [☆]

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Abstract

We prove that weak shape equivalences are monomorphisms in the shape category of uniformly pointed movable continua Sh_M. We use an example of Draper and Keesling to show that weak shape equivalences need not be monomorphisms in the shape category. We deduce that Sh_M is not balanced. We give a characterization of weak dominations in the shape category of pointed continua, in the sense of Dydak (1979). We introduce the class of pointed movable triples (X, F, Y), for a shape morphism $F: X \to Y$, and we establish an infinite-dimensional Whitehead theorem in shape theory from which we obtain, as a corollary, that for every pointed movable pair of continua (Y, X) the embedding $j: X \to Y$ is a shape equivalence iff it is a weak shape equivalence. © 1999 Elsevier Science B.V. All rights reserved.

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AMS classification: 54C56; 55P55

Dyer and Roitberg and Dydak, in [8] and [6], respectively, proved that the homotopy category of pointed path-connected CW-spaces, HCW_*, is balanced, i.e., a map $f: X \to Y$ is an equivalence in HCW_* iff it is both an epimorphism and a monomorphism in HCW_*.

Since shape theory does not modify homotopy theory on CW-complexes and ANRs then, the mentioned result can be trivially stated in the context of the shape category of pointed path-connected CW-spaces, that is a full subcategory of the pointed shape category, *Sh*.

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Movable spaces constitute a very important class in the theory of shape. Movability can be seen as a natural generalization of the property of being shape dominated by a CW-complex.

In this paper we deal with pointed continua (compact Hausdorff connected topological spaces) and pointed metric continua. One of the aims of this article is to show that we can not obtain an analogue of above theorem when we consider the shape category of uniformly pointed movable continua Sh_M.

The theorems of the papers [8] and [6] are variants of the classical Whitehead theorem. That is why our approach is based on the Whitehead theorem in shape theory.

A shape morphism $F:(X,*) \to (Y,*)$ is said to be a *weak shape equivalence* if it induces isomorphisms between all the homotopy pro-groups. On the other hand, $F:(X,*) \to (Y,*)$ is a *very weak shape equivalence* if it induces isomorphisms on the shape groups [9].

Keesling corrected, in [13], a gap in Moszyńska's proof of a variant of the Whitehead theorem in shape theory (see [21,22]). He showed that a monomorphism in the category of uniformly movable pro-groups needs not be a monomorphism in the category of pro-groups. However, Keesling proved that if $(X,*)$ and $(Y,*)$ are pointed (uniformly) movable metric continua a shape morphism $F:(X,*) \to (Y,*)$ is a weak shape equivalence iff it is a very weak shape equivalence.

It is well known, see [4], for example, that weak shape equivalences need not be shape equivalences even for pointed movable spaces. Here, the dimension of the spaces plays an important role. Nevertheless, there are infinite-dimensional Whitehead type theorems in the theory of shape that allow to conclude that a weak shape equivalence

$$F:(X,*) \to (Y,*)$$

between pointed movable metric continua is a shape equivalence provided F is a shape domination (Dydak [5]) or $Y \in$ FANRs (Edwards and Geoghegan [10]).

The authors [16] (Cuchillo, Sanjurjo and the authors [20]), defined an ultrametric (topology) on the set of shape morphisms $Sh(X,Y)$ between compacta (arbitrary topological spaces) X, Y. In [18] we proved that this metric is useful to obtain in a short and elementary way the infinite-dimensional Whitehead theorems already known.

In this paper we apply this machinery, essentially Theorem 1, to prove, as we already announced, that Sh_M is not balanced. We shall also obtain a characterization (Corollary 1) of weak dominations, in the shape category of pointed uniformly movable continua, in the sense of Dydak [5]. Finally we will prove another infinite-dimensional Whitehead theorem in shape theory. We introduce the class of pointed movable triples (X,F,Y), for a shape morphism $F:X \to Y$, and we establish Theorem 5 from which we obtain, as a corollary, that for every pointed movable pair of continua (Y,X) the embedding $j:X \to Y$ is a shape equivalence iff it is a weak shape equivalence. This article also shows that the applications of our techniques (see [16,18,19]) can be also obtained if metrizability is not required.

We will write Sh_C to represent the shape category of pointed compact topological spaces.

In order to make the paper as selfcontained as possible we will recall the basic ideas and results of [16,20,18] that we need. In a few cases when we state a known result we provide a new proof of it (if shorter).

Let (X, x_0) and (Y, y_0) be pointed compacta (compact metric spaces). We will assume Y to be embedded in the Hilbert cube Q. Let $i_\varepsilon : Y \to B(Y, \varepsilon)$ be the inclusion. For any pair $f, g : (X, x_0) \to (Q, y_0)$ of maps take $F(f,g) = \inf\{\varepsilon > 0\colon f \simeq g$ in $B(Y,\varepsilon) = Y_\varepsilon\}$ ($\simeq$ means the pointed homotopy relation).

It is clear that (pointed) approximative maps (see [2]) $\{f_k\} : (X, x_0) \to (Y, y_0)$ correspond with F-Cauchy sequences and that (pointed) homotopic approximative maps are equivalent F-Cauchy sequences.

Given $\alpha, \beta \in \mathit{Sh}((X, x_0), (Y, y_0))$ (the set of shape morphisms between the pointed spaces (X, x_0), (Y, y_0)) and F-Cauchy sequences $\{f_k\}$, $\{g_k\}$ in the classes of α, β, respectively, the formula

$$d(\alpha, \beta) = \lim_{k \to \infty} F(f_k, g_k)$$

produces a well defined complete, non-Archimedean metric in $\mathit{Sh}((X, x_0), (Y, y_0))$ such that the composition of pointed shape morphisms induces uniformly continuous maps between the spaces involved (the reader can see [24] for information about ultrametrics). This fact provides many new pointed shape invariants (see [16] for details in the unpointed case). Next proposition states the geometrical meaning of the above metric.

Proposition 1 [16]. *Given $\alpha, \beta \in \mathit{Sh}((X, x_0), (Y, y_0))$, $d(\alpha, \beta) < \varepsilon$ if and only if $S(i_\varepsilon) \circ \alpha = S(i_\varepsilon) \circ \beta$, as pointed morphisms ($S$ denotes the shape functor).*

In order to simplify notation *we will suppress base points consistently.*

In [20] the above construction is extended for arbitrary topological spaces. If X, Y are (pointed) topological spaces, let

$$\boldsymbol{p} : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, A) \quad \text{and} \quad \boldsymbol{q} : Y \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$$

be HPol$_*$-expansions of X and Y, respectively.

Denote $\boldsymbol{Sh}(X, Y) = (\mathit{Sh}(X, Y_\mu), q^*_{\mu\mu'}, M)$.

For every $\mu \in M$ and $F \in \mathit{Sh}(X, Y)$ take $V^F_\mu = \{G \in \mathit{Sh}(X, Y)$ such that $q_\mu \circ F = q_\mu \circ G$ as pointed homotopy classes to $Y_\mu\}$.

Proposition 2 [20]. *The family $\{V^F_\mu\colon F \in \mathit{Sh}(X, Y),\ \mu \in M\}$ is a base for a topology $T_{\boldsymbol{q}}$ in $\mathit{Sh}(X, Y)$. Moreover, the topology so obtained depends only on X and Y, in the sense that if $\boldsymbol{q}' : Y \to \boldsymbol{Y}' = (Y_\nu, q_{\nu\nu'}, N)$ is another* HPol$_*$*-expansion of Y, then the identity map $(\mathit{Sh}(X, Y), T_{\boldsymbol{q}}) \to (\mathit{Sh}(X, Y), T_{\boldsymbol{q}'})$ is a homeomorphism.*

In order to study the topological structure of the spaces of shape morphisms next result is useful.

Proposition 3 [20]. *Let $\boldsymbol{q}^* : \mathit{Sh}(X, Y) \to \boldsymbol{Sh}(X, Y)$ be the morphism induced by $\boldsymbol{q}$. Then, $\boldsymbol{q}^*$ is an inverse limit of $\boldsymbol{Sh}(X, Y)$ in Top.*

One of the key results is the following theorem.

Theorem 1 [18]. *Let $F: X \to Y$ be a shape morphism that is a weak shape equivalence; then, for any compact connected pointed polyhedron P, F induces an isomorphism $\boldsymbol{Sh}(P,X) \to \boldsymbol{Sh}(P,Y)$ in pro-Top.*

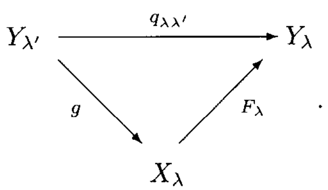
Returning to the compact metric framework, it is well known that out of pointed (compact) connected polyhedra there is a countable set $\{P_n\colon n \in \mathbb{N}\}$ containing one of each pointed homotopy type. Consider the inverse system $\{P_n, p_n\colon n \in \mathbb{N}\}$ where $p_n: P_{n+1} \to P_n$ is the constant (pointed) map. Let $(W, *)$ be the pointed internally movable connected space obtained by applying the star-construction, see [23] or [14, p. 185], to the above inverse sequence.

The space W is useful because the uniform topological type of $Sh(W,X)$ characterizes the shape of X, provided X is pointed movable. More precisely, in [18] it is shown that a shape morphism between pointed movable metric continua $F: X \to Y$ is a shape equivalence iff F induces a bi-uniform homeomorphism $F^*: Sh(W,X) \to Sh(W,Y)$.

Recall the following definition due to Dydak [5].

Definition 1. Let X, Y be pointed topological spaces, $F: X \to Y$ be a shape morphism and $\boldsymbol{p}: X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, A)$ and $\boldsymbol{q}: Y \to \boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, A)$ be HPol$_*$-expansions of X and Y, respectively and a level preserving morphism $\{F_\lambda\}_{\lambda \in A}$ representing F.

F is said to be a *weak shape domination* provided for any $\lambda \in A$ there exist $\lambda' \geqslant \lambda$ and a pointed H-map $g: Y_{\lambda'} \to X_\lambda$ such that the following diagram commutes

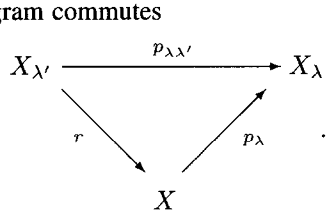


Proposition 4. *Let X, Y be pointed continua. Let $F: X \to Y$ be a shape morphism that is a weak shape domination. If X is uniformly pointed movable it follows that $F^*(Sh(Z,X))$ is a dense subspace of $Sh(Z,Y)$ for any pointed continuum Z.*

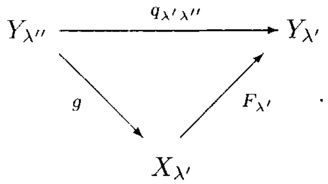
Proof. We keep the notation of above definition.

Let $\beta \in Sh(Z,Y)$ and $\lambda \in A$. Describe $\beta: Z \to Y$ as an approaching morphism $\beta: Z \to \boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, A)$ (see, for example, [14, p. 28]).

Take $\lambda' \in A$ such that there is a shape morphism $r: X_{\lambda'} \to X$ such that the following diagram commutes



Now we consider $\lambda'' \geqslant \lambda'$ and a pointed H-map $g: Y_{\lambda''} \to X_{\lambda'}$ as in the last definition, i.e., such that the following diagram commutes



Define $\alpha = r \circ g \circ \beta_{\lambda''} : Z \to X$.

We have that $F^*(\alpha) \in V_\lambda^\beta$. Indeed,

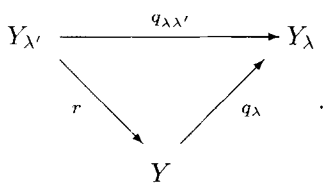
$$\begin{aligned} q_\lambda \circ \big(F^*(\alpha)\big) &= q_\lambda \circ F \circ \alpha = F_\lambda \circ p_\lambda \circ \alpha = F_\lambda \circ p_\lambda \circ r \circ g \circ \beta_{\lambda''} \\ &= F_\lambda \circ p_{\lambda\lambda'} \circ g \circ \beta_{\lambda''} = q_{\lambda\lambda'} \circ F_{\lambda'} \circ g \circ \beta_{\lambda''} \\ &= q_{\lambda\lambda'} \circ q_{\lambda'\lambda''} \circ \beta_{\lambda''} = q_{\lambda\lambda''} \circ \beta_{\lambda''} = \beta_\lambda = q_\lambda \circ \beta. \qquad \square \end{aligned}$$

Remark. Note that weak shape dominations preserve movability and uniform movability (Theorem 2.11 of [5]). Then, in the last proposition, the uniform pointed movability of X implies the uniform pointed movability of Y.

Proposition 5. *Let X, Y be pointed continua. Let $F: X \to Y$ be a shape morphism such that $F^*(Sh(W,X))$ is a dense subspace of $Sh(W,Y)$. Then F is a weak shape domination provided Y is uniformly pointed movable.*

Proof. Assume F to be represented by a level preserving morphism $\{F_\lambda\}_{\lambda \in A}$.

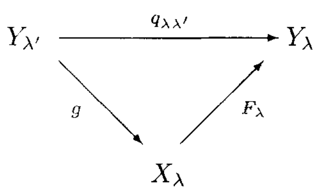
Take $\lambda \in A$. There exist $\lambda' \in A$ and a shape morphism $r: Y_{\lambda'} \to Y$ such that the following diagram commutes



Since W shape dominates every pointed finite polyhedron there are shape morphisms $i_{\lambda'}: Y_{\lambda'} \to W$ and $r_{\lambda'}: W \to Y_{\lambda'}$ such that $S(Id_{Y_{\lambda'}}) = r_{\lambda'} \circ i_{\lambda'}$.

Define $\beta = r \circ r_{\lambda'}: W \to Y$. From the density of $F^*(Sh(W,X))$ there is $\alpha \in Sh(W,X)$ such that $F^*(\alpha) \in V_\lambda^\beta$.

Take $g = p_\lambda \circ \alpha \circ i_{\lambda'}: Y_{\lambda'} \to X_\lambda$. It follows that the diagram



commutes.

Indeed,

$$F_\lambda \circ g = F_\lambda \circ p_\lambda \circ \alpha \circ i_{\lambda'} = q_\lambda \circ F \circ \alpha \circ i_{\lambda'} = q_\lambda \circ \beta \circ i_{\lambda'}$$
$$= q_\lambda \circ r \circ r_{\lambda'} \circ i_{\lambda'} = q_\lambda \circ r = q_{\lambda\lambda'}. \qquad \square$$

Corollary 1. *Let X, Y be uniformly pointed movable continua and let $F : X \to Y$ be a shape morphism. The following conditions are equivalent:*

(a) *F is a weak shape domination.*

(b) *$F^*(Sh(Z, X))$ is a dense subspace of $Sh(Z, Y)$, for every pointed continuum Z.*

(c) *$F^*(Sh(W, X))$ is a dense subspace of $Sh(W, Y)$.*

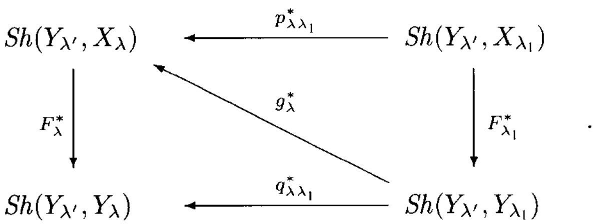
From Theorem 1, we can obtain an alternative short proof of the following theorem:

Theorem 2 (Dydak). *Let X, Y be pointed continua. Let $F : X \to Y$ be a shape morphism that is a weak shape equivalence. If Y is pointed movable it follows that F is a weak shape domination.*

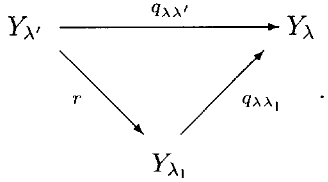
Proof. We take polyhedral expansions $\boldsymbol{p} : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, A)$ and $\boldsymbol{q} : Y \to \boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, A)$ of X and Y, respectively and a level preserving morphism $\{F_\lambda\}_{\lambda \in A}$ representing F.

Let $\lambda \in A$. Take $\lambda' \geqslant \lambda$ an associated pointed movability index.

Applying Theorem 1 ($P = Y_{\lambda'}$) and Morita's characterization of isomorphisms in pro-categories, [15], there exist $\lambda_1 \in A$, $\lambda_1 \geqslant \lambda'$, and $g^*_\lambda : Sh(Y_{\lambda'}, Y_{\lambda_1}) \to Sh(Y_{\lambda'}, X_\lambda)$ such that the following diagram commutes



Take a shape morphism $r : Y_{\lambda'} \to Y_{\lambda_1}$ such that the following diagram commutes



Let $g = g^*_\lambda(r) : Y_{\lambda'} \to X_\lambda$.

We have that

$$F_\lambda \circ g = F_\lambda \circ g^*_\lambda(r) = (F^*_\lambda \circ g^*_\lambda)(r) = q^*_{\lambda\lambda_1}(r) = q_{\lambda\lambda_1} \circ r = q_{\lambda\lambda'}.$$

Then, F is a weak shape domination. $\square$

Next corollary is an easy consequence of the last theorem (see [14, p. 186], for a proof).

Corollary 2. *Let X, Y be pointed continua. Let $F : X \to Y$ be a weak shape equivalence. If Y is pointed movable, then F is an epimorphism in the pointed shape category Sh.*

Remark. When Y is uniformly pointed movable we have the following alternative short proof:

Let $\alpha, \beta : Y \to Z$ be shape morphisms such that $\alpha \circ F = \beta \circ F$.

Since $F^*(Sh(Y, X))$ is a dense subspace of $Sh(Y, Y)$, we have that $\alpha^* = \beta^* : Sh(Y, Y) \to Sh(Y, Z)$. Then, $\alpha = \beta$.

Example. Under the conditions of the last theorem $F^*(Sh(W, X))$ can be a proper dense subspace of $Sh(W, Y)$.

Proof. Take a Kahn's (see [12] and [4]) sequence of compact connected polyhedra $\{Z_n\}_{n\in\mathbb{N}}$ and maps $h_n : Z_{n+1} \to Z_n$, $n \in \mathbb{N}$ such that for $i < j$ the map $h_i \circ \cdots \circ h_j : Z_{j+1} \to Z_i$ is essential and Z_n is $[(2p - 1) + (2p - 2)n]$-connected (p fixed odd prime).

Using the above sequence Draper and Keesling, see [4], constructed pointed movable metric continua X and Y, a continuous map $f : X \to Y$ that is a weak shape equivalence but it is not a shape equivalence.

X is the inverse limit of the sequence $\{(X_n, x_n)\}_{n\in\mathbb{N}}$ where $(X_n, x_n) = \bigvee_1^n (Z_i, z_i)$ and the bonding maps $p_{nn+1} : X_{n+1} \to X_n$ are defined by $p_{nn+1}(x) = x$ for $x \in Z_i$, $i \leqslant n$, and $p_{nn+1}(x) = h_n(x)$ for $x \in Z_{n+1}$.

Y is the inverse limit of the sequence $\{(Y_n, y_n)\}_{n\in\mathbb{N}}$ where $(Y_n, y_n) = \bigvee_1^n (Z_i, z_i)$ and the bonding maps $q_{nn+1} : Y_{n+1} \to Y_n$ are defined by $q_{nn+1}(x) = x$ for $x \in Z_i$, $i \leqslant n$, and $q_{nn+1}(x) = y_n$ for $x \in Z_{n+1}$.

The map $f : X \to Y$ is induced by the sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n : X_{n+1} \to Y_n$ defined as $f_n(x) = x$, $x \in X_n$ and $f_n(x) = y_n$ if $x \in Z_{n+1}$.

Draper and Keesling showed that the shape morphism $S(f) = F$ induces isomorphisms of all the homotopy pro-groups, then $F^*(Sh(W, X))$ is a dense subspace of $Sh(W, Y)$. On the other hand, there are shape morphisms $r : W \to Y$ and $j : Y \to W$ such that $r \circ j = S(Id_Y)$.

$r \notin F^*(Sh(W, X))$. Indeed, if there is $\alpha \in Sh(W, X)$ such that $r = F^*(\alpha) = F \circ \alpha$ one has $F \circ \alpha \circ j = r \circ j = S(Id_Y)$. Then F is a shape domination and using a theorem of Dydak [5] (see also [18] for a different proof using the ultrametric) F would be a shape equivalence. This is contradictory. □

Theorem 3. *Let X, Y be pointed continua and let $F : X \to Y$ be a weak shape equivalence, then $F^* : Sh(Z, X) \to Sh(Z, Y)$ is injective for every uniformly pointed movable continuum Z.*

Proof. Let $\alpha, \beta \in Sh(Z, X)$ such that $F \circ \alpha = F \circ \beta$.

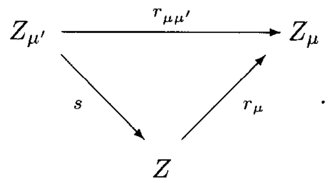
We take polyhedral expansions $\boldsymbol{r}: Z \to \boldsymbol{Z} = (Z_\mu, r_{\mu\mu'}, M)$, $\boldsymbol{p}: X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, A)$ and $\boldsymbol{q}: Y \to \boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, A)$ of Z, X and Y, respectively. Consider $\{F_\lambda\}_{\lambda \in A}$ to be a level preserving morphism representing F.

We assume α, β to be represented by $(\{\alpha_\lambda\}_{\lambda \in A}, \phi)$ and $(\{\beta_\lambda\}_{\lambda \in A}, \psi)$.

Take $\lambda \in A$.

Let $\mu \geqslant \phi(\lambda), \psi(\lambda)$ such that $F_\lambda \circ \alpha_\lambda \circ r_{\phi(\lambda)\mu} = F_\lambda \circ \beta_\lambda \circ r_{\psi(\lambda)\mu}$.

Take $\mu' \geqslant \mu$ to be the associated uniform movability index. Then there is a shape morphism $s: Z_{\mu'} \to Z$ such that the following diagram commutes



We have that $F \circ \alpha \circ s = F \circ \beta \circ s$. Theorem 1 implies that $\alpha \circ s = \beta \circ s$. Then $p_\lambda \circ \alpha \circ s = p_\lambda \circ \beta \circ s$.

Since $p_\lambda \circ \alpha = \alpha_\lambda \circ r_{\phi(\lambda)\mu} \circ r_\mu$ and $p_\lambda \circ \beta = \beta_\lambda \circ r_{\psi(\lambda)\mu} \circ r_\mu$, it follows that

$$\alpha_\lambda \circ r_{\phi(\lambda)\mu} \circ r_\mu \circ s = \beta_\lambda \circ r_{\psi(\lambda)\mu} \circ r_\mu \circ s.$$

Therefore,

$$\alpha_\lambda \circ r_{\phi(\lambda)\mu} \circ r_{\mu\mu'} = \beta_\lambda \circ r_{\psi(\lambda)\mu} \circ r_{\mu\mu'}$$

and

$$\alpha_\lambda \circ r_{\phi(\lambda)\mu'} = \beta_\lambda \circ r_{\psi(\lambda)\mu'},$$

consequently $\alpha = \beta$. $\square$

From the last theorem and Corollary 2 we obtain the following theorem

Theorem 4. *Let X, Y be uniformly pointed movable continua and let $F: X \to Y$ be a weak shape equivalence, then F is both a monomorphism and an epimorphism in the category Sh_M.*

Corollary 3. *Let X, Y be pointed movable metric continua and let $F: X \to Y$ be a very weak shape equivalence, then F is both a monomorphism and an epimorphism in the category Sh_M of pointed movable metric continua.*

The fact that Sh_M is not balanced follows from Theorem 4 and next proposition which shows something stronger.

Proposition 6. *There exist pointed movable metric continua X, Y and a weak shape equivalence $F: X \to Y$ that is not a monomorphism in the category Sh_C. Thus, there exists a shape morphism $F: X \to Y$ that is both a monomorphism and an epimorphism in the category Sh_M that is not a monomorphism in Sh_C.*

Proof. Take the inverse sequence $\{Z_n, h_n\}_{n\in\mathbb{N}}$, the metric continua X, Y and the shape morphism $F = S(f): X \to Y$ of Example 1. F is a weak shape equivalence.

Let Z be the pointed compactum defined as the inverse limit of $\{Z_n, h_n\}$.

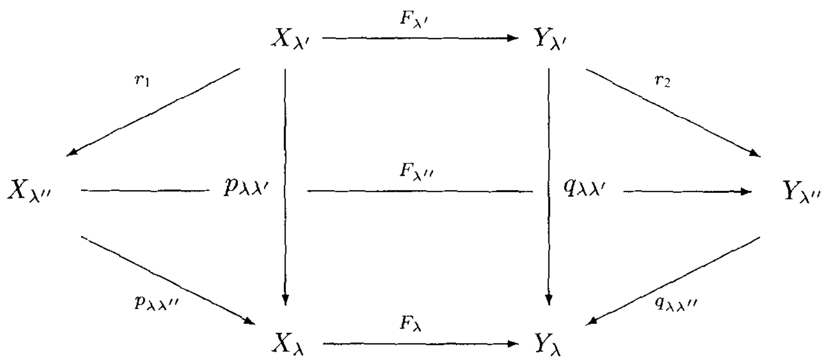
For any $n \in \mathbb{N}$, let $\alpha_n : Z_n \to X_n$ the natural inclusion and $\beta_n : Z_n \to X_n$ the trivial map.

The sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ induce continuous maps $\alpha, \beta: Z \to X$, respectively. We will denote again by α and β the corresponding shape morphisms.

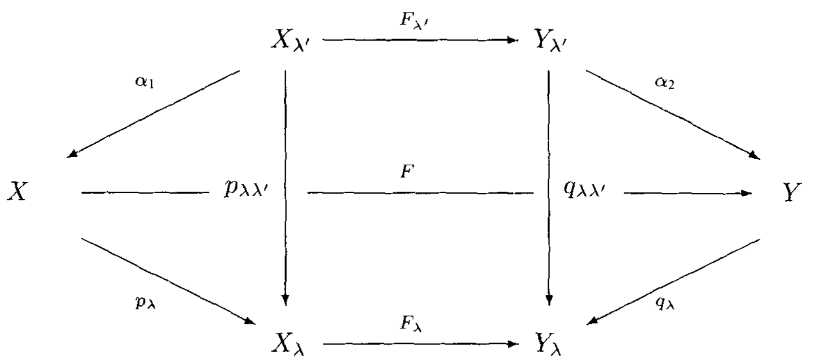
Since $f_n \circ \alpha_{n+1} = f_n \circ \beta_{n+1}$, we have that $F \circ \alpha = F \circ \beta$.

However α and β do not coincide because the composition $h_i \circ \cdots \circ h_j : Z_{j+1} \to Z_i$ is essential for $i < j$. □

Definition 2. Let X, Y be pointed topological spaces and $F: X \to Y$ a shape morphism. The triple (X, F, Y) is said to be pointed movable if there are HPol$_*$-expansions $\boldsymbol{p}: X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, A)$ and $\boldsymbol{q}: Y \to \boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, A)$ of X and Y, respectively and a level preserving morphism $\{F_\lambda\}_{\lambda\in A}$ representing F such that for every $\lambda \in A$ there exists $\lambda' \geqslant \lambda$ such that every $\lambda'' \geqslant \lambda$ admits pointed H-maps $r_1 : X_{\lambda'} \to X_{\lambda''}$ and $r_2 : Y_{\lambda'} \to Y_{\lambda''}$ such that the following diagram commutes



In a similar way (X, F, Y) is said to be uniformly pointed movable provided for every $\lambda \in A$ there exist $\lambda' \geqslant \lambda$ and shape morphisms $\alpha_1 : X_{\lambda'} \to X$ and $\alpha_2 : Y_{\lambda'} \to Y$ such that the following diagram commutes



It is obvious that the (uniform) movability of X and Y is necessary for the triple (X, F, Y) to be (uniformly) movable. It is also clear that uniform movability of triples implies movability of triples.

Theorem 5. *Let X, Y be compact connected pointed spaces and let $F: X \to Y$ be a shape morphism that is a weak shape equivalence. If the triple (X, F, Y) is pointed movable it follows that F is a shape equivalence.*

Proof. Let $\boldsymbol{p}: X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, A)$, $\boldsymbol{q}: Y \to \boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, A)$ be HPol$_*$-expansions of X and Y, respectively. Let $\{F_\lambda\}_{\lambda \in A}$ be a level preserving morphism representing F as in above definition.

Take $\lambda \in A$. Let $\lambda' \geqslant \lambda$ be a movability index for the triple (X, F, Y).

Let $R \in$ Obj(HPol$_*$).

Using Theorem 1 (see [18]), there is a morphism

$$(\{g_\lambda^*\}_{\lambda \in A}, \phi): \boldsymbol{Sh}(R, Y) \to \boldsymbol{Sh}(R, X)$$

representing the inverse of $\{F_\lambda^*\}_{\lambda \in A}$ in pro-*Top*.

Then, there is $\mu \in A$, $\mu \geqslant \lambda'$, such that the diagrams

$$\begin{array}{ccccc} Sh(R, X_{\phi(\lambda)}) & \xrightarrow{F_{\phi(\lambda)}^*} & Sh(R, Y_{\phi(\lambda)}) & \xrightarrow{g_\lambda^*} & Sh(R, X_\lambda) \\ & \nwarrow {\scriptstyle p_{\phi(\lambda)\mu}^*} & & {\scriptstyle p_{\lambda\mu}^*} \nearrow & \\ & & Sh(R, X_\mu) & & \end{array}$$

$$\begin{array}{ccccc} Sh(R, Y_{\phi(\lambda)}) & \xrightarrow{g_\lambda^*} & Sh(R, X_\lambda) & \xrightarrow{F_\lambda^*} & Sh(R, Y_\lambda) \\ & \nwarrow {\scriptstyle q_{\phi(\lambda)\mu}^*} & & {\scriptstyle q_{\lambda\mu}^*} \nearrow & \\ & & Sh(R, Y_\mu) & & \end{array}$$

commute.

Take $r_1: X_{\lambda'} \to X_\mu$ and $r_2: Y_{\lambda'} \to Y_\mu$ as in Definition 2 for $\lambda'' = \mu$.

(a) Let $h: R \to Y_{\lambda'}$ be any pointed H-map.
Consider $r_2 \circ h: R \to Y_{\lambda'} \to Y_\mu$. Let $k = (g_\lambda^* \circ q_{\phi(\lambda)\mu}^*)(r_2 \circ h): R \to X_\lambda$. We have that

$$\begin{aligned} F_\lambda \circ k &= \left(F_\lambda^* \circ g_\lambda^* \circ q_{\phi(\lambda)\mu}^*\right)(r_2 \circ h) = q_{\lambda\mu}^*(r_2 \circ h) \\ &= q_{\lambda\mu} \circ r_2 \circ h = q_{\lambda\lambda'} \circ h. \end{aligned}$$

(b) Let $u_1, u_2: R \to X_{\lambda'}$ be pointed H-maps, such that $F_{\lambda'} \circ u_1 = F_{\lambda'} \circ u_2$.
Then, $r_2 \circ F_{\lambda'} \circ u_1 = r_2 \circ F_{\lambda'} \circ u_2$.

Since $r_2 \circ F_{\lambda'} = F_\mu \circ r_1$, we have $q_{\phi(\lambda)\mu} \circ F_\mu \circ r_1 \circ u_1 = q_{\phi(\lambda)\mu} \circ F_\mu \circ r_1 \circ u_2$. Thus

$$F_{\phi(\lambda)} \circ p_{\phi(\lambda)\mu} \circ r_1 \circ u_1 = F_{\phi(\lambda)} \circ p_{\phi(\lambda)\mu} \circ r_1 \circ u_2.$$

By composing with g_λ^* we obtain

$$(g_\lambda^* \circ F_{\phi(\lambda)}^* \circ p_{\phi(\lambda)\mu}^*)(r_1 \circ u_1) = (g_\lambda^* \circ F_{\phi(\lambda)}^* \circ p_{\phi(\lambda)\mu}^*)(r_1 \circ u_2).$$

Now we have,

$$p_{\lambda\mu}^*(r_1 \circ u_1) = p_{\lambda\mu}^*(r_1 \circ u_2).$$

Therefore,

$$p_{\lambda\mu} \circ r_1 \circ u_1 = p_{\lambda\mu} \circ r_1 \circ u_2, \text{ and } p_{\lambda\lambda'} \circ u_1 = p_{\lambda\lambda'} \circ u_2.$$

Now the proof of the theorem follows from similar arguments of the Theorems 2, 3 of [14, pp. 148–149].

Corollary 4. *Let (Y, X) be a pointed movable pair of continua. The embedding $j : X \to Y$ is a shape equivalence if and only if it is a weak shape equivalence.*

Corollary 5. *Let (Y, X) be a pointed movable pair of metric continua. The embedding $j : X \to Y$ is a shape equivalence if and only if it is a very weak shape equivalence.*

Using Fox's theorem (see [7], for example), we have

Corollary 6. *Let X, Y be pointed metric continua and let $F : X \to Y$ be a shape morphism that is a very weak shape equivalence. Then F is a shape equivalence provided the pair (Z, X) of Theorem* 4.3.3 *of* [7] *is pointed movable.*

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