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The Connectivities of Leaf Graphs of 2-Connected Graphs

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Given a connected graph G, denote by $\mathscr{V}$ the family of all the spanning trees of G. Define an adjacency relation in $\mathscr{V}$ as follows: the spanning trees $\mathbf{t}$ and $\mathbf{t}'$ are said to be adjacent if for some vertex $u \in V$, $\mathbf{t} - u$ is connected and coincides with $\mathbf{t}' - u$. The resultant graph $\mathscr{G}$ is called the leaf graph of G. The purpose of this paper is to show that if G is 2-connected with minimal degree δ, then $\mathscr{G}$ is $(2\delta - 2)$-connected.

1. INTRODUCTION

Let $G = (V, E)$ be a connected graph. Let $\mathbf{t}$ be a spanning tree of G. If a vertex $u \in V$ is adjacent to only one vertex in $\mathbf{t}$, then the vertex u is called an *outer vertex* of the spanning tree $\mathbf{t}$. A vertex which is not outer is called *inner*. An edge is called "outer" if it is incident to an outer vertex and "inner" otherwise.

Let u be an outer vertex of a spanning tree $\mathbf{t}$ and assume that the edge uu_1 is included in the set $E(\mathbf{t})$ of all the edges in $\mathbf{t}$. Let $N(u) = \{u_i \in V \mid uu_i \in E\}$. For any vertex u_i in $N(u)$, $\mathbf{t}_i = (\mathbf{t} - uu_1) \cup uu_i$ is also a spanning tree of G. Since the vertex u is adjacent to only u_1 in $\mathbf{t}$, u is also adjacent to only u_i in $\mathbf{t}_i$. Thus the vertex u is an outer vertex in $\mathbf{t}_i$.

The subgraph $\mathbf{t} - uu_1$ includes two connected components because any edge in a spanning tree is a bridge. One is a singleton u and the other is

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the subgraph $\mathbf{t}-u$. It is a plain fact that $\mathbf{t}-u=\mathbf{t}_i-u$ and this graph is connected.

Let $\mathscr{V}$ be the set of all the spanning trees of G. We define an adjacency relation so that $\mathbf{t}$ and $\mathbf{s}\in\mathscr{V}$ are adjacent if and only if there exists a vertex $u\in V$ such that $\mathbf{t}-u=\mathbf{s}-u$ and this graph is connected.

The graph thus obtained is studied by H. Broersma and Li Xueliang [1]. They called it the *leaf-exchange spanning tree graph*. In this paper, we call it simply the *leaf graph* and denote it by $\mathscr{G}$.

The following fact is well known.

THEOREM 1. *The leaf graph of a* 2*-connected graph is connected.*

The definition of a leaf graph induces a natural map from the edge set of the leaf graph to the vertex set V by μ: $\mathbf{ts}\mapsto u$. A lemma found in an unpublished paper by A. J. Bondy and L. Lovász can be stated as follows.

THEOREM 2 (Bondy and Lovász). *Let G be a* 2*-connected graph and u any vertex in G. Then the induced subgraph of $\mu^{-1}(V\backslash u)$ of the leaf graph is connected.*

The *degree* of $u\in V$ is the number of vertices of $N(u)$. The minimum numbers of the degree of the vertices in G is called the *minimum degree*. In this paper, we show an extension of Theorem 1.

THEOREM 3. *Let $G=(V, E)$ be a* 2*-connected graph of minimum degree δ. Then the leaf graph of G is* $(2\delta-2)$*-connected.*

Suppose that G has a Hamiltonian path $\mathbf{t}$ with outer vertices of degree δ. Then $\mathbf{t}$ is adjacent to $2\delta-2$ spanning trees in the leaf graph of G. Thus the lower bound $2\delta-2$ is the best possible.

2. PREPARATIONS

In what follows, we often use a concept called a passage. A passage is a path in a leaf graph which is determined by a special rule. In this section, we introduce the concepts and notations used in the subsequent arguments.

Let $\mathbf{t}$ be a spanning tree of a connected graph G. Because a spanning tree does not include a cycle, there exists exactly one path between any vertices u and $v\in V$. We denote the path by $P_{\mathbf{t}}(u, v)$. Similarly, there exists uniquely the shortest path between any disjoint subtrees $\mathbf{a}$ and $\mathbf{b}$ of $\mathbf{t}$, denoted by $P_{\mathbf{t}}(\mathbf{a}, \mathbf{b})$. A *simple path* is a path in a spanning tree $\mathbf{t}$ which includes an outer vertex of $\mathbf{t}$ and any inner vertices of which are of degree two in $\mathbf{t}$.

As explained in Section 1, any edge of the leaf graph determines a vertex in V. Given an oriented path $\mathscr{P}$ in the leaf graph $\mathscr{G}$, the vertex in G

which is determined by the initial edge of $\mathscr{P}$ in the leaf graph is called the *departing point* of $\mathscr{P}$.

Assume that $u_1 u_n \in E$ is not contained in $\mathbf{t}$ and let $P_{\mathbf{t}}(u_1, u_n) = (u_1, u_2, \ldots, u_n)$ be the path in $\mathbf{t}$. We define a path in the leaf graph $\mathscr{G}$ from $\mathbf{t}$ to the spanning tree $\mathbf{t}' = (\mathbf{t} - u_i u_{i+1}) \cup u_1 u_n$, called a short-cut passage or a round passage.

Let us fix $i < n$. The subgraph $\mathbf{t} - u_i u_{i+1}$ has two connected components. We denote by A_i the connected component which includes the vertex u_1 and the other by B_i. Let X_i be the set of all the outer vertices of $\mathbf{t}$ included in $A_i \backslash u_1$. Let

$$P_i^0 = (u_1, u_2, \ldots, u_i, u_{i+1}).$$

We want to transfer the outer edge $u_i u_{i+1}$ of P_i^0 to $u_1 u_n$ by a sequence of movements of outer edges.

First of all consider the simplest case where X_i is an empty set. Then P_i^0 is a simple path. See Fig. 1. Since the vertex u_1 is an outer vertex, there exists a natural path from $\mathbf{t}$ to $\mathbf{t}'$ in $\mathscr{G}$ which is determined by P_i^0.

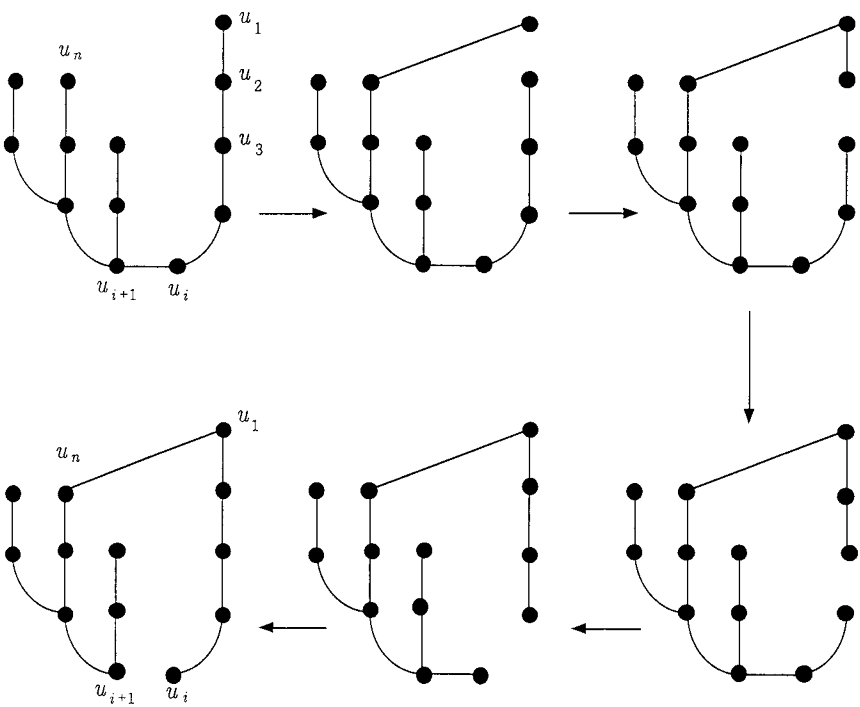


FIGURE 1

In fact, let $\mathbf{r}_1 = (\mathbf{t} - u_1 u_2) \cup u_1 u_n$ and

$$\mathbf{r}_j = (\mathbf{r}_{j-1} - u_j u_{j+1}) \cup u_j u_{j-1}$$

for any $j \leqslant i$. Then $\mathbf{r}_j$ is a spanning tree of G and $\mathbf{r}_j$ is adjacent to $\mathbf{r}_{j-1}$ in the leaf graph for any $j \leqslant i$. If $j \neq h$, then $\mathbf{r}_j \neq \mathbf{r}_h$. Thus

$$\mathscr{P}_i^0 = (\mathbf{t}, \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_i = \mathbf{t}')$$

is a path between $\mathbf{t}$ and $\mathbf{t}'$ in the leaf graph. We call the path the *short-cut passage* determined by the edge $u_1 u_n$ and the path P_i^0. The departing point of the short-cut passage is u_1.

Next suppose that X_i is not an empty set and any vertices in X_i are adjacent to B_i. Let us define a path from $\mathbf{t}$ to $\mathbf{t}'$ by a sequence of short-cut passages as follows.

Let $X_i = \{x_1^1, x_2^1, ..., x_l^1\}$. Let y_j be any vertex in $N(x_j^1) \cap B_i$ for $j \leqslant l$. We subdivide A_i into the set of paths as follows. Let

$$P_i^1 = P_{\mathbf{t}}(x_1^1, P_i^0) = (x_1^1, x_1^2, ..., x_1^{m_1})$$

be the shortest path from the outer vertex x_1^1 to the path P_i^0 in the spanning tree $\mathbf{t}$. Similarly, we recursively define the path

$$P_i^j = P_{\mathbf{t}}\left(x_j^1, \bigcup_{h=0}^{j-1} P_i^h\right) = (x_j^1, x_j^2, ..., x_j^{m_j})$$

from the outer vertex x_j^1 to the subtree $\bigcup_{h=0}^{j-1} P_i^h$ for any $j \leqslant l$. See Fig. 2i.

Clearly the path P_i^l is simple. Thus there exists a short-cut passage determined by $x_l^1 y_l$ and P_i^l. It is a path between $\mathbf{t}$ and

$$\mathbf{r}_l = (\mathbf{t} - x_l^{m_l - 1} x_l^{m_l}) \cup x_l^1 y_l$$

in the leaf graph $\mathscr{G}$. See Figs. 2i–2ii. We denote the short-cut passage by $\mathscr{P}_i^l$.

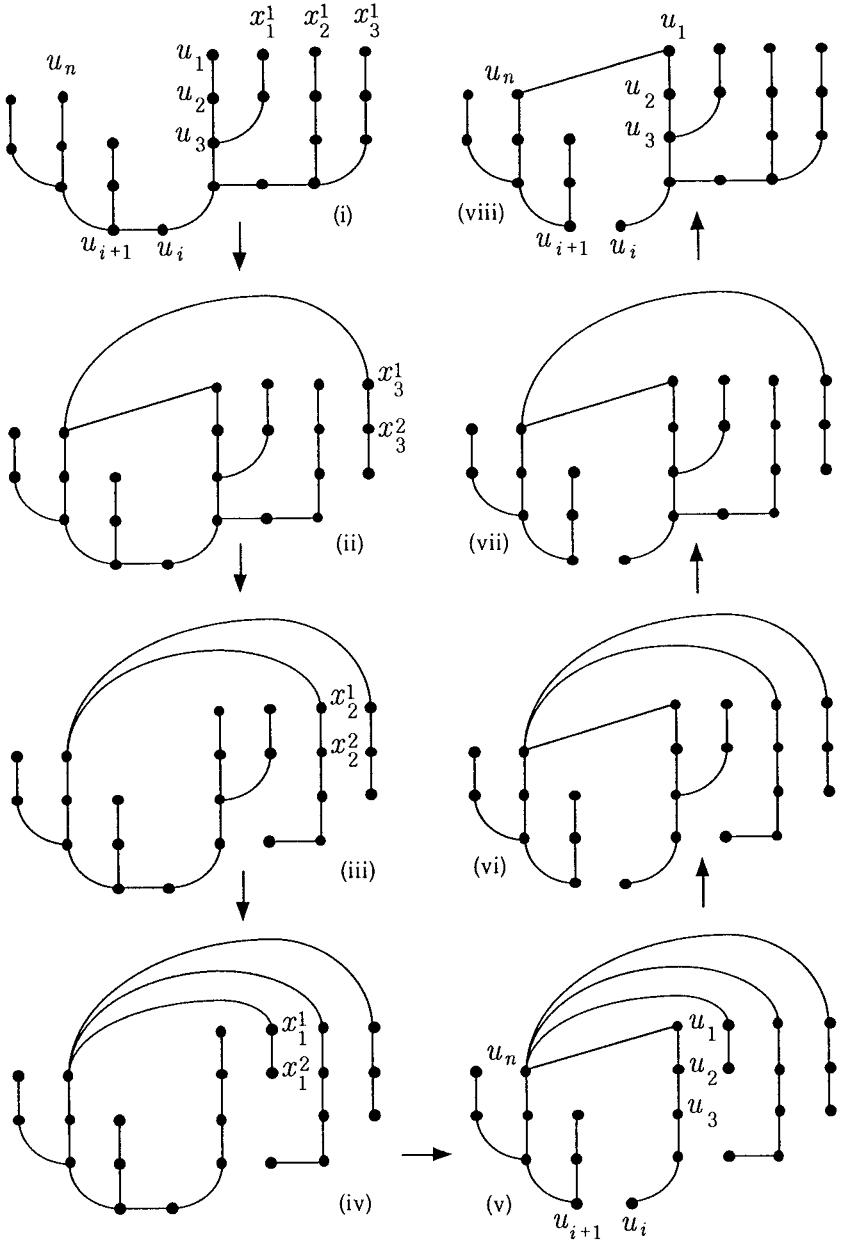
If P_i^{l-1} admits an inner vertex of degree three in $\mathbf{t}$, then the vertex is an outer vertex of P_i^l. Since the spanning tree $\mathbf{r}_l$ does not include the edge $x_l^{m_l - 1} x_l^{m_l}$, the path P_i^{l-1} is simple in $\mathbf{r}_l$. Thus there also exists a short-cut passage $\mathscr{P}_i^{l-1}$ determined by $x_{l-1}^1 y_{l-1}$ and P_i^{l-1}. It is a path from $\mathbf{r}_l$ to

$$\mathbf{r}_{l-1} = (\mathbf{t} - x_{l-1}^{m_{l-1} - 1} x_l^{m_{l-1}}) \cup x_{l-1}^1 y_{l-1}$$

in the leaf graph. See Figs. 2ii–2iii.

We recursively define a short-cut passage $\mathscr{P}_i^j$ determined by $x_j^1 y_j$ and P_i^j which is a path in the leaf graph from $\mathbf{r}_{j+1}$ to

$$\mathbf{r}_j = (\mathbf{r}_{j+1} - x_j^{m_j - 1} x_j^{m_j}) \cup x_j^1 y_j$$



x_1^1 x_2^1 x_3^1
u_n
u_1
u_2
u_3
u_{i+1} u_i
(i)
x_3^1
x_3^2
(ii)
x_2^1
x_2^2
(iii)
x_1^1
x_1^2
(iv)
u_n
u_1
u_2
u_3
u_{i+1} u_i
(v)
(vi)
(vii)
u_1
u_n
u_2
u_3
(viii)
u_{i+1} u_i

FIGURE 2

for $0 \leqslant j < l-1$. Finally let

$$\mathbf{r}_0 = (\mathbf{r}_1 - u_i u_{i+1}) \cup u_1 u_n$$

See Fig. 2v.

Now we have moved the edge $u_i u_{i+1}$ to $u_1 u_n$. Next let us move the edge $x_j^1 y_j$ back to the original place $x_j^{m_j-1} x_j^{m_j}$ for all $j > 0$. Define paths

$$\tilde{P}_i^j = (y_j, x_j^1, x_j^2, ..., x_j^{m_j-1})$$

for any $j > 0$ in the spanning tree $\mathbf{r}_0$. Clearly $\tilde{P}_i^1$ is a simple path in $\mathbf{r}_0$. Therefore there exists a short-cut passage $\tilde{\mathscr{P}}_i^1$ determined by $x_1^{m_i-1} x_1^{m_i}$ and P_i^1 which is a path from $\mathbf{r}_0$ to the spanning tree

$$\tilde{\mathbf{r}}_1 = (\tilde{\mathbf{r}}_0 - x_1^1 y_1) \cup x_1^{m_1-1} x_1^{m_1}$$

in the leaf graph $\mathscr{G}$. See Figs. 2v–2vi.

Let us define spanning trees

$$\tilde{\mathbf{r}}_j = (\tilde{\mathbf{r}}_{j-1} - x_j^1 y_j) \cup x_j^{m_j-1} x_j^{m_j}$$

recursively for any $j > 1$. As before, the path $\tilde{P}_i^j$ is a simple path in $\tilde{\mathbf{r}}_{j-1}$. Let $\tilde{\mathscr{P}}_i^j$ be the short-cut passage determined by the edge $x_j^{m_j-1} x_j^{m_j}$ and the simple path $\tilde{P}_i^j$ which is a path from $\tilde{\mathbf{r}}_{j-1}$ to $\tilde{\mathbf{r}}_j$.

Especially

$$\tilde{\mathbf{r}}_l = (\mathbf{r}_{l-1} - x_l^1 y_l) \cup x_l^{m_l-1} x_l^{m_l} = (\mathbf{t} - u_i u_{i+1}) \cup u_1 u_n = \mathbf{t}'$$

Thus the sequence of the short-cut passage

$$(\mathscr{P}_i^l, \mathscr{P}_i^{l-1}, ..., \mathscr{P}_i^2, \mathscr{P}_i^1, \mathscr{P}_i^0, \tilde{\mathscr{P}}_i^1, \tilde{\mathscr{P}}_i^2, ..., \tilde{\mathscr{P}}_i^{l-1}, \tilde{\mathscr{P}}_i^l)$$

is a path between $\mathbf{t}$ and $\mathbf{t}'$ in the leaf graph $\mathscr{G}$, which is called a *round passage with respect to* $\{y_j\}$. The departing point of the round passage is x_l^1.

3. THE PROOF OF THE THEOREM

Assume for contradiction that there exists a cut set $\mathscr{S}$ of $\mathscr{G}$ which contains at most $2\delta - 3$ vertices. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be the connected components of $\mathscr{G} - \mathscr{S}$. Let $\mathbf{t}$ be a vertex in $\mathscr{S}$ and $\mathbf{t}_{a,1} \in \mathscr{C}_1$ and $\mathbf{t}_{1,b} \in \mathscr{C}_2$ be adjacent to $\mathbf{t}$. Let us find out internally disjoint $2\delta - 2$ paths from $\mathbf{t}_{a,1}$ to $\mathbf{t}_{1,b}$ in $\mathscr{G}$.

Let u be the vertex such that $\mathbf{t} - u = \mathbf{t}_{a,1} - u$ and v be the vertex such that $\mathbf{t} - v = \mathbf{t}_{1,b} - v$. Notice that $u \neq v$, for otherwise $\mathbf{t}_{a,1}$ and $\mathbf{t}_{1,b}$ are adjacent.

Let $N(u)=\{u_i\}$ and $N(v)=\{v_i\}$ and edges uu_1 and vv_1 be included in $E(\mathbf{t})$. We define a subgraph

$$\mathbf{t}_{i,j}=(\mathbf{t}-uu_1-vv_1)\cup uu_i\cup vv_j$$

for any i and j. If $uu_i\neq vv_j$, then $\mathbf{t}_{i,j}$ is a spanning tree of G.

If $uv\notin E$, then always $\mathbf{t}_{i,j}$ is a spanning tree of G. Since u and v are outer vertices of $\mathbf{t}_{i,j}$, we have $(\mathbf{t}_{i,j}-vv_j)\cup vv_1=\mathbf{t}_{i,1}$ and $(\mathbf{t}_{i,j}-uu_i)\cup uu_1=\mathbf{t}_{1,j}$ for any i and j. See Fig. 3.

Thus there exist paths

$$\mathscr{P}_i=(\mathbf{t}_{a,1},\mathbf{t}_{i,1},\mathbf{t}_{i,b},\mathbf{t}_{1,b})$$

$$\mathscr{Q}_j=(\mathbf{t}_{a,1},\mathbf{r}_{a,j},\mathbf{t}_{1,j},\mathbf{t}_{1,b})$$

in the leaf graph. If $i\neq 1, a$ and $j\neq 1, b$, then $\mathscr{P}_i\neq\mathscr{Q}_j$. But

$$\mathscr{P}_1=(\mathbf{t}_{a,1},\mathbf{t}_{1,1},\mathbf{t}_{1,b},\mathbf{t}_{1,b})=(\mathbf{t}_{a,1},\mathbf{t}_{1,1},\mathbf{t}_{1,b})=(\mathbf{t}_{a,1},\mathbf{t}_{a,1},\mathbf{t}_{1,1},\mathbf{t}_{1,b})=\mathscr{Q}_1$$

$$\mathscr{P}_a=(\mathbf{t}_{a,1},\mathbf{t}_{a,1},\mathbf{t}_{a,b},\mathbf{t}_{1,b})=(\mathbf{t}_{a,1},\mathbf{t}_{a,b},\mathbf{t}_{1,b})=(\mathbf{t}_{a,1},\mathbf{t}_{a,b},\mathbf{t}_{1,b},\mathbf{t}_{1,b})=\mathscr{Q}_b$$

Therefore we have found out the following internally disjoint $2\delta-2$ paths in the case $uv\notin E$.

$$\{\mathscr{P}_1,\mathscr{P}_2,...,\mathscr{P}_\delta,\mathscr{Q}_1,\mathscr{Q}_2,...,\mathscr{Q}_\delta\mid\mathscr{P}_1=\mathscr{Q}_1,\mathscr{P}_a=\mathscr{Q}_b\}.$$

We call the path $\mathscr{P}_i$ or $\mathscr{Q}_j$ a *short path* from $\mathbf{t}_{a,1}$ to $\mathbf{t}_{1,b}$.

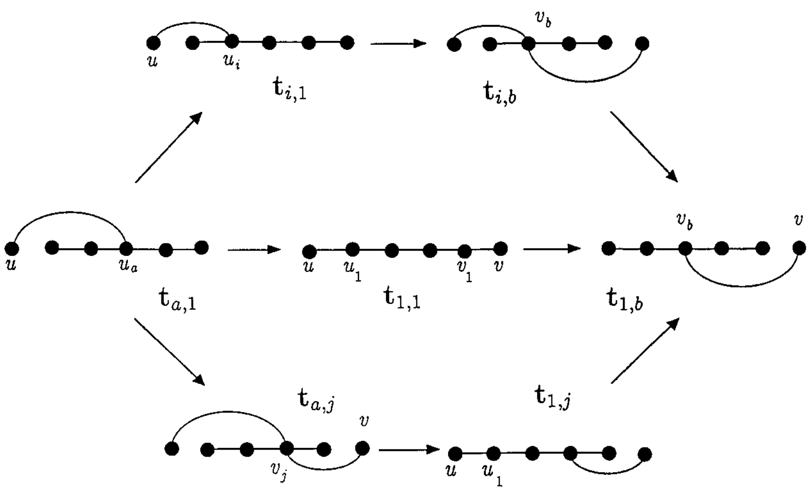


FIGURE 3

If $uv \in E$, then v_i and $v_j = u$ for some i and j. Without loosing generality, we may label the index of adjacent vertices of u and v so that $u_\delta = v$ and $v_\delta = u$. Then $uu_\delta = uv = v_\delta v$ and $\mathbf{t}_{\delta,\delta}$ is not a spanning tree. Since v is not outer in $\mathbf{t}_{\delta,j}$, the spanning tree $\mathbf{t}_{\delta,j}$ is not adjacent to $\mathbf{t}_{\delta,1}$. Similarly $\mathbf{t}_{i,\delta}$ is not adjacent to $\mathbf{t}_{1,\delta}$. Thus some of the short paths constructed above do not constitute a path in the leaf graph if $uv \in E$.

We will construct paths between $\mathbf{t}_{a,1}$ and $\mathbf{t}_{1,b}$ in order to compensate the loss of short paths. Special attention will be paid to the departing points of paths in order to show the internal disjointness of supplemented paths and the other paths. The departing point of a short path is u or v. If the departing point x of a path is not u and v, then we use the following convention of the index of $N(x) = \{x_i\}$.

(1) If $x_i = u$, then $i = \delta$.

(2) If $x_i = v$ and $\delta > 2$, then $i = \delta - 1$.

Notice the following two properties of short paths. A spanning tree which is an inner vertex of a short path does not include the edge uv. Because we do not move an edge in $\mathbf{t}$ except those which are incident to u or v, the above spanning tree does not include also the edge $e \in E \backslash E(\mathbf{t})$ which is not incident to u or v.

Now we divide the argument into the following three cases.

Case 1. $a = \delta$ and $b \neq \delta$.

Case 2. $a \neq \delta$ and $b \neq \delta$.

Case 3. $a = \delta$ and $b = \delta$.

Case 1. $a = \delta$ and $b \neq \delta$.

(1) At first let us count the number of the existing short paths in the present case. If $i < \delta$, then the vertex u_i is not v. Since the vertex v_b is not u, the spanning tree $\mathbf{t}_{i,b}$ is adjacent to $\mathbf{t}_{1,b}$. The spanning tree $\mathbf{t}_{i,1}$ is adjacent to $\mathbf{t}_{i,b}$ for $i < \delta$. Thus the short paths

$$\{\mathscr{P}_1, \mathscr{P}_2, ..., \mathscr{P}_{\delta-1}\}$$

exist.

(2) Next we shall find out a path $\mathscr{R}$ from $\mathbf{t}_{\delta,1}$ to $\mathbf{t}_{1,\delta}$ which is internally disjoint from the above short paths. Since $\mathbf{t} = (\mathbf{t}_{\delta,1} - uv) \cup uu_1$ and $\mathbf{t}_{1,\delta} = (\mathbf{t} - vv_1) \cup vu$, the spanning tree $(\mathbf{t}_{\delta,1} - vv_1) \cup uu_1$ is $\mathbf{t}_{1,\delta}$.

Let $P = P_{\mathbf{t}_{\delta,1}}(u_1, v)$. The vertex v is adjacent to exactly u and v_1 in $\mathbf{t}_{\delta,1}$. Since the vertex u is outer in $\mathbf{t}_{\delta,1}$, one of the outer edges of P is $v_1 v$.

If $V(P) = V \backslash u$, then P is a simple path in $\mathbf{t}_{\delta,1}$. Therefore there exists a short-cut passage as explained in Section 2. See Fig. 4. The short-cut

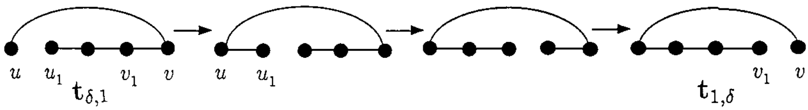


FIGURE 4

passage is a path from $\mathbf{t}_{\delta,1}$ to $\mathbf{t}_{1,\delta}=(\mathbf{t}_{\delta,1}-vv_1)\cup uu_1$ because one of the outer edges of P is v_1v. We define $\mathscr{R}$ to be the short-cut passage. Notice that any inner vertex of the short-cut passage includes uv. Because a departing point x of the short-cut passage is u_1, any inner vertex of the short-cut passage includes the edge $xx_\delta=u_1u$ by the convention (1).

If $V(P)\neq V\backslash u$, then there exists an outer vertex of $\mathbf{t}_{\delta,1}$ which is not in $V(P)\cup u$. Let X be the set of all such outer vertices.

If all outer vertices in X are adjacent to u in G, then there exists a round passage with respect to u. The round passage is a path from $\mathbf{t}_{\delta,1}$ to $\mathbf{t}_{1,\delta}=(\mathbf{t}_{\delta,1}-vv_1)\cup uu_1$ since one of the outer edges of P is v_1v. The round passage includes the edge uv. Let x be the departing point of the round passage. Any inner vertex of the round passage includes the edge $xx_\delta=xu$ by the convention (1). We choose the round passage as $\mathscr{R}$.

If $\delta>2$ and there exists an outer vertex $x\in X$ which is not adjacent to u in G, then $x_\delta\in N(x)$ is not u and v by the convention (1) and (2). Thus the edge xx_δ is not incident to u and v.

The spanning tree $\mathbf{r}_\delta^1=(\mathbf{t}_{\delta,1}-xx_1)\cup xx_\delta$ is adjacent to $\mathbf{t}_{\delta,1}$ because v_1 is not x. See Fig. 5. Since the vertex x_δ is not u, the spanning tree $\mathbf{r}_\delta^2=(\mathbf{r}_\delta^1-uu_\delta)\cup uu_1$ is adjacent to $\mathbf{r}_\delta^1$. Since $x_\delta\neq v$, $\mathbf{r}_\delta^3=(\mathbf{r}_\delta^1-vv_1)\cup vv_\delta$ is adjacent to $\mathbf{r}_\delta^2$. The spanning tree $\mathbf{r}_\delta^3$ is adjacent to $\mathbf{t}_{1,\delta}$ since v_δ is not x. Therefore we found out the path

$$\mathscr{R}=(\mathbf{t}_{\delta,1},\mathbf{r}_\delta^1,\mathbf{r}_\delta^2,\mathbf{r}_\delta^3,\mathbf{r}_{1,\delta})$$

in the leaf graph. The departing point of the path is x.

For all the above cases, the inner vertex of $\mathscr{R}$ either includes both the edges uv and xx_δ or else includes the edge xx_δ which is not incident to u and v. Thus the path $\mathscr{R}$ is internally disjoint from the short paths.

If $\delta=2$ and there exists a vertex in X which is not adjacent to u and v, then we can find out a path $\mathscr{R}$ as above. If $\delta=2$ and all vertices in X are adjacent to u or v, then there exists the round passage with respect to $\{u,v\}$. We define $\mathscr{R}$ to be the round passage. Clearly $\mathscr{R}$ is internally disjoint from the short paths.

Especially the following Claim is shown.

Claim. *There exists a path $\mathscr{R}$ from $\mathbf{t}_{\delta,1}$ to $\mathbf{t}_{1,\delta}$ in the leaf graph of any 2-connected graph which does not pass through $\mathbf{t}_{i,j}$ for any i and j.*

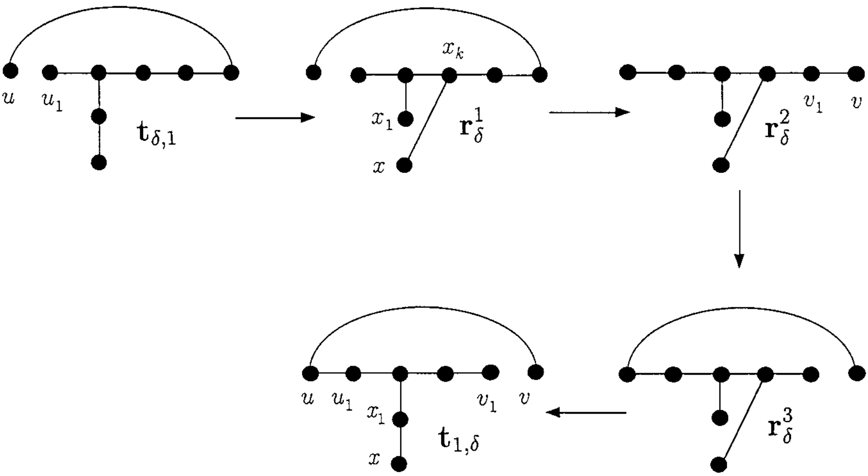


FIGURE 5

By Claim, there exist paths

$$\mathscr{P}_1 = (\mathbf{t}_{a,1}, \mathbf{t}, \mathbf{t}_{1,b})$$

$$\tilde{\mathscr{R}} = (\mathbf{t}_{a,1}, \mathbf{t}_{\delta,1}, \mathscr{R}, \mathbf{t}_{1,\delta}, \mathbf{t}_{1,b})$$

in the leaf graph of any 2-connected graph. Thus we showed our theorem for the case $\delta = 2$. In what follows, we assume that $\delta \geqslant 3$.

(3) We shall find out a path $\mathscr{L}$ from $\mathbf{t}_{\delta,1}$ to $\mathbf{t}_{\delta,b}$ which is internally disjoint from the short paths and $\mathscr{R}$. Let $P' = P_{\mathbf{t}_{\delta,1}}(v_b, v)$. One of the outer edges of P' is $v_1 v$ because the vertex v is adjacent to exactly u and v_1 in $\mathbf{t}_{\delta,1}$. If $V(P') = V \backslash u$, then P' is a simple path. Thus there exists a short-cut passage which is a path between $\mathbf{t}_{\delta,1}$ and $\mathbf{t}_{\delta,b} = (\mathbf{t}_{\delta,1} - vv_1) \cup vv_b$. See Fig. 6. Any inner vertex of the short-cut passage includes the edge uv. Since the outer vertices of $\mathbf{t}_{\delta,1}$ are u and v_b, the departing point x of the short-cut passage is v_b. Thus the inner vertex of the short-cut passage includes the edge $xx_{\delta-1} = v_b v$ by the convention (2). We define $\mathscr{L}$ to be this short-cut passage.

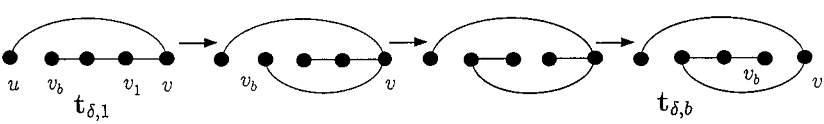


FIGURE 6

If $V(P') \neq V \backslash u$, then there exist an outer vertex in $\mathbf{t}_{\delta,1}$ which is not in $V(P') \cup u$. Let X' be the set of all such outer vertices.

If all outer vertices $x \in X'$ are adjacent to v, then there exists a round passage with respect to v. It is a path from $\mathbf{t}_{\delta,1}$ to $\mathbf{t}_{\delta,b} = (\mathbf{t}_{\delta,1} - vv_1) \cup vv_b$ since one of the outer edges of P' is $v_1 v$. Let x be the departing point of the round passage. The inner vertex of the round passage includes the edge $xx_{\delta-1} = xv$ by the convention (2). We choose the round passage as $\mathscr{L}$.

If there exists an outer vertex $x \in X'$ which is not adjacent to v in G, then the vertex $x_{\delta-1} \in N(x)$ is not u and v by the convention (1) and (2). Let $\mathbf{r}^1_{\delta-1} = (\mathbf{t}_{\delta,1} - xx_1) \cup xx_{\delta-1}$. See Fig. 7. Since $x_{\delta-1}$ is not u, the spanning tree $\mathbf{r}^2_{\delta-1} = (\mathbf{r}^1_{\delta-1} - uu_\delta) \cup uu_1$ is adjacent to $\mathbf{r}^1_{\delta-1}$. The spanning tree $\mathbf{r}^3_{\delta-1} = (\mathbf{r}^2_{\delta-1} - vv_1) \cup vv_b$ is adjacent to $\mathbf{r}^2_{\delta-1}$ because x is not adjacent to v. The spanning tree $\mathbf{r}^4_{\delta-1} = (\mathbf{r}^3_{\delta-1} - uu_1) \cup uu_\delta$ is adjacent to $\mathbf{t}_{\delta,b}$ because $x_{\delta-1} \neq u$. Therefore we have found out the path

$$\mathscr{L} = (\mathbf{t}_{\delta,1}, \mathbf{r}^1_{\delta-1}, \mathbf{r}^2_{\delta-1}, \mathbf{r}^3_{\delta-1}, \mathbf{r}^4_{\delta-1}, \mathbf{t}_{\delta,b})$$

in the leaf graph. The inner vertex of the path includes the edge $xx_{\delta-1} \in E \backslash E(\mathbf{t})$ which is not incident to u and v. The inner vertex of $\mathscr{L}$ includes both the edges uv and $xx_{\delta-1}$ or else includes the edge $xx_{\delta-1}$ which is not incident to u and v. Thus the path $\mathscr{L}$ is internally disjoint from the short paths and $\mathscr{R}$.

The case when $\delta = 3$ is shown because we have found out the four paths $\mathscr{P}_1$ and $\mathscr{P}_2$ and $\tilde{\mathscr{R}}$ and

$$\tilde{\mathscr{L}} = (\mathbf{t}_{a,1} = \mathbf{t}_{\delta,1}, \mathscr{L}, \mathbf{t}_{\delta,b}, \mathbf{t}_{1,b})$$

Finally we shall find out remaining paths for $\delta > 3$.

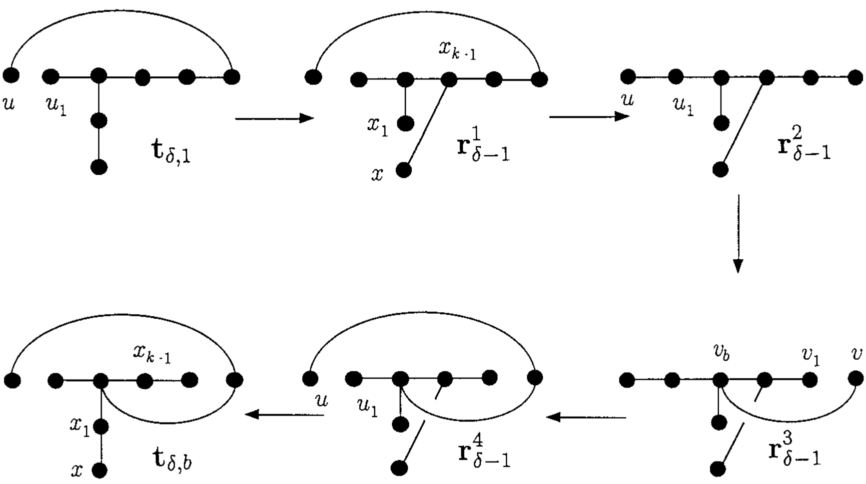


FIGURE 7

(4) In this paragraph we fix once and for all an outer vertex $x \neq u$ of $\mathbf{t}_{\delta,1}$. Notice that, by the convention (1) and (2), $x_i \in N(x)$ is not u and v for any $2 \leqslant i \leqslant \delta - 2$. There exist at least $\delta - 3$ vertices in $N(v)$ which is not v_1 and v_b and x. Choose a permutation σ so that $v_{\sigma(i)}$ is not v_1 and v_b and x for any $2 \leqslant i \leqslant \delta - 2$. If v is not adjacent to x and $b = \delta - 1$, then we choose the trivial permutation as σ.

We find out $\delta - 3$ paths of which the first step is $\mathbf{r}_i^1 = (\mathbf{t}_{\delta,1} - xx_1) \cup xx_i$ for $2 \leqslant i \leqslant \delta - 2$. There exists a vertex $u_p \in N(u)$ which is not u_1 and $u_\delta = v$ and x. Let $\mathbf{r}_i^2 = (\mathbf{r}_i^1 - uu_\delta) \cup uu_p$. Since $x_i \neq v$, the spanning tree $\mathbf{r}_i^3 = (\mathbf{r}_i^2 - vv_1) \cup vv_{\sigma(i)}$ is adjacent to $\mathbf{r}_i^2$. Since $v_{\sigma(i)} \neq x$, $\mathbf{r}_i^3$ is adjacent to $\mathbf{t}_{p,\sigma(i)}$.

If $\sigma(i) = \delta$, then we also have to find out a long path in the leaf graph. But there exist $\delta - 3$ vertices in $N(v)$ which is not v_1 and v_b and v_δ. We prepare a permutation as follows. Let ρ be the permutation such that

(1) $v_{\rho(i)}$ is not v_1 and v_b and v_δ for any $2 \leqslant i \leqslant \delta - 2$.

(2) If $\rho(i) = \sigma(j)$, then $i = j$ for any $2 \leqslant i, j \leqslant \delta - 2$.

Clearly $\mathbf{t}_{p,\sigma(i)}$ is adjacent to $\mathbf{t}_{p,\rho(i)}$ in the leaf graph. Furthermore $\mathbf{t}_{p,\rho(i)}$ is adjacent to $\mathbf{t}_{1,\rho(i)}$ because $v_{\rho(i)} \neq v_\delta = u$. Thus we can define the path

$$\mathcal{Q}_i' = (\mathbf{t}_{\delta,1}, \mathbf{r}_i^1, \mathbf{r}_i^2, \mathbf{r}_i^3, \mathbf{t}_{p,\sigma(i)}, \mathbf{t}_{p,\rho(i)}, \mathbf{t}_{1,\rho(i)}, \mathbf{t}_{1,b})$$

for any $2 \leqslant i \leqslant \delta - 2$. See Fig. 8.

Any inner vertex in the first half of $\mathcal{Q}_i'$ includes the edge xx_i which is not incident to u and v. Therefore the other paths already constructed do not intersect the first half of $\mathcal{Q}_i'$. Furthermore because the paths $\mathcal{R}$ and $\mathcal{L}$ do not pass through $\mathbf{t}_{i,j}$, the paths are internally disjoint from $\mathcal{Q}_i'$. The set of

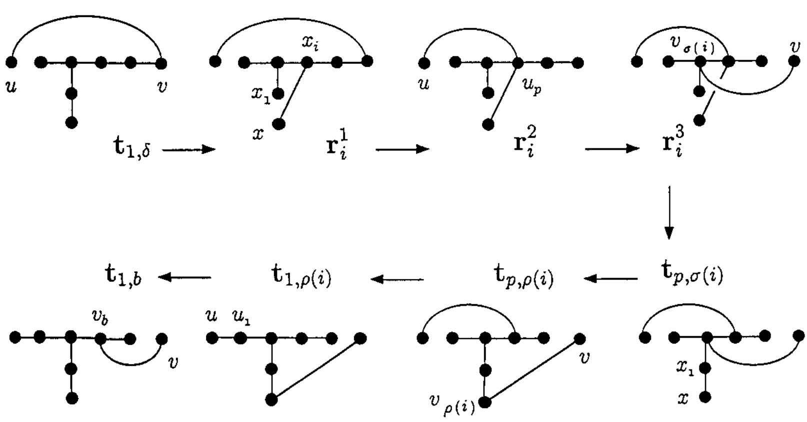


FIGURE 8

all the vertices which are traversed by the short paths except the outer vertices is

$$\{\mathbf{t}_{i,1}, \mathbf{t}_{i,b} \mid 1 \leqslant i \leqslant \delta - 1\}.$$

The paths $\mathscr{Q}_i'$ do not pass through this set. Thus $\mathscr{Q}_i'$ is internally disjoint from the short paths.

Finally, we have found out the $2\delta - 2$ internally disjoint paths from $\mathbf{t}_{a,1} = \mathbf{t}_{\delta,1}$ to $\mathbf{t}_{1,b} \neq \mathbf{t}_{1,\delta}$.

$$\{\mathscr{P}_1, \mathscr{P}_2, ..., \mathscr{P}_{\delta-1}, \tilde{\mathscr{R}}, \tilde{\mathscr{L}}, \mathscr{Q}_2', \mathscr{Q}_3', ..., \mathscr{Q}_{\delta-2}'\}.$$

Case 2. $a \neq \delta$ and $b \neq \delta$.

If $\mathbf{t}_{a,1} \neq \mathbf{t}_{\delta,1}$ and $\mathbf{t}_{1,b} \neq \mathbf{t}_{1,\delta}$, then the short paths except $i = \delta$ exist. That is, we have the set of short paths

$$\{\mathscr{P}_1, \mathscr{P}_2, ..., \mathscr{P}_{\delta-1}, \mathscr{Q}_1, \mathscr{Q}_2, ..., \mathscr{Q}_{\delta-1} \mid \mathscr{P}_1 = \mathscr{Q}_1, \mathscr{P}_a = \mathscr{Q}_b\}.$$

Let us find out remaining two paths.

In Case 1, we constructed the path $\mathscr{L}$ between $\mathbf{t}_{\delta,1}$ and $\mathbf{t}_{\delta,b}$. Because $\mathscr{L}$ does not pass through the vertex $\mathbf{t}_{i,j}$, the path

$$\tilde{\mathscr{L}} = (\mathbf{t}_{a,1}, \mathbf{t}_{\delta,1}, \mathscr{L}, \mathbf{t}_{\delta,b}, \mathbf{t}_{1,b})$$

is internally disjoint from the short paths. Next we shall find out a path $\mathscr{R}'$ from $\mathbf{t}_{1,\delta}$ to $\mathbf{t}_{a,\delta}$. Notice that the spanning tree $\mathbf{t}_{a,\delta}$ coincides with $(\mathbf{t}_{1,\delta} - uu_1) \cup uu_a$.

We also use the same method as we used to find out $\mathscr{R}$ or $\mathscr{L}$. Let $P'' = P_{\mathbf{t}_{1,\delta}}(u_a, u)$. One of the outer edges of P'' is the edge $u_1 u$ since v is outer in $\mathbf{t}_{1,\delta}$ and u is adjacent to exactly two vertices u_1 and v.

If $V(P'') = V \backslash v$, then P'' is a simple path. Thus there exists a short-cut passage which is a path between $\mathbf{t}_{1,\delta}$ and $\mathbf{t}_{a,\delta} = (\mathbf{t}_{1,\delta} - uu_1) \cup uu_a$. See Fig. 9. We choose the short-cut passage as $\mathscr{R}'$.

If $V(P'') \neq V \backslash v$, then there exists an outer vertex of the spanning tree $\mathbf{t}_{1,\delta}$ which is not in $V(P'') \cup v$. Let X'' be the set of all such outer vertices.

If all the vertices in X'' are adjacent to u, then there exists a round passage with respect to u. It is a path between $\mathbf{t}_{1,\delta}$ and $\mathbf{t}_{a,\delta} = (\mathbf{t}_{1,\delta} - uu_1) \cup uu_a$ because $u_1 u$ is outer in P''.

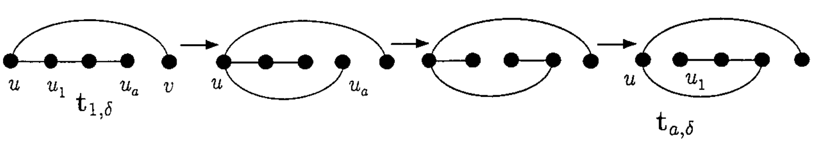


FIGURE 9

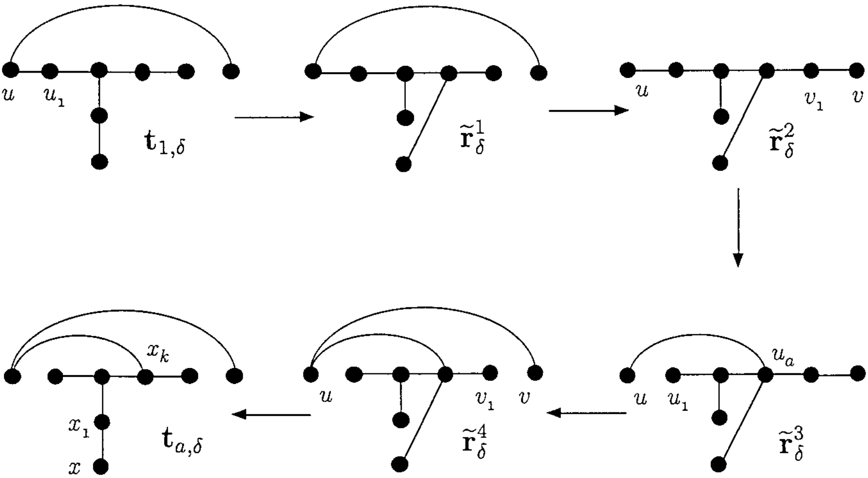


FIGURE 10

If there exists an outer vertex $x \in X''$ which is not adjacent to u in G, then x_δ is not u and v by the convention (1) and (2). The spanning tree $\mathbf{t}_{1,\delta}$ is adjacent to $\tilde{\mathbf{r}}_\delta^1 = (\mathbf{t}_{1,\delta} - xx_1) \cup xx_\delta$ in the leaf graph. See Fig. 10. Because $x_\delta \neq v$, $\tilde{\mathbf{r}}_\delta^1$ is adjacent to $\tilde{\mathbf{r}}_\delta^2 = (\tilde{\mathbf{r}}_\delta^1 - vv_\delta) \cup vv_1$. Furthermore $\tilde{\mathbf{r}}_\delta^2$ is adjacent to $\tilde{\mathbf{r}}_\delta^3 = (\tilde{\mathbf{r}}_\delta^2 - uu_1) \cup uu_a$ since v_1 is not u. Since $u_a \neq v$, the spanning tree is adjacent to $\tilde{\mathbf{r}}_\delta^4 = (\tilde{\mathbf{r}}_\delta^3 - vv_1) \cup vv_\delta$. Because x is not adjacent to u, we have that u_a is not x. Thus $\tilde{\mathbf{r}}_\delta^4$ is adjacent to $\mathbf{t}_{a,\delta}$. Therefore we found out the path

$$\mathscr{R}' = (\mathbf{t}_{1,\delta}, \tilde{\mathbf{r}}_\delta^1, \tilde{\mathbf{r}}_\delta^2, \tilde{\mathbf{r}}_\delta^3, \tilde{\mathbf{r}}_\delta^4, \mathbf{t}_{a,\delta})$$

in the leaf graph.

We can show that $\mathscr{R}'$ is internally disjoint from the short paths and $\mathscr{L}$ by comparing the existence of the edge uv or a set of adjacent vertices of u or v in spanning trees.

Let

$$\tilde{\mathscr{R}}' = (\mathbf{t}_{a,1}, \mathbf{t}_{a,\delta}, \mathscr{R}', \mathbf{t}_{1,\delta}, \mathbf{t}_{1,b}).$$

We get the internally disjoint $2\delta - 2$ paths as follows.

$$\{\mathscr{P}_1, \mathscr{P}_2, ..., \mathscr{P}_{\delta-1}, \mathscr{Q}_1, \mathscr{Q}_2, ..., \mathscr{Q}_{\delta-1}, \tilde{\mathscr{L}}, \tilde{\mathscr{R}}' \mid \mathscr{P}_1 = \mathscr{Q}_1, \mathscr{P}_a = \mathscr{Q}_b\}$$

Case 3. $a = \delta$ and $b = \delta$.

The induced subgraphs of $\{\mathbf{t}_{i,1}\}$ and $\{\mathbf{t}_{1,i}\}$ are respectively complete graphs because $\mathbf{t}_{i,1} - u = \mathbf{t}_{j,1} - u$ and $\mathbf{t}_{1,i} - v = \mathbf{t}_{1,j} - v$ for any i and j. Therefore if $\mathscr{C}_1 \cap \{\mathbf{t}_{i,1} \in \mathscr{V} \mid i \leqslant \delta\} = \mathbf{t}_{\delta,1}$ and $\mathscr{C}_2 \cap \{\mathbf{t}_{1,i} \in \mathscr{V} \mid i \leqslant \delta\} = \mathbf{t}_{1,\delta}$, then

$$\mathscr{S} = \{\mathbf{t}, \mathbf{t}_{2,1}, \mathbf{t}_{3,1}, ..., \mathbf{t}_{\delta-1,1}, \mathbf{t}_{1,2}, \mathbf{t}_{1,3}, ..., \mathbf{t}_{1,\delta-1}\}$$

since $\mathscr{S}$ includes at most $2\delta - 3$ vertices. By Claim, there exists a path $\mathscr{R}$ from $\mathbf{t}_{\delta,1}$ to $\mathbf{t}_{1,\delta}$ which does not pass through $\mathbf{t}_{i,j}$. Thus the path $\mathscr{R}$ does not pass through the cut set. That is, there exists $h < \delta$ such that $\mathbf{t}_{h,1} \in \mathscr{C}_1$ or $\mathbf{t}_{1,h} \in \mathscr{C}_2$. But this case was already treated in Case 1. The proof is now complete.

It is a natural question to ask whether the leaf graph of a 2-connected graph has a Hamiltonian cycle or not. However, the leaf graph of the complete bipartite graph $K_{2,3}$ is not Hamiltonian. Therefore the authors conjecture that if a graph is 2-connected with minimum degree three, then the leaf graph is Hamiltonian.

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