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# Differential Equations with Discontinuous Right-Hand Sides*

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## 1. INTRODUCTION

Many phenomena from physics, control theory, and economics have been successfully described by differential equations with discontinuous functions. This in turn leads to a great need to develope a theory for such differential equations. However, as far as existence and qualitative behavior are concerned, such a theory is far from being complete. Even existence results are available only for very special discontinuous function $f$ (as the discontinuous part in equation), e.g., see [5] for piecewise continuous $f$, [9] for $f$ having one discontinuous point, [8, 12] for $f$ of bounded variation.

In this paper, we are going to establish some existence results for differential equations with discontinuous right-hand sides with great generality. Some of the results in this paper can be stated and proved for high order ordinary and partial differential equations. For example, we may consider the following problem

$$-\Delta u(x) + cu(x) = f(u(x)) \quad \text{for} \quad x \in \Omega$$
$$u(x) = 0 \quad \text{for} \quad x \in \partial\Omega,$$

where $\Omega$ is a bounded domain in $R^n$, which has a boundary $\partial\Omega$ of class $C^3$ (if $n \geqslant 2$), $f: R \to R$ needs not be continuous. But in this paper we only consider the following problems in $X = R^n$.

$$\left.\begin{aligned} x' &= f(x) \\ x(0) &= x_0, \end{aligned}\right\} \tag{1}$$

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where $x_0 \in X$, $f\colon X \to X$ may not be continuous, associated to which is the following differential inclusion, the so-called regularization of (1),

$$\left.\begin{aligned} x' &\in F(x) \\ x(0) &= x_0, \end{aligned}\right\} \tag{2}$$

where $F(x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{con}}\, f(x + B_\varepsilon)$, $B_\varepsilon = \{x \in X\colon |x| \leqslant \varepsilon\}$, $\overline{\mathrm{con}}\, A$ denotes the closed convex hull of $A$. Let $J = [0, a]$ with $a > 0$, and call $x\colon J \to X$ a solution of problem (1) or (2) if $x(t)$ is absolutely continuous (AC for short), $x(0) = x_0$ and $x'(t) = f(x(t))$ or $x'(t) \in F(x(t))$ a.e. on $J$. We denote the set of all the solutions of problem (1) or (2) by $S(f, x_0)$ or by $S(F, x_0)$, respectively. Hence, we always have $S(f, x_0) \subset S(F, x_0)$, and in general $S(f, x_0) \neq S(F, x_0)$. As a convention we always assume that $f$ is bounded; and satisfies the following condition if we expect global solutions,

$$|f(x)| \leqslant M(1 + |x|) \qquad \text{on } X \tag{3}$$

for some $M > 0$. The following is known.

LEMMA 1.1 [1]. *For any $f$ satisfying* (3) *and any $x_0 \in X$, $S(F, x_0) \neq \varnothing$.*

Most of the articles in the literature concerning differential equations with discontinuities (like (1)) simply study their regularizations (like (2)) and the solutions of which are called the solutions of the original problem (see [9, 12]). But this consideration may be misleading. In fact, an $x \in S(F, x_0)$ may violate $x'(t) = f(x(t))$ at every $t$. To see this, define $f(x) = 1$ on $x \leqslant 0$ and $= -1$ on $x > 0$, and take $x_0 = 0$, and $x(t) \equiv 0$. This suggests that we should not take an $x \in S(F, x_0)$ as a solution of (1) for granted. But instead, we should take advantage of Lemma 1.1 and ask when $S(f, x_0) = S(F, x_0)$ or $S(f, x_0) \cap S(F, x_0) \neq \varnothing$ is true since in either case the existence of a true solution of problem (1) which comes from practice has been obtained. Furthermore, since the structure of the solutions of (2) is fairly clear, the structure of all the solutions of (1) will be known once $S(F, x_0) = S(f, x_0)$ for every $x_0$.

In Section 2 we are going to establish certain relations between $S(f, x_0)$ and $S(F, x_0)$, hence some existence results for problem (1) can be derived as consequences. When $X = R$, some further results on such relations as well as the solvability of problem (1) are proved in Section 3. Nonautomous problems are briefly discussed in the last section. The main theorems of the paper include

THEOREM A. *If $f$ is continuous on $X$ except on a countable subset $\sigma$, then $S(f, x_0) = S(F, x_0)$ for any $x_0 \in X \Leftrightarrow x \in \sigma$ and $0 \in F(x)$ implies $f(x) = 0$.*

THEOREM B. *Let $\sigma$ be a Borel set in $R$ with $m(\sigma)=0$ ($m$ denotes the Lebesgue measure in $R$) and $f: R \to R$ be continuous except on $\sigma$. If $x \in \sigma$ and $0 \in F(x) \Rightarrow f(x)=0$, then*

$$S(F, x_0) = S(f, x_0).$$

## 2. RELATIONS BETWEEN $S(f, x_0)$ AND $S(F, x_0)$

LEMMA 2.1 [7]. *If $I \subset R$ is not countable, then $I$ can be expressed as*

$$I = I_1 \cup I_2,$$

*where $I_1$ is countable, and, for any $x \in I_2$,*

$$x \in \operatorname{cl}\{y \in I_2 : y > x\} \cap \operatorname{cl}\{y \in I_2 : y < x\},$$

*where $\operatorname{cl}(A)$ denotes the closure of $A$.*

Let $X = R^n$, $\sigma = \{x_i : i \geqslant 1\} \subset X$. We have

THEOREM 2.2. *Suppose that $f$ is continuous at every $x \in X \backslash \sigma$. Then, $S(f, x_0) = S(F, x_0)$ for any $x_0 \in X \Leftrightarrow x \in \sigma$ and $0 \in F(x)$ implies $f(x)=0$.*

*In particular, if $x \in \sigma$ and $0 \in F(x) \Rightarrow f(x)=0$, then the problem (1) has a solution for every $x_0 \in X$.*

*Proof.* Sufficiency. For any given $x \in S(F, x_0)$, set $I_i = \{t \in J : x(t) = x_i\}$ and $I = \bigcup_{i=1}^{\infty} I_i$. For any fixed $i$, if $m(I_i) \neq 0$, then Lemma 2.1 implies that $x'(t)=0$ a.e. on $I_i$. Hence $0 \in F(x(t))$ a.e. on $I_i$. Thus, $x'(t) = f(x(t))$ a.e. on $I_i$. Consequently, $x'(t) = f(x(t))$ a.e. on $I$. Evidently, $F(x(t)) = \{f(x(t))\}$ for all $t \in J \backslash I$. Therefore, $x'(t) = f(x(t))$ a.e. on $J$ and $x \in S(f, x_0)$. This proves $S(f, x_0) = S(F, x_0)$.

Necessity. Suppose that $0 \in F(x_i)$ but $f(x_i) \neq 0$ for some $i \geqslant 1$. Set $x(t) \equiv x_i$ on $J$, then $x \in S(F, x_i)$ but $x \notin S(f, x_i)$, hence $S(f, x_i) \neq S(F, x_i)$ and the proof is complete. ∎

*Remark* 2.3. We can prove Theorem 2.2 and some other results in this section only when $\sigma$ is denumerable. When $X = R$, this restriction on $\sigma$ will be greatly relaxed in the next section.

Suppose that $K$ is any given cone in $R^n$. We write $y \geqslant x$ (or $x \leqslant y$) if $y - x \in K$, $y \gg x$ (or $x \ll y$) if $y - x \in \mathring{K}$ (the interior of $K$). $x: J \to X$ is said to be strictly monotone if $t < s$ implies $x(t) \ll x(s)$ (or $x(t) \gg x(s)$).

COROLLARY 2.4. *Suppose that $f$ is continuous at every $x \in X \backslash \sigma$, and for each $i$, there is an $\varepsilon_i > 0$ such that*

$$0 \notin \operatorname{cl}\{f(x_i + B_{\varepsilon_i})\} \subset K. \tag{4}$$

*Then, $S(f, x_0) = S(F, x_0) \neq \varnothing$.*

We next study the question as when $S(f, x_0) \neq \emptyset$ even if $S(f, x_0) \neq S(F, x_0)$.

THEOREM 2.5. *Assume that $f$ is continuous at every $x \in X \backslash \sigma$. Then an $x \in S(F, x_0)$ is also in $S(f, x_0)$ if one of the following conditions is true*

$$m\{t : x'(t) = 0\} = 0 \tag{5}$$

$$J = \bigcup_{n=1}^{\infty} [a_n, b_n] \tag{6}$$

*and $x(t)$ is strictly monotone on each $[a_n, b_n]$.*

*Proof.* If we adopt the same notation as in Theorem 2.2, then $m(I) = 0$ under (5) or (6), hence $x \in S(f, x_0)$. ▮

Note that condition (6) does not imply condition (5), in fact we have the following.

EXAMPLE 2.6. For any given $0 < \varepsilon < 1$, there is a continuously differential function $x: [0, 1] \to R$, which is strictly increasing, such that $x(0) = 0$, $x(1) = 1$, and

$$m\{t \in [0, 1] : x'(t) = 0\} = \varepsilon.$$

To see this, we construct a Cantor-like set $C_\varepsilon \subset [0, 1]$ with $m(C_\varepsilon) = \varepsilon$. We first remove an open interval, denoted by $P_1$, with width $(1 - \varepsilon)/3$, from the middle of $[0, 1]$. Then remove two open intervals, the union of which is denoted by $P_2$, each with width $(1 - \varepsilon)/3^2$ and from one of the two middle points of the two intervals $[0, 1] \backslash P_1$, respectively. By this process, we will remove an open set $P = \bigcup_{n=1}^{\infty} P_n$ with $m(P) = 1 - \varepsilon$. Let $C_\varepsilon = [0, 1] \backslash P$ and define a function $y: [0, 1] \to R$ such that $y(t) \equiv 0$ on $C_\varepsilon$, $y$ is positive and continuous on each $\mathrm{cl}(P_n)$, and furthermore,

$$\lim_{n \to \infty} \max_{P_n} |y(t)| = 0.$$

We finally define $x(t) = \int_0^t y(s)\, ds$. Then $x(0) = 0$, $x(t)$ is strictly increasing and $\{t \in [0, 1] : x'(t) = 0\} = C_\varepsilon$. Clearly, by a suitable choice of the values of $y(t)$ on $P_n$, we can have $x(t)$ also satisfy $x(1) = 1$.

For our next theorem we need the following definition.

DEFINITION [3]. $f$ is said to be $K$-continuous on $X$ iff, for any $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(y) - f(x)| < \varepsilon$$

whenever $|y - x| < \delta$ and $x < y$.

THEOREM 2.7. *Assume that $f$ is K-continuous on $X$, an $x \in S(F, x_0)$ is such that, for almost every $t \in J$, there is a $\delta = \delta(t) > 0$ such that*

$$x(t) \leqslant x(s) \tag{7}$$

*on $(t-\delta, t)$ or $(t, t+\delta)$. Then we also have $x \in S(f, x_0)$.*

*Proof.* Since $x'(t)$ is measurable on $J$, it is easily proved (see [3]) that, for almost all $t \in J$, there are sequences $t_k$ and $s_k$ such that $s_k < t < t_k$, $s_k$, $t_k \to t$, $x'(t_k) \in F(x(t_k))$, $x'(s_k) \in F(x(s_k))$, $x'(t) \in F(x(t))$, and $x'(t_k) \to x'(t)$, $x'(s_k) \to x'(t)$ as $k \to \infty$. Suppose $t$ is such a point and also satisfies (7). Without loss of generality we may assume that $x(t_k) \gg x(t)$ for all $k \geqslant 1$, hence, there is a sequence $\delta_k > 0$ such that $\delta_k \to 0$ and $x(t_k) + B_{\delta_k} - x(t) \subset K$. Using the $K$-continuity of $f$ at $x(t)$, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(y) - f(x)| < \varepsilon$$

for all $y \in X$ which satisfies $y \geqslant x(t)$ and $|y - x| < \delta$. Choose a $k_0$ such that

$$|x(t_k) - x(t)| + \delta_k < \delta$$

whenever $k \geqslant k_0$. We then have

$$F(x(t_k)) \subset \{x \in X : |x - f(x(t))| < \varepsilon\}$$

for $k \geqslant k_0$. Since $x'(t_k) \to x'(t)$ and $\varepsilon > 0$ is arbitrary, we conclude that $x'(t) = f(x(t))$. Thus, $x'(t) = f(x(t))$ a.e. on $J$, the proof is complete. ∎

## 3. PROBLEMS (1) AND (2) IN $R$

We concentrate in this section the problems (1) and (2) in the case $X = R$. An immediate consequence of Theorem 2.2 is that $S(f, x_0) \neq \varnothing$ if $f$ is continuous on $T \backslash \sigma$ and

$$\inf_{x \in R} f(x) > 0. \tag{8}$$

However, this is not true if we replace (8) by $f(x) > 0$ for every $x$. This can be seen by taking $f(x) = 1$ on $x \leqslant 0$, and $= x$ on $x > 0$, and $x_0 = 0$. Also, by taking $f(x) = 1$ on $x < 0$, and $= -1$ on $x \geqslant 0$, and $x_0 = 0$, we see that $S(f, x_0) = \varnothing$ even for some one-sided continuous functions. Nevertheless, we have the following Theorem 3.1 which can be implied from Theorem 4.1 [2]. Since the author came to know Ref. [2] after he got his own proof that may still be interesting, this proof is presented here.

THEOREM 3.1. *Assume that $f$ is right (left) continuous on $R$ and*

$$f(x) \geqslant 0 \qquad (\leqslant 0) \tag{9}$$

*at every discontinuous point $x$ of $f$. Then $S(f, x_0) \neq \varnothing$.*

LEMMA 3.2 [7]. *For any function $f: R \to R$, we have*

$$\underline{f}(x+0) = \underline{f}(x-0) \leqslant f(x) \leqslant \bar{f}(x+0) = \bar{f}(x-0) \tag{10}$$

*at every $x$ except at points of an enumerable set, where $\underline{f}(x+0) = \liminf_{y \to x+} f(y)$, etc.*

*Proof of Theorem* 3.1. Suppose $f$ is right continuous on $R$. By Lemma 3.2, $f(x)$ is continuous on $R$ except on a countable set, hence Theorem 2.2 is applicable. For each $\varepsilon > 0$ we consider the following subsidiary problem

$$\left.\begin{aligned} x' &= f(x) + \varepsilon \\ x(0) &= x_0. \end{aligned}\right\} \tag*{$(1)_\varepsilon$}$$

We claim that $(1)_\varepsilon$ has a solution $x_\varepsilon(t)$ such that, for any $t$, if $f$ is discontinuous at $x_\varepsilon(t)$, there is a $\delta = \delta(t) > 0$ such that

$$x_\varepsilon(t) < x_\varepsilon(s) \tag{11}$$

whenever $0 < s - t < \delta$. In order to prove this we first prove the local solvability of $(1)_\varepsilon$ with this requirement for any $x_0 \in R$. Consider the following three possibilities.

(i) If $f$ is continuous at $x_0$ and $f(x_0) + \varepsilon = 0$, then $x_\varepsilon(t) \equiv x_0$ is a solution of $(1)_\varepsilon$;

(ii) If $f$ is continuous at $x_0$ and $f(x_0) + \varepsilon > 0$ $(<0)$, then there is an $\eta > 0$ such that $f(x) + \varepsilon > \eta$ $(< -\eta)$ whenever $x \in [x_0 - \eta, x_0 + \eta]$. Hence, Theorem 2.2 implies that $(1)_\varepsilon$ has a local solution in this case;

(iii) If $f$ is discontinuous at $x_0$, then $f(x_0) + \varepsilon > 0$ by (9). By the right continuity of $f$ at $x_0$, there is an $\eta > 0$ such that $f(x) + \varepsilon > \eta$ whenever $x \in [x_0, x_0 + \eta]$. Define $g: R \to R$ by

$$g(x) = \begin{cases} f(x) + \varepsilon, & x \geqslant x_0 \\ f(x_0) + \varepsilon, & x < x_0 \end{cases}$$

and consider

$$\left.\begin{aligned} x' &= g(x) \\ x(0) &= x_0 \end{aligned}\right\} \tag{12}$$

Theorem 2.2 can be used again to conclude that (11) has a local solution $x(t)$. Clearly, $x(t) \geqslant x_0$ for all $t$, and hence for sufficiently small $t > 0$, $x_\varepsilon(t) \equiv x(t)$ is a local solution of $(1)_\varepsilon$. Furthermore, $x_\varepsilon(t) > x_0$ for all such $t$'s. Now we define the following class of functions

$$\Omega = \{x_b : x \text{ is a solution of } (1)_\varepsilon \text{ on } [0, b) \text{ satisfying } (11)\} \tag{13}$$

and for $x_b$, $y_d \in \Omega$, we define $x_b \leqslant y_d$ iff $b \leqslant d$ and $y_d|_{[0,b]} = x_b$. Then $\Omega$ becomes a partially ordered set. Given any chain $S \subset \Omega$, there must be a sequence $x_{b_n} \in S$ such that

$$\lim_{n \to \infty} b_n = \sup\{b : \text{there is an } x_b \in S\} = d.$$

Define

$$x(t) = x_{b_n}(t) \qquad \text{for} \quad t \in [0, d).$$

Thus $x \in \Omega$ and $y \leqslant x$ for all $y \in S$, hence $S$ has an upper bound in $\Omega$. Zorn's lemma implies that $\Omega$ has a maximal element $x_\varepsilon$. Evidently, $x_\varepsilon(t)$ is a solution of $(1)_\varepsilon$ satisfying (11) on $[0, a)$ because of (3). For the same reason $x_\varepsilon(t)$ can be defined on the whole $J$.

If we use $D^r$ to denote right derivative, next we claim that, for the so obtained $x_\varepsilon(t)$, $D^r x_\varepsilon(t)$ exists, and

$$D^r x_\varepsilon(t) = f(x_\varepsilon(t)) + \varepsilon \tag{14}$$

at every $t \in [0, a)$. In fact, (14) is obviously true if $f$ is continuous at $x_\varepsilon(t)$. If, however, $f$ is discontinuous at $x_\varepsilon(t)$, then

$$D^r x_\varepsilon(t) = \lim_{s \to t+} \frac{1}{s - t} \int_t^s [f(x_\varepsilon(\xi)) + \varepsilon] \, d\xi = f(x_\varepsilon(t)) + \varepsilon$$

because of (11) and the right continuity of $f$ at $x_\varepsilon(t)$.

Now we want to prove that if an $x_\varepsilon(t)$ has been so obtained for every $\varepsilon > 0$, then the functions $\{x_\varepsilon(t)\}$ form a decreasing sequence as $\varepsilon \to 0+$. In fact, if this is not true, then there are $x_\varepsilon(t)$ and $x_\delta(t)$ with $\varepsilon > \delta > 0$ with the following properties: there are $0 \leqslant t_2 < t_1 \leqslant a$ such that $x_\varepsilon(t_2) = x_\delta(t_2)$ and $x_\varepsilon(s) < x_\delta(s)$ whenever $s \in (t_2, t_1]$. Set $m(t) = x_\delta(t) - x_\varepsilon(t)$, then

$$D^r m(t_2) \geqslant 0. \tag{15}$$

On the other hand, however,

$$D^r m(t_2) = D^r x_\delta(t_2) - D^r x_\varepsilon(t_2) = \delta - \varepsilon < 0. \tag{16}$$

It is a contradiction.

Now choose a decreasing sequence $\varepsilon_n \to 0+$, accordingly we obtain a decreasing sequence of $x_{\varepsilon_n}$. By letting $n \to \infty$ in the following relation

$$x_{\varepsilon_n}(t) = x_0 + \int_s^t [f(x_{\varepsilon_n}(s)) + \varepsilon_n] \, ds \tag{17}$$

and because of the right continuity of $f$, we obtain

$$x(t) = x_0 + \int_s^t f(x(s)) \, ds \tag{18}$$

for some continuous $x: J \to R$, which is a solution of the problem (1). In case $f$ is left continuous, the proof is similar. Thus, the proof is complete. ▮

We now come to the question of the relations between problems (1) and (2) again. The following lemmas are needed.

LEMMA 3.3 [10]. *A necessary and sufficient condition that a continuous, strictly increasing function $x(t)$ be absolutely continuous is that*

$$m\{x(t): t \text{ is in the domain and } x'(t) = +\infty\} = 0. \tag{19}$$

LEMMA 3.4 [10]. *If $y$ is continuous and of bounded variation on $J$, then $y$ is AC iff $y$ has (N) property, i.e., $N \subset J$ and $m(N) = 0 \Rightarrow m(y(N)) = 0$. If $y$ is continuous and has (N) property, then $y(S)$ is (Lebesgue) measurable if $S \subset J$ is.*

*Remark* 3.5. Lemma 3.3 implies that the inverse of $x(t)$ from Example 2.5 is not AC. Hence this together with Lemma 3.4 implies that for AC function $x(t)$ and measurable $S \subset R$, $x^{-1}(S)$ may not be measurable.

LEMMA 3.6. *Let $y$ be AC on $J$, $g$ be measurable and finite-valued on $y(J)$, $S \subset J$ be measurable. Suppose $h(t) = g(y(t)) \, y'(t)$ is integrable on $J$. Then*

(i) $\displaystyle\int_{y(a)}^{y(b)} g \, ds = \int_a^b h \, dt$ *if* $S = [a, b]$,

(ii) $\displaystyle\int_{y(S)} g \, ds = \int_S h \, dt$ *if* $y$ *is increasing*

(iii) $\displaystyle\int_{y(S)} g \, ds \leqslant \int_S g(y(t)) \, |y'(t)| \, dt$ *if* $g \geqslant 0$ *on* $y(J)$.

*Proof.* The proof of (i) is given in [11], and (ii) in [2]; the proof of (iii) is as follows.

For any given $\varepsilon > 0$, there is a $\delta > 0$ s.t.

$$\int_e g(y(t))\,|y'(t)|\,dt < \varepsilon$$

for any measurable $e \subset J$ with $m(e) < \delta$. Now we take a sequence of disjoint intervals $(a_i, b_i) \subset J$ s.t. $S \subset A = \bigcup_{i=1}^{\infty} [a_i, b_i]$ and $m(A \backslash S) < \delta$. Let $\bar{a}_i, \bar{b}_i \in [a_i, b_i]$ be such that

$$y(\bar{a}_i) = \min\{y(t) : a_i \leqslant t \leqslant b_i\}$$

and

$$y(\bar{b}_i) = \max\{y(t) : a_i \leqslant t \leqslant b_i\}.$$

Then, (i) $\Rightarrow$

$$\int_{y(\bar{a}_i)}^{y(\bar{b}_i)} g\,ds = \int_{\bar{a}_i}^{\bar{b}_i} h\,dt, \qquad \forall i \geqslant 1.$$

Hence,

$$\begin{aligned}\int_{y(S)} g\,ds &\leqslant \int_{y(A)} g\,ds \leqslant \sum_{i=1}^{\infty} \int_{y([a_i, b_i])} g\,ds \\ &= \sum_{i=1}^{\infty} \int_{y(\bar{a}_i)}^{y(\bar{b}_i)} g\,ds \leqslant \sum_{i=1}^{\infty} \left| \int_{\bar{a}_i}^{\bar{b}_i} g(y(t))\,|y'(t)|\,dt \right| \\ &\leqslant \sum_{i=1}^{\infty} \left| \int_{a_i}^{b_i} g(y(t))\,|y'(t)|\,dt \right| \leqslant \int_A g(y(t))\,|y'(t)|\,dt \\ &\leqslant \int_S g(y(t))\,|y'(t)|\,dt + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (iii) is thus proved. ∎

As an immediate consequence of Lemma 3.6, we have

LEMMA 3.7 [10]. *For any $x: J \to R$ which is* AC, *we have*

$$m\{x(t) : t \in J \text{ s.t. } x'(t) = 0\} = 0. \tag{20}$$

*Proof.* Let $M = \{x(t) : t \in J \text{ s.t. } x'(t) = 0\}$ and $S = x^{-1}(M) = (x')^{-1}(o)$, then $S$ is measurable. If we take $g = 1$ and $y = x$, then Lemma 3.6(iii) implies

$$m(M) = \int_{x(S)} ds \leqslant \int_S |x'(t)|\,dt = 0. \quad ∎$$

COROLLARY 3.8. *If $G: J \to 2^R \backslash \varnothing$ is such that $G(x) = \{0\}$ a.e., then any solution of $x'(t) \in G(x(t))$ is a constant solution $x(t) \equiv c$ s.t. $0 \in G(c)$.*

Lemma 3.2 [2] shows that any solution of the problem (1) in $R$ is monotone, therefore if $S(F, x_0) \cap S(f, x_0) \neq \varnothing$, $S(F, x_0)$ must contain monotone functions. This observation leads to the following results.

THEOREM 3.9. *Let $\sigma$ be any Borel set in $R$ with $m(\sigma) = 0$. Assume that $f$ is continuous on $R$ except on $\sigma$, and*

$$x \in \sigma \quad \text{and} \quad 0 \in F(x) \Rightarrow f(0) = 0. \tag{21}$$

*Then, any monotone $x$ of $S(F, x_0)$ is also in $S(f, x_0)$.*

*Proof.* Set $S = x^{-1}(\sigma)$, then $S$ is measurable. Put $g = 1$ and $y = x$ in Lemma 3.6(ii) we obtain

$$0 = m(\sigma) = \int_\sigma ds = \pm \int_S x'(t)\, dt. \tag{22}$$

Since $x'(t)$ keeps one sign a.e., we must have $x'(t) = 0$ a.e. on $S$. Thus, $x'(t) = f(x(t))$ on $S$, hence $x'(t) = f(x(t))$ a.e. on $J$. ▮

Note that given an monotone AC function $x: R \to R$ and a Borel set $\sigma$ in $R$ with $m(\sigma) = 0$, we may not have $m\{x^{-1}(\sigma)\} = 0$ in general, but we always have $x'(t) = 0$ a.e. on $\{x^{-1}(\sigma)\}$ as is shown in the proof above.

LEMMA 3.10. *Assume that $f$ is continuous except on a subset $\sigma = \sigma_1 + \sigma_2$ of $R$, where $\sigma_1$ is a zero-measure set but $\sigma_2$ is arbitrary (may not be measurable). Assume furthermore that*

$$\left.\begin{aligned} &x \in \sigma_1 \quad \text{and} \quad 0 \in F(x) \Rightarrow f(x) = 0, \\ &x \in \sigma_2 \Rightarrow 0 \notin F(x). \end{aligned}\right\} \tag{23}$$

*Then any $x(t)$ of $S(F, x_0)$ is monotone.*

*Proof.* Let $S_0 = \{t \in J: x'(t) \notin F(x(t))\}$ and $\Lambda_0 = x(S_0) \cup \sigma_1$, then $m(S_0) = m(\Lambda_0) = 0$, by Lemma 3.4. Suppose there are $t_1, t_2 \in J$ such that $x(t_1) = x(t_2) = \alpha$ and

$$\beta = \max\{x(t): t_1 \leqslant t \leqslant t_2\} > \alpha. \tag{24}$$

Let the maximum be attained at $t = s$ and set $\Lambda = [\alpha, \beta] \backslash \Lambda_0$. For any $\lambda \in \Lambda \backslash \{\beta\}$, let

$$s_1 = \max\{t: t < s, x(t) = \lambda\}$$

and

$$s_2 = \min\{t : t > s,\ x(t) = \lambda\}.$$

Then, $s_1 < s < s_2$ and $s_1, s_2 \in [t_1, t_2] \backslash S_0$.

Hence

$$0 \leqslant x'(s_1) \in F(\lambda) \qquad \text{and} \qquad 0 \geqslant x'(s_2) \in F(\lambda).$$

This implies $0 \in F(\lambda)$ since $F(\lambda)$ is convex. Thus, $\lambda \notin \sigma$, hence $\lambda$ is a continuous point of $f$. Therefore, $x'(s_1) = x'(s_2) = 0$, and consequently,

$$m\{x(t) : t \in [t_1, t_2] \text{ s.t. } x'(t) = 0\} = \beta - \alpha$$

a contradiction to Theorem 3.7. A similar procedure can also give a contradiction if instead of (24) we have

$$\beta = \min\{x(t) : t_1 \leqslant t \leqslant t_2\} < \alpha.$$

Hence the lemma is proved. ▮

A combination of Theorem 3.9 and Lemma 3.10 immediately implies the following theorem which is a generalization of Theorem 2.2 in $X = R$.

THEOREM 3.11. *Let $\sigma$ be a Borel set in $R$ with $m(\sigma) = 0$. Suppose that $f$ is continuous except on $\sigma$, and* (21) *holds. Then,*

$$S(F, x_0) = S(f, x_0).$$

Let $C(J) = \{x : J \to R \text{ continuous}\}$ and $S : R \to 2^{C(J)}$ be defined by $S(x) = S(f, x)$ for any $x \in R$. Then because of the properties of $S(F, x)$ (see [1]), Theorem 3.11 implies

COROLLARY 3.12. *Under the conditions of Theorem* 3.11, *we have*

(i) $S(f, x)$ *is a nonempty compact continuum in* $C(J)$ *for every* $x \in R$,

(ii) $S : R \to 2^{C(J)}$ *is upper semi-continuous.*

The following theorem is a consequence of [2, Theorem 4.1], here we supply a different proof.

THEOREM 3.13. *Assume that $f : R \to R$ is continuous almost everywhere and there are two constants $0 < c < d$ such that*

$$c \leqslant f(x) \leqslant d \qquad (or \quad -d \leqslant f(x) \leqslant -c) \tag{25}$$

*on $R$. Then $S(f, x_0) \neq \varnothing$ for every $x_0 \in R$.*

*Proof.* Note first that $f$ is Lebesgue measurable. We assume here $c \leqslant f(x) \leqslant d$ on $R$, the other case can be treated analogously. For each $n \geqslant 1$ define $x_n(t)$ on $[-1, 0]$ to be any function satisfying $x_n(0) = x_0$ and

$$c \leqslant \frac{x_n(t) - x_n(s)}{t - s} \leqslant d \tag{26}$$

for all $t, s \in [-1, 0]$. By Lemma 3.3, both $x_n(t)$ and $x_n^{-1}(t)$ (the inverse of $x_n$) are absolutely continuous, hence have property (N). Thus, $f(x_n(t))$ is measurable on $[-1, 0)$. Repeating the same argument, we can recursively define $x_n(t)$ on $[0, a)$ by

$$x_n(t) = x_0 + \int_0^{t - 1/n} f(x_n(s))\, ds. \tag{27}$$

Suppose that $x_{n_k}(t) \to x(t)$ uniformly on $J$ with some continuous $x$ on $J$. Evidently, $x(t)$ satisfies condition (26), hence $x^{-1}$ has property (N). Thus, for $J_0 = \{t \in J : f \text{ is discontinuous at } x(t)\}$, we have

$$m(J_0) = 0. \tag{28}$$

Therefore, by letting $n \to \infty$ in (27), we obtain

$$x(t) = x_0 + \int_0^t f(x(s))\, ds \tag{29}$$

hence $x \in S(f, x_0) \neq \varnothing$. The proof is thus complete. ∎

*Remark* 3.14. The following example shows that the problem (1) may not have a solution if $f$ is too irregular.

EXAMPLE 3.15. Take a function $g: [0, 1] \to [0, 1]$ (see [6]), which is strictly increasing and continuous, $g(0) = 0$, $g(1) = 1$ and $A = g^{-1}(B)$ is not measurable for some $B \subset C$, where $C$ is the Cantor set in $[0, 1]$. Define $f: R \to R$ by

$$f(x) = \begin{cases} 1, & x \notin [0, 1] \\ g(x) + 1, & x \in [0, 1]. \end{cases} \tag{30}$$

We claim that $S(f, 0) = \varnothing$. Otherwise suppose there is an $x \in S(f, 0)$, then $x'(t) = f(x(t))$ is measurable. On the other hand, $x(t)$ certainly satisfies (26) with $c = 1$, $d = 2$, hence $x'(t) = g(x(t)) + 1$ on some $[0, p]$ with $x(p) = 1$ and $x(t)$ has property (N). Therefore,

$$\{t : g(x(t)) + 1 \in B + 1\} = \{t : x(t) \in A\}$$

is not measurable since otherwise it would be against the property (N). This contradiction shows that $S(f, 0) = \varnothing$.

## 4. Nonautonomous Cases

Suppose that $f: J \times X \to X$ is Lebesgue measurable in its first variable. Consider

$$\left.\begin{aligned} x'(t) &= f(t, x(t)) \\ x(0) &= x_0 \end{aligned}\right\} \tag{1}^*$$

and

$$\left.\begin{aligned} x'(t) &\in F(t, x(t)) \\ x(0) &= x_0, \end{aligned}\right\} \tag{2}^*$$

where $F(t, x) = \bigcap_{\varepsilon > 0} \overline{\text{con}}\, f(t, x + B_\varepsilon)$. It can be easily checked that some of the previous results can be carried over to problem $(1)^*$ and $(2)^*$ if we assume that there is a measurable $J_0 \subset J$ with $m(J_0) = m(J)$ such that, for every $t \in J_0$, $f(t, \cdot)$ is continuous at every $x \in X \backslash \sigma$, where $\sigma$ is some given countable set. Unfortunately, $\sigma$, the set of all the points at which $f$ is discontinuous with some $t \in J_0$, has to be independent of $t \in J_0$. The following example shows this point.

Example 4.1. $f: R \times R \to R$ is defined by

$$f(t, x) = \begin{cases} \frac{1}{2}, & t \leqslant x \\ 2, & t > x. \end{cases} \tag{31}$$

Then $f(t, \cdot)$ is right continuous at every $(t, x)$, and even $\inf f(t, x) = \frac{1}{2} > 0$. However, $(1)^*$ with $x_0 = 0$ does not have a solution since otherwise, suppose $x$ is a solution, then $x(t)$ cannot be identically equal to $t$, and by the definition of $f$ we cannot have $x(t) > t$ or $x(t) < t$ at any $t$ either. It is a contradiction.

It is known that $f: J \times X \to X$ is Caratheodóry iff, for any $\varepsilon > 0$, there is a measurable $J_0 \subset J$ such that $m(J - J_0) < \varepsilon$ and $f: J_0 \times X \to X$ is continuous. Example 4.1 also shows that an analogous result for those functions $f$ that is measurable in $t$ and right continuous in $x$ is not true. This is indeed the reason why the set $\sigma$ has to be independent of $t \in J_0$ in order to carry the results of Sections 2 and 3 to problems $(1)^*$ and $(2)^*$.

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