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The Liouville Equation with Singular Data: A Concentration-Compactness Principle via a Local Representation Formula[1]

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For a bounded domain $\Omega \subset \mathbb{R}^2$, we establish a concentration-compactness result for the following class of "singular" Liouville equations:

$$-\Delta u = e^u - 4\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j} \qquad \text{in } \Omega,$$

where $p_j \in \Omega$, $\alpha_j > 0$ and δ_{p_j} denotes the Dirac measure with pole at point p_j, $j = 1, \ldots, m$. Our result extends Brezis–Merle's theorem (*Comm. Partial Differential Equations* **16** (1991) 1223–1253) concerning solution sequences for the "regular" Liouville equation, where the Dirac measures are replaced by $L^p(\Omega)$-data $p > 1$. In some particular case, we also derive a mass-quantization principle in the same spirit of Li–Shafrir (*Indiana Univ. Math. J.* **43** (1994) 1255–1270). Our analysis was motivated by the study of the Bogomol'nyi equations arising in several self-dual gauge field theories of interest in theoretical physics, such as the Chern–Simons theory ("Self-dual Chern–Simons Theories," Lecture Notes in Physics, Vol. 36, Springer-Verlag, Berlin, 1995) and the Electroweak theory ("Selected Papers on Gauge Theory of Weak and Electromagnetic Interactions," World Scientific, Singapore).

1. INTRODUCTION

Motivated by the study of the Bogomol'nyi equations for self-dual field theories of interest in theoretical physics such as the Chern–Simons Theory [16–18], analyzed in [7, 9, 26, 28, 32, 37] and the Electroweak Theory [21], discussed in [1, 33] and references therein, we investigate the *singular*

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Liouville equation on a bounded open domain $\Omega \subset \mathbb{R}^2$ given as follows:

$$-\Delta u = e^u - 4\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j} \qquad \text{in } \Omega, \tag{1}$$

where $\alpha_j > 0$ and $p_j \in \Omega,\ j = 1, \ldots, m.$

Here δ_p denotes the Dirac measure with pole at the point p, and by keeping the physical notations, we shall refer to the given points $p_j,\ j = 1, \ldots, m$, as the *vortex* points.

Our work concerns two aspects of problem (1). Firstly, we shall derive an explicit local representation formula for solutions of (1), see Theorem 1. In this direction, we shall pursue further the analysis of Chou–Wan [13] relative to the Liouville equation in the punctured disk, and extend the well-known Liouville formula [25] valid for solutions of (1) in case $\alpha_j = 0,\ \forall j = 1, \ldots, m.$

Secondly, we shall take advantage of such a "local" representation formula, to derive a concentration-compactness principle in the same spirit of the result derived by Brezis–Merle [6] for "regular" Liouville-type equations, where the sum of Dirac measures in (1) is replaced by a more regular L^p-function, $p > 1$. In this direction, we have:

THEOREM. *Let u_n be a solution sequence of* (1) *such that*

$$\int_\Omega e^{u_n} \leqslant C, \qquad \forall n \in \mathbb{N}, \tag{2}$$

for some $C > 0$. Along a subsequence u_{k_n} one of the following alternative holds:

(i) $\forall K \subset\subset \Omega$, *there exist a constant $C_K > 0$*: *$sup_K\, |u_{k_n}(x) - 2\sum_{j=1}^m \alpha_j \ln|x - p_j|| \leqslant C_K$.*

(ii) $\forall K \subset\subset \Omega$, $\sup_K\{u_{k_n}(x) - 2\sum_{j=1}^m \alpha_j \ln|x - p_j|\} \to -\infty$.

(iii) *There exist a finite and nonempty set $S = \{q_1, \ldots, q_l\} \subset \Omega$, $l \in \mathbb{N}$, and corresponding sequences $\{x_n^1\}_{n\in\mathbb{N}}, \ldots, \{x_n^l\}_{n\in\mathbb{N}} \subset \Omega\backslash\{p_1, \ldots, p_l\}$, such that $x_n^i \to q_i$ and $u_{k_n}(x_n^i) \to \infty$ for $i \in \{1, \ldots, l\}$. Furthermore, $\sup_K\{u_{k_n}(x) - 2\sum_{j=1}^m \alpha_j \ln|x - p_j|\} \to -\infty$ on any compact set $K \subset \Omega\backslash S$, and $e^{u_{k_n}} \to \sum_{i=1}^l \beta_i \delta_{q_i}$ weakly in the sense of measures on Ω, with $\beta_i \in 8\pi\mathbb{N}$ if $q_i \neq p_j$ and $\beta_i \geqslant 8\pi$ if $q_i = p_j$ for some $j = 1, \ldots, m$.*

Clearly (i)–(iii) state a concentration-compactness principle for the sequence e^{u_n}.

If $\alpha_j = 0,\ \forall j = 1, \ldots, m$, then the theorem above reduces to the Brezis–Merle result in [6] as completed by Li–Shafrir [23]. In fact, if more generally we knew that the set of blow up points S does not contain any of the vortex points $p_j,\ j = 1, \ldots, m$, or equivalently that the integral condition in (2) could be strengthened to the condition: $\int_\Omega \exp(u_n - 2\sum_{j=1}^m \alpha_j \ln|x-p_j|) \leqslant C$, then Brezis–Merle's analysis would be sufficient to guarantee (i)–(iii).

Thus, the true delicate case for us to analyze concerns the case where the sequence u_n admits a blow up point that coincides with one of the given vortex points.

Under no circumstances, this situation can be fitted into Brezis–Merle's assumptions. Indeed, by setting

$$v_n(x) = u_n(x) - 2\sum_{j=1}^m \alpha_j \ln|x-p_j|,$$

then v_n defines the regular part of u_n, and satisfies

$$\begin{cases} -\Delta v_n = V_n(x)e^{v_n} & \text{in } \Omega, \\ \int_\Omega V_n(x)e^{v_n} \leqslant C, \end{cases} \tag{3}$$

with $V_n(x) = \prod_{j=1}^m |x-p_j|^{2\alpha_j}$. It is possible to check that u_n and v_n admit the same set of blow up points. Thus, if u_n blows up at a vortex point so does v_n. In this case, by a standard blow up argument, we see that necessarily we must have, $\int_\Omega e^{v_n} \to \infty$ as $n \to \infty$. Consequently, the Brezis–Merle assumptions fail to apply to v_n in this situation and furthermore, it is not at all clear that the sequence e^{u_n} should be subject to a concentration phenomenon.

The analysis of this situation is the goal of our main Theorem 2, whose proof is also interesting in itself as it illustrates in a clear way the origin of the concentration-compactness principle stated above. Concerning alternative (iii), the values β_i relative to each concentration point q_i, $i = 1, \ldots, l$, have been investigated by Li–Shafrir [23]. In our setting, the Li–Shafrir result states that each concentration point q_i which does not coincide with a vortex point, carries a "mass" $\beta_i = 8\pi m_i$, $m_i \in \mathbb{N}$ (mass quantization principle). Unfortunately, in case the concentration point coincides with a vortex point, we will be able to obtain an analogous "mass quantization" property only in certain cases, by means of the Alexandrov–Bol inequality (cf. [2]) as proved by Suzuki [36].

We hope that our analysis will be relevant to the understanding of related problems. For instance, we mention the following mean-field equation:

$$-\Delta\phi = \lambda\frac{e^\phi}{\int_M e^\phi} - 4\pi\sum_{j=1}^m \alpha_j\delta_{p_j} \qquad \text{in } M \tag{3_λ}$$

with M a Reimannian compact 2-manifold without boundary, which bares significant applications towards the Bogomol'nyi equations mentioned above. Note that the condition $\lambda = 4\pi\sum_{j=1}^m \alpha_j\delta_{p_j}$ is *necessary* to the solvability of $(3)_\lambda$. Problem $(3)_\lambda$ suites particularly well our framework. Indeed, if we were to investigate a solution sequence ϕ_n for $(3)_{\lambda_n}$ with $\lambda_n \to \lambda$, we could use the transformation:

$$u_n = \phi_n + \ln \lambda_n - \ln \int_M e^{\phi_n},$$

and reduce to consider a solution sequence of (1) satisfying (2).

Thus, on the ground of the Brezis–Merle and Li–Shafrir results, quite interesting existence results have been derived for problem $(3)_\lambda$ in the "regular" case, where the measure $\sum_{j=1}^m \alpha_j \delta_{p_j}$ is replaced by a $L^p(M)$-function $p > 1$, (cf. [8, 14, 20, 22, 24, 35]). On the contrary, much less is known about the *singular* problem $(3)_\lambda$ with the exception of the (coercive) case where $\lambda = 4\pi \sum_{j=1}^m \alpha_j$ is assumed to satisfy $\lambda \in (0, 8\pi)$. Indeed, in this case, the presence of the singular datum does not affect in any significant way the analysis of $(3)_\lambda$. Thus, as for the regular case, problem $(3)_\lambda$ can be set in a *coercive* variational framework, and a solution may be derived by direct minimization. In the particular case where $\lambda = 4\pi$, an explicit solution for $(3)_{\lambda=4\pi}$ has been obtained by Olesen [29]. Olesen has treated $(3)_\lambda$ over the flat two torus with a single vortex point, i.e. $m = 1$, having multiplicity $\alpha_1 = 1$, and derived his solution in term of the Weierstrass $\mathscr{P}$-function. In this way, he was able to claim the presence of Abrikosov mixed states of 1-vortex type for the Chern–Simons model proposed in [17, 18].

We conclude by mentioning that the situation where $\lambda = 8\pi$ is already more delicate to analyze and problem $(3)_{\lambda=8\pi}$ has attracted interest also in the context of the assigned Gauss curvature problem over the two sphere $M = S^2$, see [10, 11, 19] and references therein. For manifolds M other than the sphere problem $(3)_{\lambda=8\pi}$ has been treated in [15, 27]. Quite more involved is the case $\lambda > 8\pi$, and we refer to our forthcoming paper [5] for some recent progress in this direction. Other partial results concerning this problem are contained in [3, 4].

2. A LOCAL REPRESENTATION FORMULA

In this section, we derive a local representation formula for the solutions of (1) around each one of the vortex points. So, without loss of generality, we take the origin as such vortex point and let $D_r = \{x \in \mathbb{R}^2 \colon |x| < r\}$, $r > 0$. We set $D = D_{r=1}$, and for $\alpha > 0$, consider the problem:

$$-\Delta u = e^u - 4\pi\alpha\delta_{p=0} \qquad \text{in } D. \tag{4}$$

Since e^u is smooth away from zero, and $e^u = O(|x|^{2\alpha})$ as $|x| \to 0$, we can also assure that every solution of (4) satisfies

$$\int_D e^{u(x)}\,dx < \infty. \tag{5}$$

By introducing complex notations we set: $z = x + iy$, for $(x, y) \in D$. The following Liouville-type representation formula holds for the solution of (4).

THEOREM 1. *Any solution $u(z)$ of problem (4) can be decomposed as $u(z) = 2\alpha \ln |z| + v(z)$ and for some function ψ analytic in D with $\psi(0) \neq 0$, $v(z)$ may take only one of the following forms*:

$$v(z) = \ln \frac{|(1+\alpha)\psi(z) + z\psi'(z)|^2}{(1 + |z^{\alpha+1}\psi(z) + a|^2)} + \ln 8, \tag{6}$$

with $z^{\alpha+1}\psi(z)$ locally univalent in D^*, $a \in \mathbb{C}$ and $a = 0$ if and only if $\alpha \notin \mathbb{N}$; or,

$$v(z) = \ln \frac{|-(1+\alpha)\psi(z) + z\psi'(z)|^2}{(|z^{\alpha+1}|^2 + |\psi(z)|^2)^2} + \ln 8, \tag{7}$$

with $z^{-(\alpha+1)}\psi(z)$ locally univalent in D^*;
or, limited to the case where $\alpha = m - \frac{1}{2}$ for some $m \in \mathbb{N}$,

$$v(z) = \ln \frac{|e^{z^{m+\frac{1}{2}}+\psi(z)}((m+\frac{1}{2})\psi(z) + z\psi'(z))|^2}{(1 + |e^{z^{m+\frac{1}{2}}}\psi(z)|^2)^2} + \ln 8, \tag{8}$$

with $z^{m+\frac{1}{2}}\psi(z)$ locally univalent in D^*.

Proof. In the punctured disk $D^* = D \setminus \{0\}$, the function:

$$u(z) = v(z) + 2\alpha \ln |z|, \tag{9}$$

solves the problem:

$$\begin{cases} -\Delta u = e^u & \text{in } D^*, \\ \int_D e^u < \infty. \end{cases} \tag{10}$$

A result due to Chou and Wan, see [13, Theorem 3], gives the following Liouville-type representation formula for u:

$$u(z) = \ln 8 \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}, \tag{11}$$

where $f(z)$ is a meromorphic locally univalent function in D^* such that:

Case A.

$$\begin{cases} f(z) = g(z)z^{\beta}, & \beta \in \mathbb{R}, \\ \text{with } g(z) \text{ single-valued analytic in } D^*, \end{cases} \tag{12}$$

or,

Case B.

$$\begin{cases} f(z) = \phi(\sqrt{z}), & \phi(z)\phi(-z) = 1, \\ \text{with } \phi(z) \text{ single-valued analytic in } D^*. \end{cases} \tag{13}$$

We will analyze Cases A and B separately. In particular, we will show that Case A leads to (6) or (7), while Case B leads to (8).

Case A. Since $\int_D e^u < +\infty$, the function $g(z)$ in (12) cannot have an essential singularity at the origin, see Lemma 4 in [13]. Thus, for some $n \in \mathbb{Z}$, we may write

$$g(z) = z^{-n}\psi(z),$$

with $\psi(z)$ holomorphic in D and $\psi(0) \neq 0$. Hence,

$$f(z) = z^{\beta-n}\psi(z), \tag{14}$$

and inserting (14) into (11) we get

$$u(z) = \ln 8 \frac{|(\beta-n)z^{\beta-n-1}\psi(z) + z^{\beta-n}\psi'(z)|^2}{(1+|z^{\beta-n}\psi(z)|^2)^2}.$$

Since,

$$u(z) = 2\alpha \ln|z| + O(1), \qquad \text{as } z \to 0, \tag{15}$$

we have the following three possibilities:

(A.1) $\beta - n > 0$. In this case by (15) we have that necessarily $\beta - n = \alpha + 1$, and

$$u(z) = \ln 8 \frac{|(1+\alpha)\psi(z) + z\psi'(z)|^2}{(1+|z^{\alpha+1}\psi(z)|^2)^2} + \ln|z|^{2\alpha},$$

which leads to (6) with $a = 0$.

(A.2) $\beta - n = 0$. In this case (15) requires that ψ' admits a zero of order α at the origin. Since ψ is analytic, this situation can occur only for $\alpha \in \mathbb{N}$. Therefore for some $m \in \mathbb{N}$, we must have that $\alpha = m$ and $\psi(z) = z^{m+1}\varphi(z) +$

a, for some $a \in \mathbb{C}\setminus\{0\}$ and $\varphi(0) \neq 0$. Hence, by reading φ in place of ψ, this case yields to (6) when $\alpha \in \mathbb{N}$ and $a \neq 0$.

(A.3) $\beta - n < 0$. In this case using again (15) we have that necessarily $|\beta - n| = 1 + \alpha$. Thus,

$$u(z) = \ln 8 \frac{|-(1+\alpha)\psi(z) + z\psi'(z)|^2}{(|z^{\alpha+1}|^2 + |\psi(z)|^2)^2} + \ln |z|^{2\alpha},$$

which leads to (7).

Case B. As in Case A, using Lemma 4 in [13], we see that the function $\phi(z)$ in (13) cannot have an essential singularity at the origin, and for some $n \in \mathbb{Z}$, we may write

$$\phi(z) = z^{-n}\varphi(z), \tag{16}$$

with φ holomorphic in D and $\varphi(0) \neq 0$. On the other hand, in view of the condition

$$\phi(z)\phi(-z) = 1, \tag{17}$$

we have that $\varphi(z)\varphi(-z) = (-1)^n z^{2n}$. This, together with the fact that $\varphi(0) \neq 0$, implies that necessarily $n = 0$ (i.e. $\phi = \varphi$). Thus, ϕ is holomorphic in D and $\phi(0) = \pm 1$. Furthermore, $\phi(z) \neq 0$, $\forall z \in D$ as it easily follows by (17). Consequently, $h(z) = \ln \phi(z)$ is a well-defined holomorphic function on D. Also note that, by (17), h is odd in D. Thus, for some $m \in \mathbb{N} \cup \{0\}$, h takes the form:

$$h(z) = z^{2m+1} \sum_{k=0}^{\infty} h_k z^{2k}.$$

Setting $\beta = 2m + 1$, we have

$$f(z) = \phi(\sqrt{z}) = e^{h(\sqrt{z})} = \exp\left(z^{\frac{\beta}{2}} \sum_{k=0}^{\infty} h_k z^k \right) = \exp(z^{\frac{\beta}{2}}\psi(z)), \tag{18}$$

for some $\psi(z)$ holomorphic in D, with $\psi(0) \neq 0$. Inserting expression (18) into (11) and using (15), we find that necessarily $\frac{\beta}{2} - 1 = \alpha$, that is $\alpha = m - \frac{1}{2}$, for some $m \in \mathbb{N}$, and (8) immediately follows.

3. THE CONCENTRATION PHENOMENA: THE CASE OF BLOW UP AT A VORTEX POINT

In this section, we analyze the situation where a solution sequence for (4) admits a blow up point at a vortex point. Localizing our analysis around such a vortex point, taken for simplicity to be the origin, we consider a sequence u_n satisfying

$$\begin{cases} -\Delta u_n = e^{u_n} - 4\pi\alpha_n\delta_{p=0} & \text{in } D, \\ \int_D e^{u_n(x)}\,dx \leqslant C, \end{cases} \tag{19}$$

for suitable $C > 0$. Furthermore, to express the fact that the origin is the *only* blow up point for u_n in D, we assume that, for any $r \in (0, 1)$,

$$\exists C_r > 0\colon \max_{r \leqslant |x| \leqslant 1} u_n \leqslant C_r, \tag{20}$$

$$\max_{\bar{D}_r} u_n \to +\infty, \qquad \text{as } n \to \infty. \tag{21}$$

We have:

THEOREM 2. *Let u_n satisfy* (19) *with $\alpha_n \to \alpha > 0$ and assume* (20) *and* (21). *There exist a subsequence u_{k_n} of u_n such that*

(a) $\max_K u_{k_n} \to -\infty$, *as $n \to \infty$, for every compact set $K \subset D\setminus\{0\}$*

(b) $e^{u_{k_n}} \to \beta\delta_{p=0}$, *weakly in the sense of measures on D, with $\beta \geqslant 8\pi$.*

Furthermore, if

(c) $\limsup_{n\to\infty} \int_D e^{u_n} < 16\pi$,

then, either $\beta = 8\pi$ or $\beta \geqslant 8\pi(1+\alpha)$.

REMARK 1. Note first that from parts (a) and (b) of Theorem 2, it follows that, if the sequence u_n satisfies (19) and $\lim_{n\to\infty} \int_D e^{u_n} < 8\pi$, then u_n cannot blow up in D. Indeed, while it is a well-known fact (cf. [23]) that u_n cannot admit a blow up point in $D\setminus\{0\}$, Theorem 2 excludes the possibility of blow up at zero.

REMARK 2. If u_n satisfies (19) with $\alpha_n \geqslant 1$ and we assume (c), then necessarily $\beta = 8\pi$.

Proof of Theorem 2. We start with some general observations. Set,

$$v_n(z) = u_n(z) - 2\alpha_n \ln|z|, \quad z \in D, \tag{22}$$

which satisfies,

$$\begin{cases} -\Delta v_n = |z|^{2\alpha_n} e^{v_n} & \text{in } D, \\ \int_D |z|^{2\alpha_n} e^{v_n} \leqslant C. \end{cases} \tag{23}$$

Taking into account (22) and (23), assumptions (20)–(21) can be restated *equivalently* in terms of v_n as follows:

for any $r \in (0, 1)$,

$$\exists C_r > 0: \max_{r \leqslant |x| \leqslant 1} v_n \leqslant C_r, \tag{24}$$

$$\max_{\bar{D}_r} v_n \to +\infty, \qquad \text{as } n \to \infty. \tag{25}$$

Set $z_n \in \bar{D}$: $v_n(z_n) = \max_{\bar{D}}$, and so by (24) and (25),

$$z_n \to 0 \text{ and } v_n(z_n) \to +\infty, \qquad \text{as } n \to \infty. \tag{26}$$

A standard blow up argument shows that, in view of (23) and (26) necessarily, $\int_D e^{v_n} \to +\infty$. So the situation analyzed here can never fulfil the assumptions of the Brezis–Merle result, but must be handled directly.

We start with property (a). Note that by (23) and (24) we can use Harnack's inequality to reduce to prove that for every $r \in (0, 1)$, along a subsequence, v_n satisfies

$$\min_{|x|=r} v_n \to -\infty, \qquad \text{as } n \to \infty. \tag{27}$$

By Theorem 1, along a subsequence which for simplicity we still denote by v_n, we may assume that v_n takes either one of the form (6), (7) or (8) with $\alpha = \alpha_n > 0$ and $\psi = \psi_n$ holomorphic on D, $\psi_n(0) \neq 0$. We consider separately each one of these different cases.

Case 1.

$$v_n(z) = \ln \frac{|(1 + \alpha_n)\psi_n(z) + z\psi_n'(z)|^2}{(1 + |z^{1+\alpha_n}\psi_n(z) + a_n|^2)^2} + \ln 8 \tag{28}$$

with $a_n \in \mathbb{C}$ and $a_n \neq 0$ if and only if $\alpha_n \in \mathbb{N}$.

By (26) we have that $|(1 + \alpha_n)\psi_n(z_n) + z_n\psi_n'(z_n)|^2 \to \infty$. Since $z^{1+\alpha_n}\psi_n(z)$ is locally univalent in D^* and $\psi_n(0) \neq 0$, we derive that $g_n(z) = (1 + \alpha_n)\psi_n(z) + z\psi_n'(z)$ is holomorphic and never vanishes in D. So, $\ln |g_n(z)|^2$ defines an harmonic function in D. Fix $r_1 \in (r, 1)$, and set $\gamma_n = \min_{|z| \leqslant r_1} |g_n(z)|^2$. If $\limsup_{n\to\infty} \gamma_n = 0$, from (28), we immediately derive (27). Hence, suppose that there exist $\gamma > 0$ such that, along a subsequence, we have $\gamma_n \geqslant \gamma$. In this situation, we may apply Harnack's inequality to the positive harmonic function $\ln \frac{|g_n(z)|^2}{\gamma_n}$ on D_{r_1}. Hence, for $0<\varepsilon<r_1-r$, we find a constant $\tau \geqslant 1$ (depending on ε and r only) such that

$$\max_{|z|\leqslant r+\varepsilon} \ln \frac{|g_n(z)|^2}{\gamma_n} \leqslant \tau \min_{|z|\leqslant r+\varepsilon} \ln \frac{|g_n(z)|^2}{\gamma_n}.$$

Consequently,

$$\min_{|z|\leqslant r+\varepsilon} |g_n(z)|^{2\tau} \geqslant \gamma_n^{\tau-1} \max_{|z|\leqslant r+\varepsilon} |g_n(z)|^2 \geqslant \gamma^{\tau-1} \max_{|z|\leqslant r+\varepsilon} |g_n(z)|^2.$$

Setting $\beta = \gamma^{\tau-1}$, we may conclude,

$$\begin{aligned} \min_{|z|\leqslant r+\varepsilon} |(1+\alpha_n)\psi_n(z) + z\psi_n'(z)|^{2\tau} &\geqslant \beta \max_{|z|\leqslant r+\varepsilon} |(1+\alpha_n)\psi_n(z) + z\psi_n'(z)|^2 \\ &\geqslant \beta \max_{|z|\leqslant r+\varepsilon} \frac{|(1+\alpha_n)\psi_n(z) + z\psi_n'(z)|^2}{(1+|z^{1+\alpha_n}\psi_n(z)+a_n|^2)^2} \to +\infty, \\ &\quad \text{as } n \to \infty. \end{aligned}$$

That is,

$$\min_{|z|\leqslant r+\varepsilon} |(1+\alpha_n)\psi_n(z) + z\psi_n'(z)| \to +\infty, \qquad \text{as } n \to \infty. \tag{29}$$

On the other hand, for $\varepsilon > 0$ sufficiently small, by assumption we have

$$\max_{r-\varepsilon\leqslant|z|\leqslant r+\varepsilon} \frac{|(1+\alpha_n)\psi_n(z) + z\psi_n'(z)|^2}{(1+|z^{1+\alpha_n}\psi_n(z)+a_n|^2)^2} \leqslant C_{r,\varepsilon}.$$

Setting, $f_n(z) = z^{\alpha_n+1}\psi_n(z) + a_n$, by (29) we conclude that $\min_{r-\varepsilon\leqslant|z|\leqslant r+\varepsilon} |f_n(z)| \to +\infty$. Hence, for every z: $|z| = r$, we may use Cauchy's integral formula to derive

$$\begin{aligned} \frac{|(1+\alpha_n)\psi_n(z) + z\psi_n'(z)|}{1+|z^{1+\alpha_n}\psi_n(z)+a_n|^2} &\leqslant \frac{1}{r^{\alpha_n}} \left| \frac{f_n'(z)}{f_n(z)^2} \right| = \frac{1}{r^{\alpha_n}} \left| \frac{d}{dz} \frac{1}{f_n(z)} \right| \\ &\leqslant \frac{1}{2\pi r^{\alpha_n}} \int_{|z-\xi|=\varepsilon} \frac{1}{|(\xi - z)|^2} \frac{1}{|f_n(\xi)|} |d\xi| \\ &\leqslant C_{\varepsilon,r} \frac{1}{\min_{r-\varepsilon\leqslant|z|\leqslant r+\varepsilon} |f_n(z)|} \to 0, \end{aligned}$$

and (27) is established.

Case 2.

$$v_n(z) = \ln \frac{|-(1+\alpha_n)\psi_n(z) + z\psi_n'(z)|^2}{(|z^{1+\alpha_n}|^2 + |\psi_n(z)|^2)^2} + \ln 8. \tag{30}$$

By hypothesis $v_n(z_n) \to \infty$, as $n \to \infty$. In case $\max_{|z| \leqslant r} |-(1+\alpha_n)\psi_n(z) + z\psi_n'(z)| \to +\infty$, we can proceed exactly as in the previous case to derive the desired conclusion. Hence we suppose that, along a subsequence, still denoted by ψ_n, there exist a constant $C > 0$, such that $\max_{|z| \leqslant r} |-(1+\alpha_n)\psi_n(z) + z\psi_n'(z)| = \max_{|z|=r} |-(1+\alpha_n)\psi_n(z) + z\psi_n'(z)| \leqslant C$. If $\limsup_{n\to\infty} \max_{|z|\leqslant r} |\psi_n| = +\infty$, by (30), we immediately get (27). So we are left to consider the case where we have

$$\max_{\bar{D}_r} |\psi_n| \leqslant C, \tag{31}$$

for suitable $C > 0$. Since ψ_n is a sequence of holomorphic functions in D, in this case, passing to a subsequence, we may conclude that, ψ_n converges to ψ uniformly in $\bar{D}_{r/2}$ together with its derivatives. By (26) and (30), we get that necessarily, $\psi(0) = 0$. Thus,

$$\psi_n(0) \to 0, \qquad \text{as } n \to \infty,$$

and, consequently,

$$v_n(0) \to +\infty, \qquad \text{as } n \to \infty. \tag{32}$$

At this point, the following claim together with (32), yields (27) and we conclude the proof in this case as well.

Claim. *For any* $r \in (0,1)$, *we have*

$$v_n(0) + \min_{\partial D_r} v_n \leqslant 2 \ln \frac{8(1+\alpha_n)^2}{r^{2(1+\alpha_n)}}.$$

Proof. The idea of our proof is inspired by that of Theorem 2 given by Shafrir [31]. For given $r \in (0,1)$, define

$$\sigma_n(z) = \ln 8 \frac{|-(1+\alpha_n)\psi_n(z) + z\psi_n'(z)|^2}{(r^{2(1+\alpha_n)} + |\psi_n(z)|^2)^2}.$$

Clearly, σ_n is superharmonic in D_r and so,

$$\sigma_n(0) \geqslant \inf_{|z|=r} \sigma_n = \inf_{|z|=r} v_n. \tag{33}$$

On the other hand,

$$\sigma_n(0) = \ln 8 \frac{(1+\alpha_n)^2 |\psi_n(0)|^2}{(r^{2(1+\alpha_n)} + |\psi_n(0)|^2)^2} \leqslant \ln \frac{8(1+\alpha_n)^2}{r^{4(1+\alpha_n)}} + \ln |\psi_n(0)|^2. \tag{34}$$

Using (33) and (34), we obtain

$$v_n(0) = \ln \frac{8(1+\alpha_n)^2}{|\psi_n(0)|^2} \leqslant \ln \frac{(8(1+\alpha_n)^2)^2}{r^{4(1+\alpha_n)}} - \sigma_n(0) \leqslant 2 \ln \frac{8(1+\alpha_n)^2}{r^{2(1+\alpha_n)}} - \inf_{|z|=r} v_n,$$

which gives the desired estimates. ∎

Case 3. Suppose $\alpha + 1 = m + \frac{1}{2}$, for some $m \in \mathbb{N}$ and

$$v_n(z) = \ln \frac{|e^{z^{m+\frac{1}{2}}\psi_n(z)}((m+\frac{1}{2})\psi_n(z) + z\psi_n'(z))|^2}{(1+|e^{z^{m+\frac{1}{2}}\psi_n(z)}|^2)^2}. \tag{35}$$

For fixed $r \in (0,1)$, we claim that we can find $\bar{z}_n$: $|\bar{z}_n| = r$, such that, along a subsequence, we have

$$\max_{|z|=r} |(m+\tfrac{1}{2})\psi_n(z) + z\psi_n'(z)| = |(m+\tfrac{1}{2})\psi_n(\bar{z}_n) + \bar{z}_n\psi_n'(\bar{z}_n)| \to +\infty. \tag{36}$$

Indeed, if this was not the case, we would have

$$\max_{|z|\leqslant r} |(m+\tfrac{1}{2})\psi_n(z) + z\psi_n'(z)| = \max_{|z|=r} |(m+\tfrac{1}{2})\psi_n(z) + z\psi_n'(z)| \leqslant C,$$

for suitable $C > 0$. By (26), this implies that necessarily,

$$|e^{z_n^{(m+\frac{1}{2})}\psi_n(z_n)}| \to +\infty, \qquad \text{as } n \to \infty.$$

But this is impossible, since (35) would imply that $v_n(z_n) \to -\infty$, as $n \to \infty$, in contradiction with (26). Hence, along a subsequence, we may assume that (36) holds and also that $|e^{\bar{z}_n^{(m+\frac{1}{2})}\psi_n(\bar{z}_n)}|$ admits a limit as $n \to \infty$. Recalling that

$$\max_{|z|=r} \frac{|e^{z^{m+\frac{1}{2}}\psi_n(z)}((m+\frac{1}{2})\psi_n(z) + z\psi_n'(z))|^2}{(1+|e^{z^{m+\frac{1}{2}}\psi_n(z)}|^2)^2} \leqslant C_r \qquad \forall n \in \mathbb{N}, \tag{37}$$

only one of the following two situations are possible:

Case A.

$$|e^{\bar{z}_n^{(m+\frac{1}{2})}\psi_n(\bar{z}_n)}| \to 0, \qquad \text{as } n \to \infty,$$

or

Case B.

$$|e^{\bar{z}_n^{(m+\frac{1}{2})}\psi_n(\bar{z}_n)}| \to \infty, \qquad \text{as } n \to \infty.$$

Case A. Possibly extracting a subsequence, assume that $\bar{z}_n \to z_0$, with $|z_0| = r$. We *claim* that there exist sufficiently small $\delta > 0$ and $C > 0$, independent of n, such that

$$|e^{z^{m+\frac{1}{2}}\psi_n(z)}|^2 \leqslant C, \qquad \forall |z - z_0| \leqslant \delta. \tag{38}$$

Indeed, take $\delta > 0$ sufficiently small so that $\bar{B}_{2\delta}(z_0) \subset D \setminus \{0\}$. Consider the function $\varphi_n(z) = 2\ln(1 + |e^{z^{m+\frac{1}{2}}\psi_n}|^2)$ which satisfies

$$-\Delta\varphi_n = |z|^{2(m-\frac{1}{2})} e^{v_n}, \qquad \text{in } B_{2\delta}(z_0),$$

$$\varphi_n \geqslant 0 \qquad \text{in } \overline{B_{2\delta}(z_0)}.$$

In view of (24), we have $0 \leqslant f_n(z) := |z|^{2(m-\frac{1}{2})} e^{v_n} \leqslant C$ in $B_{2\delta}(z_0)$. With the help of Harnack's inequality, we obtain suitable constants $\beta > 1$ and $C > 0$ (depending on $\delta > 0$ only) such that

$$\max_{|z-z_0|\leqslant\delta} |e^{z^{m+\frac{1}{2}}\psi_n(z)}| \leqslant C\left(\min_{|z-z_0|\leqslant\delta} |e^{z^{m+\frac{1}{2}}\psi_n(z)}|^\beta + 1\right). \tag{39}$$

Hence, taking into account the hypothesis of Case A, we derive

$$\max_{|z-z_0|\leqslant\delta} |e^{z^{m+\frac{1}{2}}\psi_n(z)}| \leqslant C(|e^{\bar{z}_n^{m+\frac{1}{2}}\psi_n(\bar{z}_n)}|^\beta + 1) = C(1 + o(1)), \qquad \text{as } n \to \infty, \tag{40}$$

and (38) follows. Therefore, in $B_\delta(z_0) \subset D \setminus \{0\}$, we can apply Harnack's inequality to the bounded harmonic function $\ln|e^{z^{m+\frac{1}{2}}\psi_n(z)}|^2$ to obtain suitable constants $\beta \in (0, 1)$ and $A > 0$ (depending on $\delta > 0$ only), such that

$$\max_{|z-z_0|\leqslant\frac{\delta}{2}} \ln|e^{z^{m+\frac{1}{2}}\psi_n(z)}| \leqslant \beta \min_{|z-z_0|\leqslant\frac{\delta}{2}} \ln A|e^{z^{m+\frac{1}{2}}\psi_n(z)}|. \tag{41}$$

Whence, as $\bar{z}_n \to z_0$, under the hypothesis of Case A, from (41) we derive

$$\begin{aligned}\max_{|z-z_0|=\frac{\delta}{2}} |e^{z^{m+\frac{1}{2}}\psi_n(z)}| &= \max_{|z-z_0|\leqslant\frac{\delta}{2}} |e^{z^{m+\frac{1}{2}}\psi_n(z)}| \\ &\leqslant A^\beta |e^{\bar{z}_n^{(m+\frac{1}{2})}\psi_n(\bar{z}_n)}|^\beta \to 0, \qquad \text{as } n \to \infty.\end{aligned}$$

At this point, we can conclude our proof as in Case 1 above. Indeed, by Cauchy integral formula, we have

$$r^{(m+\frac{1}{2})}\left|e^{\bar{z}_n^{(m+\frac{1}{2})}\psi_n(\bar{z}_n)}\left(\left(m+\frac{1}{2}\right)\psi(\bar{z}_n)+\bar{z}_n\psi'(\bar{z}_n)\right)\right|$$

$$=\left|\bar{z}_n^{(m+\frac{1}{2})}\left(e^{\bar{z}_n^{(m+\frac{1}{2})}\psi_n(\bar{z}_n)}\left(\left(m+\frac{1}{2}\right)\psi(\bar{z}_n)+z\psi'(\bar{z}_n)\right)\right)\right|=\left|\frac{d}{dz}\,e^{z^{m+\frac{1}{2}}\psi_n(z)}|_{z=\bar{z}_n}\right|$$

$$=\left|\frac{1}{2\pi\imath}\int_{|z-z_0|=\frac{\delta}{2}}\frac{e^{z^{m+\frac{1}{2}}\psi_n(z)}}{(z-\bar{z}_n)^2}\,dz\right|\leqslant\max_{|z-z_0|=\frac{\delta}{2}}\left|e^{z^{m+\frac{1}{2}}\psi_n(z)}\right|\frac{O(1)}{\delta}\to 0.$$

This immediately implies $v_n(\bar{z}_n)\to-\infty$, and (38) is established in Case A.

Case B. This case is analogous to Case 1 above. We can complete the proof exactly in the same way with the help of Harnack's inequality and Cauchy integral formula.

So (a) has been established.

In order to obtain (b), by (5), we can further extract a subsequence so that $e^{u_n}\to v$, weakly in the sense of measure in D. In view of (a), the measure v is supported at zero, that is $v=\beta\delta_{p=0}$ for some $\beta>0$.

To show that $\beta\geqslant 8\pi$, we use a blow up argument for the sequence u_n. Let $x_n\in\bar{D}$: $\max_{\bar{D}} u_n=u_n(x_n)$. As for v_n, by (20) and (21), we have that

$$\begin{cases} u_n(x_n)\to+\infty,\\ x_n\to 0.\end{cases}\tag{42}$$

Set

$$\delta_n=\exp\left(-\frac{u_n(x_n)}{2}\right)\to 0,\qquad \text{as } n\to\infty,$$

and define

$$t_n=\max\{\delta_n,|x_n|\}\to 0,\qquad \text{as } n\to\infty.$$

The sequence,

$$\xi_n(x)=u_n(t_nx)+2\ln t_n,$$

defined on the set $B_n=D_{1/t_n}$, satisfies

$$\begin{cases} -\Delta\xi_n=e^{\xi_n}-4\pi\alpha_n\delta_{p=0} & \text{in } B_n,\\ \int_{B_n} e^{\xi_n(x)}\,dx\leqslant C,\end{cases}\tag{43}$$

for some $C>0$. Set $y_n=\frac{x_n}{t_n}$, and note that $|y_n|\leqslant 1$, so by taking a subsequence, we can assume $y_n\to y_0\in\mathbb{R}^2$. Furthermore,

$$\max_{\bar{B}_n}\ \xi_n=\xi_n(y_n)=u_n(x_n)+2\ln t_n\geqslant u_n(x_n)+2\ln\delta_n=0.$$

We distinguish two cases:

Case A: $\xi_n(y_n)\leqslant C,\ \ \forall n\in\mathbb{N}$.

Case B: $\limsup_{n\to\infty}\xi_n(y_n)=+\infty$.

Concerning the Case A, we claim that,

$$\liminf_{n\to\infty}\int_D e^{u_n}\geqslant 8\pi(1+\alpha). \tag{44}$$

Indeed, in this case,

$$0\leqslant\max_{\bar{B}_n}\ \xi_n=\xi_n(y_n)\leqslant C, \tag{45}$$

and so, if we write $\xi_n(x)=2\alpha_n\ln|x|+\phi_n(x)$, then $\phi_n(x)$ defines the regular part of ξ_n, and for every $R>0$, it satisfies:

$$\begin{cases}-\Delta\phi_n=|x|^{2\alpha_n}\,e^{\phi_n} & \text{on } D_R,\\ \max_{\partial D_R}\ |\phi_n|\leqslant C_R,\end{cases} \tag{46}$$

for suitable $C_R>0$. Since $e^{\xi_n}=|x|^{2\alpha_n}e^{\phi_n}$ is uniformly bounded in D_R, by standard elliptic estimates, we derive that $|\phi_n|$ is uniformly bounded in D_R. Therefore, we can use elliptic regularity theory together with a diagonal process to conclude that, along a subsequence, $\phi_n\to\phi$ in $C^{1,\delta}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\delta\in(0,1)$. Furthermore, ϕ satisfies

$$\begin{cases}-\Delta\phi=|x|^{2\alpha}e^{\phi} & \text{in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2}\ |x|^{2\alpha}e^{\phi}<+\infty.\end{cases} \tag{47}$$

By the results in [12], see also [30], necessarily $\int_{\mathbb{R}^2}|x|^{2\alpha}e^{\phi}=8\pi(1+\alpha)$. So,

$$\liminf_{n\to+\infty}\int_D e^{u_n}=\liminf_{n\to+\infty}\int_{B_n}e^{\xi_n}=\liminf_{n\to+\infty}\int_{B_n}|x|^{2\alpha_n}\,e^{\phi_n}$$
$$\geqslant\int_{\mathbb{R}^2}|x|^{2\alpha}e^{\phi}=8\pi(1+\alpha).$$

Hence, (44) holds and $\beta\geqslant 8\pi(1+\alpha)$ in this case.

Case B. In this case, necessarily $t_n=|x_n|$ (along a subsequence) and consequently $|y_0|=1$ (recall $y_0=\lim_{n\to\infty}\frac{x_n}{t_n}$).

Hence, in this situation, ξ_n admits a blow up point at $y_0\neq 0$ and we can apply the Li–Shafrir result [23] to the sequence ξ_n in any small neighborhood of y_0 away from zero, and obtain that

$$\lim_{n\to\infty} \int_{B_\delta(y_0)} e^{\xi_n} = 8\pi m, \qquad \text{for some } m \in \mathbb{N}, \tag{48}$$

for every $\delta > 0$ sufficiently small. As above, this yields $\beta \geqslant 8\pi m$, and in any event, the desired conclusion that $\beta \geqslant 8\pi$ is established.

We now turn to the proof of the last part of the statement. For this purpose, we recall the Alexandrov–Bol inequality (cf. [2]), as derived by Suzuki [36].

The Alexandrov–Bol inequality. Let $p \in C^2(\Omega) \cap C^0(\Omega)$ satisfy the elliptic inequality:

$$-\Delta \log p \leqslant p \qquad \text{in } \Omega \subset \mathbb{R}^2.$$

Then,

$$l^2(\partial\Omega) \geqslant \tfrac{1}{2}(8\pi - m(\Omega))m(\Omega),$$

where

$$l(\Omega) = \int_{\partial\Omega} p^{\frac{1}{2}}\, ds \qquad \text{and} \qquad m(\Omega) = \int_\Omega p\, dx.$$

We need to prove that, if

$$\beta = \lim_{n\to\infty} \int_D e^{u_n} < 8\pi(1+\alpha), \tag{49}$$

then necessarily $\beta = 8\pi$.

In view of (44), if (49) holds, then Case A can be ruled out and we have that necessarily Case B must occur.

Furthermore, since $\beta \leqslant \limsup_{n\to\infty} \int_D e^{u_n} < 16\pi$, then for the sequence ξ_n, (48) must hold with $m = 1$ and consequently,

$$\limsup_{n\to\infty} \int_A e^{\xi_n} < 8\pi, \qquad \text{for every open regular set } A \subset\subset \mathbb{R}^2 \setminus \{y_0\}. \tag{50}$$

At this point, we may use Remark 1, and conclude that y_0 is the only blow up point for the sequence ξ_n in D_R, for every $R > 1$. In particular, around the origin, ξ_n is uniformly bounded from above, and so the presence of the Dirac measure in (43) is no longer problematic for the use of the Brezis–Merle analysis to derive,

$$\max_{|x|=R} \xi_n \to -\infty, \qquad \text{as } n \to \infty,$$

$$\lim_{n\to+\infty} \int_{|x|\leqslant R} e^{\xi_n} = \lim_{n\to+\infty} \int_{B_\delta(y_0)} e^{\xi_n} = 8\pi,$$

for every $R > 1$. Going back to the original coordinates, those conditions read as follows:

$$\max_{|x|=R|x_n|} u_n + 2\ln|x_n| \to -\infty, \tag{51}$$

$$\lim_{n\to+\infty} \int_{|x|\leqslant R|x_n|} e^{u_n} = 8\pi, \tag{52}$$

for every given $R > 1$. Thus, in the set

$$\Omega_{n,\,r} = D_r \setminus D_{R|x_n|},$$

u_n satisfies,

$$\begin{cases} -\Delta u_n = e^{u_n} & \text{in } \Omega_{n,\,r}, \\ \limsup_{n\to\infty} \int_{\Omega_{n,\,r}} e^{u_n(x)}\,dx = \mu < 8\pi. \end{cases} \tag{53}$$

So, for n large, we can apply the Alexandrov–Bol inequality to u_n in $\Omega_{n,\,r}$ and conclude that

$$\left(\int_{\partial\Omega_{n,\,r}} e^{u_n/2}\,ds\right)^2 \geqslant \frac{1}{2}\left(8\pi - \int_{\Omega_{n,\,r}} e^{u_n}\right)\left(\int_{\Omega_{n,\,r}} e^{u_n}\right), \tag{54}$$

that is,

$$\left(\int_{\partial\Omega_{n,\,r}} e^{u_n/2}\right)^2 \geqslant \frac{1}{2}(8\pi - \mu + o(1))\int_{\Omega_{n,\,r}} e^{u_n}, \qquad \text{as } n \to \infty. \tag{55}$$

We will show that the l.h.s. of (55) goes to zero as $n \to +\infty$. In turn, the r.h.s goes to zero, and since $\mu < 8\pi$, we conclude

$$\int_{\Omega_{n,\,r}} e^{u_n} \to 0. \tag{56}$$

Indeed,

$$\int_{\partial\Omega_{n,\,r}} e^{u_n/2}\,ds = \int_{\partial D_r} e^{u_n/2}\,ds + \int_{|x|=R|x_n|} e^{u_n/2}\,ds.$$

Using the already established property (a) with $K = \partial D_r$, we find

$$\int_{\partial D_r} e^{u_n/2}\,ds \to 0, \qquad \text{as } n \to \infty.$$

On the other hand, using (51), we have

$$\int_{|x|=R|x_n|} e^{u_n/2}\, ds \leqslant 2\pi R|x_n| \max_{|x|=R|x_n|} e^{\frac{1}{2}u_n(x)}$$
$$= 2\pi R \max_{|x|=R|x_n|} e^{\frac{1}{2}(u_n(x)+2\ln|x_n|)} \to 0, \qquad \text{as } n \to \infty.$$

Thus, (56) holds and by (52) we conclude

$$\beta = \lim_{n\to\infty} \int_{D_r} e^{u_n} = \lim_{n\to\infty} \int_{|x|\leqslant R|x_n|} e^{u_n} + \lim_{n\to\infty} \int_{\Omega_{n,\,r}} e^{u_n} = 8\pi. \quad \blacksquare$$

From Theorem 2, we immediately derive the following version of the Brezis–Merle result [6] for a solution sequence u_n satisfying (19).

THEOREM 3. *Let u_n be a solutions sequence for problem* (19) *with $\alpha_n \to \alpha > 0$. There exists a subsequence u_{k_n} of u_n for which one of the following alternative holds*:

(i) $\sup_K |u_{k_n}(x) - 2\alpha_{k_n} \ln|x|| \leqslant C_K$ $\forall K \subset\subset D$, *and suitable constant* $C_K > 0$.

(ii) $\sup_K \{u_{k_n}(x) - 2\alpha_{k_n} \ln|x|\} \to -\infty$, $\forall K \subset\subset D$.

(iii) *There exist a finite and nonempty set $S = \{q_1, \ldots, q_l\} \subset D$, $l \in \mathbb{N}$, corresponding sequences $\{x_n^1\}_{n\in\mathbb{N}}, \ldots, \{x_n^l\}_{n\in\mathbb{N}} \subset D$, such that $x_n^i \to q_i$ and $u_{k_n}(x_n^i) \to \infty$ for $i \in 1, \ldots, l$. Furthermore*, $\sup_K \{u_{k_n}(x) - 2\alpha_n \ln|x|\} \to -\infty$ *on any compact set $K \subset D \setminus S$, and $e^{u_{k_n}} \to \sum_{i=1}^{l} \beta_i \delta_{q_i}$ weakly in the sense of measures on D, with $\beta_i \in 8\pi\mathbb{N}$ if $q_i \neq 0$ and $\beta_i \geqslant 8\pi$ if $q_i = 0$ for some $i = 1, \ldots, l$.*

At this point, the more general version of Theorem 3, as stated in the Introduction, may be easily derived.

Proof. As above, we shall work with the sequence v_n defined in (22). Note that in any subdomain $D' \subset\subset D \setminus \{0\}$, we have

$$\int_{D'} e^{v_n} \leqslant C_{D'}, \tag{57}$$

with $C_{D'} > 0$ a suitable constant depending on D' only. Recall that the blow up set S of v_n in D is defined as follows:

$$S = \{x \in D\colon \exists \{x_n\} \subset D \text{ such that } x_n \to x \text{ and } v_n(x_n) \to +\infty\}.$$

In view of (21), the solutions sequence v_n satisfies to all assumptions of the Brezis–Merle Theorem in $D\setminus\bar{D}_\delta$ for every $\delta > 0$ sufficiently small. Recalling that $v_n(x) = u_n(x) - 2\,\alpha_n \ln|x|$ in D, we may conclude that $S' = S\setminus\{0\}$ is a finite set, and along a subsequence, $u_n(x) - 2\alpha_n \ln|x|$ satisfies one of the alternatives (i)–(iii) above with D replaced by $D' = D\setminus\{0\}$ and S replaced by S'. Obviously, each blow up point for v_n in S' (when not empty) is also a blow up point for u_n. Hence, we are left to analyze what happens around zero. Observe first that the point $x = 0$ is a blow up point for v_n if and only if it is a blow up point for u_n. At this point, we may conclude our proof by observing that, in case zero is *not* a blow up point for v_n (and hence for u_n), that is $S = S'$, then v_n is uniformly bounded above in a small neighborhood of zero. This, combined with (57), gives that v_n satisfies to *all* assumptions of the Brezis–Merle Theorem in the set D, and so we immediately derive the desired conclusion in this case. If zero *is* a blow up point for v_n, and hence for u_n, then $S = S' \cup \{0\}$. Thus, u_n satisfies to all assumptions of Theorem 2 in a ball $B_{r_0}(0)$. For $S' \neq \emptyset$, take $r_0 > 0$ sufficiently small so that $B_{r_0}(0) \cap S' = \emptyset$. Thus, the conclusion follows in this case as well, by combining the Brezis-Merle result applied to v_n on $D\setminus\{0\}$ with Theorem 2 applied to u_n in $B_{r_0}(0)$. ∎

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