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A new extension of q-Euler numbers and polynomials related to their interpolation functions

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Abstract

In this work, by using a p-adic q-Volkenborn integral, we construct a new approach to generating functions of the (h, q)-Euler numbers and polynomials attached to a Dirichlet character χ. By applying the Mellin transformation and a derivative operator to these functions, we define (h, q)-extensions of zeta functions and l-functions, which interpolate (h, q)-extensions of Euler numbers at negative integers.

Keywords: p-adic Volkenborn integral; Twisted q-Euler numbers and polynomials; Zeta and l-functions

1. Introduction, definitions and notation

Let p be a fixed odd prime number. Throughout this work, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex numbers field and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is considered in many ways, e.g. as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1-q|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_q \leqslant 1$; cf. [3,2,5–7,4,11,14,16,1]. We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q},$$

where $\lim_{q\to 1} [x]_q = x$; cf. [5].

Let $UD\left(\mathbb{Z}_p\right)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD\left(\mathbb{Z}_p\right)$, Kim [3] originally defined the p-adic invariant q-integral on $\mathbb{Z}_p$ as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) \mathrm{d}\mu_q(x) = \lim_{N\to\infty} \frac{1}{\left[p^N\right]_q} \sum_{x=0}^{p^N-1} f(x) q^x,$$

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where N is a natural number and p is an odd prime number. The q-deformed p-adic invariant integral on $\mathbb{Z}_p$, in the fermionic sense, is defined by

$$I_{-q}(f) = \lim_{q\to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu_{-q}(x), \quad \text{cf. [3,5,6,4].}$$

Recently, twisted (h, q)-Bernoulli and Euler numbers and polynomials were studied by several authors (see [10,2,15,16,9,8,13,1]).

By definition of $\mu_{-q}(x)$, we see that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{cf. [5],} \tag{1.1}$$

where $f_1(x) = f(x+1)$.

In this study, we define new (h, q)-extension of Euler numbers and polynomials. By using a derivative operator on these functions, we derive (h, q)-extensions of zeta functions and l-functions, which interpolate (h, q)-extensions of Euler numbers at negative integers.

2. A new approach to q-Euler numbers

In this section, we define (h, q)-extension of Euler numbers and polynomials. Substituting $f(x) = q^{hx}\mathrm{e}^{tx}$, with $h \in \mathbb{Z}$, into (1.1) we have

$$F_q^h(t) = I_{-1}(q^{hx}\mathrm{e}^{tx}) = \frac{2}{q^h\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}\frac{t^n}{n!}, \qquad |h\log q + t| < \pi, \tag{2.1}$$

where $E_{n,q}^{(h)}$ is called the (h, q)-extension of Euler numbers. $\lim_{q\to 1} E_{n,q}^{(h)} = E_n$, where E_n is the classical Euler numbers. That is

$$\frac{2}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!} \quad \text{cf. [8,4,12,17].}$$

(h, q)-extensions of Euler polynomials, $E_{n,q}^{(h)}(x)$, are defined by the following generating function:

$$F_q^h(t, x) = F_q^h(t)\mathrm{e}^{tx} = \frac{2\mathrm{e}^{tx}}{q^h\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x)\frac{t^n}{n!}. \tag{2.2}$$

By using the Maclaurin series of e^{tx} in (2.1), we have

$$\int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} q^{hx}\frac{t^n x^n}{n!}\mathrm{d}\mu_{-1}(x) = \sum_{n=0}^{\infty} E_{n,q}^{(h)}\frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ on either side of the above equation, we obtain the Witt formula, which is given by the following theorem.

Theorem 1 (*Witt Formula*). *For $h \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$,*

$$\int_{\mathbb{Z}_p} q^{hx}x^n\mathrm{d}\mu_{-1}(x) = E_{n,q}^{(h)}, \tag{2.3}$$

and

$$\int_{\mathbb{Z}_p} q^{hy}(x+y)^n\mathrm{d}\mu_{-1}(y) = E_{n,q}^{(h)}(x).$$

From (2.2), we have

$$\sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!}.$$

By the Cauchy product, we see that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} E_{k,q}^{(h)} \frac{t^k}{k!} x^{n-k} \frac{t^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$, we arrive at the following theorem:

Theorem 2. *Let $n \in \mathbb{Z}_+ = \mathbb{Z} \cup \{0\}$. Then we have*

$$E_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} E_{k,q}^{(h)}. \tag{2.4}$$

Let d be a fixed integer. For any positive integer N, we set

$$\mathbb{X} = \mathbb{X}_d = \varprojlim_N \left(\mathbb{Z}/dp^N\mathbb{Z}\right), \qquad \mathbb{X}_1 = \mathbb{Z}_p, \qquad \mathbb{X}^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} \left(a + dp^N \mathbb{Z}_p\right),$$

$$a + dp^N \mathbb{Z}_p = \left\{x \in \mathbb{X} : x \equiv a \left(\bmod dp^N\right)\right\},$$

where $a \in \mathbb{Z}$ with $0 \leqslant a < dp^N$ (cf. [3]). It is known that

$$\int_{\mathbb{Z}_p} f(x) \mathrm{d}\mu_{-1}(x) = \int_{\mathbb{X}} f(x) \mathrm{d}\mu_{-1}(x), \quad \text{cf. [3]}.$$

From this we note that

$$\int_{\mathbb{X}} (x+t)^k q^{ht} \mathrm{d}\mu_{-1}(t) = d^k \sum_{a=0}^{d-1} (-1)^a q^{ha} \int_{\mathbb{Z}_p} \left(t + \frac{a+x}{d}\right)^k \left(q^d\right)^{ht} \mathrm{d}\mu_{-1}(t), \tag{2.5}$$

where d is an odd positive integer. From (2.2) and (2.5), we obtain the following theorem.

Theorem 3 (*Distribution Relation*). *For d an odd positive integer, $k \in \mathbb{Z}_+$, we have*

$$E_{k,q}^{(h)}(x) = d^k \sum_{a=0}^{d-1} (-1)^a q^{ha} E_{k,q^d}^{(h)} \left(\frac{x+a}{d}\right).$$

By (1.1), Kim [5] defined the following integral equation:

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \tag{2.6}$$

where $n \in \mathbb{N}$, $f_n(x) = f(x+n)$.

Let d be an odd positive integer and χ be the Dirichlet character with conductor d; substituting $f(x) = q^{hx}\chi(x)\mathrm{e}^{tx}$, for $h \in \mathbb{Z}$, into (2.6), we obtain

$$F_q^h(t,\chi) = \frac{2\sum_{a=0}^{d-1} (-1)^a \chi(a) \mathrm{e}^{ta} q^{ha}}{q^{hd}\mathrm{e}^{td} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h)} \frac{t^n}{n!}, \qquad |t + h \log q| < \frac{\pi}{d}, \tag{2.7}$$

where $E_{n,\chi,q}^{(h)}$ denote (h,q)-extensions of generalized Euler numbers.

From (2.7), we see that

$$\int_{\mathbb{X}} \chi(x) q^{hx} x^n \mathrm{d}\mu_{-1}(x) = d^n \sum_{a=0}^{d-1} \chi(a) q^{ha} (-1)^a \int_{\mathbb{Z}_p} \left(q^d\right)^{hx} \left(\frac{a}{d} + x\right)^n \mathrm{d}\mu_{-1}(x). \tag{2.8}$$

By Theorem 1 and (2.8), we obtain the following theorem.

Theorem 4. *Let d be an odd positive integer and χ be Dirichlet's character with conductor d. Then we have*

$$E_{n,\chi,q}^{(h)} = d^n \sum_{a=0}^{d-1} \chi(a) q^{ha} (-1)^a E_{n,q^d}^{(h)} \left(\frac{a}{d}\right).$$

From (2.6), we also note that

$$F_q^h(t, x, \chi) = \frac{2 \sum\limits_{a=0}^{d-1} (-1)^a \chi(a) \mathrm{e}^{t(a+x)} q^{ha}}{q^{hd} \mathrm{e}^{td} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h)}(x) \frac{t^n}{n!}, \tag{2.9}$$

where $h \in \mathbb{Z}$, $E_{n,\chi,q}^{(h)}(x)$ are called generalized (h, q)-extensions of Euler polynomials attached to χ and $F_q^h(t, x, \chi) = F_q^h(t, \chi) \mathrm{e}^{tx}$.

By (2.9), we easily see that

$$\int_{\mathbb{X}} (x + y)^n \chi(y) q^{hy} \mathrm{d}\mu_{-1}(y) = E_{n,\chi,q}^{(h)}(x). \tag{2.10}$$

By using (2.10), we arrive at the following theorem.

Theorem 5. *Let d be an odd integer. Then we have*

$$E_{n,\chi,q}^{(h)}(x) = d^n \sum_{a=0}^{d-1} (-1)^a \chi(a) q^{ha} E_{n,q^d}^{(h)} \left(\frac{a + x}{d}\right).$$

3. A new approach to the (h, q)-Euler zeta function

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. By using a geometric series in (2.2), we obtain

$$2\mathrm{e}^{xt} \sum_{n=0}^{\infty} q^{hn} \mathrm{e}^{tn} (-1)^n = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!}.$$

By applying the derivative operator $\frac{\mathrm{d}^k}{\mathrm{d}t^k} |_{t=0}$ to the above equation, we have

$$E_{k,q}^{(h)}(x) = 2 \sum_{n=0}^{\infty} (-1)^n q^{hn} (x + n)^k. \tag{3.1}$$

By (3.1), we define new extensions of Hurwitz type (h, q)-Euler zeta functions as follows:

Definition 1. For $h \in \mathbb{Z}$, $s \in \mathbb{C}$ and $0 < x \le 1$, we define

$$\zeta_{E,q}^{(h)}(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{hn}}{(n + x)^s}. \tag{3.2}$$

$\zeta_{E,q}^{(h)}(s, x)$ is an analytic function on the whole complex s-plane. If $x = 1$, then we define the (h, q)-Euler zeta function as follows:

$$\zeta_{E,q}^{(h)}(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{hn}}{n^s}.$$

For $s = -k$, $k \in \mathbb{Z}_+$ in (3.2) and using (3.1), we arrive at the following theorem.

Theorem 6. *For $k \in \mathbb{Z}_+$, we have*

$$\zeta_{E,q}^{(h)}(-k, x) = E_{k,q}^{(h)}(x). \tag{3.3}$$

Remark 1. By applying the Mellin transformation to the generating function of (h, q)-Euler polynomials, for $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)} \int_0^{\infty} F_q^h(-t, x) t^{s-1} \mathrm{d}t = \zeta_{E,q}^{(h)}(s, x).$$

By substituting $s = -n$, $n \in \mathbb{Z}_+$ and using the Cauchy residue theorem, we obtain another proof of Theorem 6.

By using (2.7) we have with $\chi(a + d) = \chi(a)$, where d is an odd positive integer,

$$2\sum_{m=0}^{\infty} (-1)^m \chi(m) \mathrm{e}^{tm} q^{hm} = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h)} \frac{t^n}{n!}. \tag{3.4}$$

By applying the derivative operator $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,|_{t=0}$ to the above equation, we have

$$E_{k,\chi,q}^{(h)} = 2\sum_{m=0}^{\infty} (-1)^m q^{hm} \chi(m) m^k. \tag{3.5}$$

By using (3.5), we define new extensions of (h, q)-Euler l-functions as follows:

Definition 2. Let $s \in \mathbb{C}$. We define

$$l_{E,q}^{(h)}(s, \chi) = 2\sum_{m=1}^{\infty} \frac{(-1)^m q^{hm} \chi(m)}{m^s}. \tag{3.6}$$

$l_{E,q}^{(h)}(s, x)$ is an analytic function on the whole complex s-plane. From (3.5) and (3.6), we arrive at the following theorem.

Theorem 7. *For $k \in \mathbb{Z}_+$, we have*

$$l_{E,q}^{(h)}(-k, \chi) = E_{k,\chi,q}^{(h)}. \tag{3.7}$$

Remark 2.

$$\frac{1}{\Gamma(s)} \int_0^{\infty} F_{q,\chi}^h(-t) t^{s-1} \mathrm{d}t = l_{E,q}^{(h)}(s, \chi).$$

By using the Cauchy residue theorem we obtain another proof of Theorem 7.

By substituting $m = a + dn$, $a = 1, \ldots, d$, d is odd, $n = 0, 1, 2, \ldots$, into (3.6), we have

$$\begin{aligned} l_{E,q}^{(h)}(s, \chi) &= 2\sum_{a=1}^{d} \sum_{m=0}^{\infty} \frac{(-1)^{a+dm} q^{dhm+ha} \chi(dm + a)}{(a + dm)^s} \\ &= d^{-s} \sum_{a=1}^{d} (-1)^a \chi(a) q^{ha} \sum_{m=0}^{\infty} \frac{2(-1)^m q^{dhm}}{\left(m + \frac{a}{d}\right)^s} \\ &= d^{-s} \sum_{a=1}^{d} (-1)^a \chi(a) q^{ha} \zeta_{E,q^d}^{(h)}\left(s, \frac{a}{d}\right). \end{aligned}$$

By substituting $s = -n$, $n \in \mathbb{Z}_+$, into the above equation, we have

$$\begin{aligned} l_{E,q}^{(h)}(-n,\chi) &= d^n \sum_{a=1}^{d} (-1)^a \chi(a) q^{ha} \zeta_{E,q^d}^{(h)}\left(-n, \frac{a}{d}\right) \\ &= d^n \sum_{a=1}^{d} (-1)^a \chi(a) q^{ha} E_{n,q^d}^{(h)}\left(\frac{a}{d}\right). \end{aligned} \tag{3.8}$$

By using (2.4), (3.7) and (3.8), we obtained the following theorem.

Theorem 8 (*Distribution Relations for the Generalized* (h, q)*-Extension of Euler Numbers*). *Let d be an odd integer. Then we have*

$$E_{n,\chi,q}^{(h)} = \sum_{a=1}^{d} \sum_{k=0}^{n} \binom{n}{k} (-1)^a \chi(a) q^{ha} a^{n-k} d^k E_{k,q^d}^{(h)}.$$

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References

[1] M. Cenkci, M. Can, V. Kurt, p-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers, Adv. Stud. Contemp. Math. 9 (2) (2004) 203–216.
[2] L.C. Jang, S.-D. Kim, D.-W. Park, Y.S. Ro, A note on Euler number and polynomials, J. Inequal. Appl. (2006) 5 pp. Art. ID 34602.
[3] T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 9 (2002) 288–299.
[4] T. Kim, On Euler–Barnes multiple zeta functions, Russ. J. Math. Phys. 10 (3) (2003) 261–267.
[5] T. Kim, On the analogs of Euler numbers and polynomials associated with p-adic q-integral on Z_p at $q = -1$, J. Math. Anal. Appl. 331 (2007) 779–792.
[6] T. Kim, On the q-extension of Euler and Genocchi numbers, J. Math. Anal. Appl. 326 (2007) 1458–1465.
[7] T. Kim, A new approach to q-zeta function, J. Comput. Anal. Appl. 9 (2007) 395–400.
[8] T. Kim, q-Euler numbers and polynomials associated with p-adic q-integrals, J. Nonlinear Math. Phys. 14 (1) (2007) 15–27.
[9] T. Kim, The modified q-Euler numbers and polynomials. arxivmath.NT/0702523.
[10] T. Kim, A note on p-adic invariant integral in the rings of p-adic integers, Adv. Stud. Contemp. Math. 13 (1) (2006) 95–99.
[11] T. Kim, An invariant p-adic q-integral on $\mathbb{Z}_p$, Appl. Math. Lett., in press (doi:10.1016/j.aml.2006.11.011).
[12] H. Ozden, Y. Simsek, S.-H. Rim, I.N. Cangul, A note on p-adic q-Euler Measure, Adv. Stud. Contemp. Math. 14 (2) (2007) 233–239.
[13] S-H. Rim, T. Kim, A note on q-Euler numbers associated with the basic q-zeta function, Appl. Math. Lett. 20 (4) (2007) 366–369.
[14] M. Schork, Ward's "calculus of sequences", q-calculus and the limit q to -1, Adv. Stud. Contemp. Math. 13 (2) (2006) 131–141.
[15] Y. Simsek, q-analogue of twisted l-series and q-twisted Euler numbers, J. Number Theory 110 (2) (2005) 267–278.
[16] Y. Simsek, Twisted (h, q)-Bernoulli numbers and polynomials related to twisted (h, q)-zeta function and L-function, J. Math. Anal. Appl. 324 (2006) 790–804.
[17] H.M. Srivastava, T. Kim, Y. Simsek, q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series, Russ. J. Math. Phys. 12 (2) (2005) 241–268.