Single-valued extension property at the points of the approximate point spectrum

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Abstract

A localized version of the single-valued extension property is studied at the points which are not limit points of the approximate point spectrum, as well as of the surjectivity spectrum. In particular, we shall characterize the single-valued extension property at a point \(\lambda_0 \in \mathbb{C}\) in the case that \(\lambda_0 I - T\) is of Kato type. From this characterizations we shall deduce several results on cluster points of some distinguished parts of the spectrum.

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1. Introduction

Let \(T \in L(X)\) denote a bounded operator on an infinite-dimensional complex Banach space \(X\). Recall that \(T \in L(X)\) is said to be bounded below if \(T\) is injective and has closed range \(T(X)\). The classical approximate point spectrum \(\sigma_{\text{ap}}(T)\) is defined as the set of all
\( \lambda \in \mathbb{C} \) for which \( \lambda I - T \) is not bounded below. Note that if \( T \) is bounded below then \( T \in \Phi_+(X) \), where
\[
\Phi_+(X) := \{ T \in L(X): \alpha(T) := \dim \ker T < \infty \text{ and } T(X) \text{ is closed} \},
\]
is the class of all \textit{upper semi-Fredholm} operators. The \textit{surjectivity spectrum} \( \sigma_{su}(T) \) is defined as the set of all \( \lambda \in \mathbb{C} \) for which \( \lambda I - T \) is not surjective. Note that if \( T \) is surjective then \( T \in \Phi_-(X) \) where
\[
\Phi_-(X) := \{ T \in L(X): \beta(T) := \text{codim} T(X) < \infty \}
\]
is the class of all \textit{lower semi-Fredholm} operators. The class of all semi-Fredholm operators is defined by \( \Phi^\pm(X) := \Phi_+(X) \cup \Phi_-(X) \), while the class of all Fredholm operators is defined by \( \Phi(X) := \Phi^+_\pm(X) \). Note that if \( T \in \Phi_\pm(X) \) then \( T(X) \) is closed. For an arbitrary operator \( T \in L(X) \) the \textit{approximate point spectrum} and the \textit{surjectivity spectrum} are dual to each other, in the sense that \( \sigma_{ap}(T) = \sigma_{su}(T^*) \) and \( \sigma_{su}(T) = \sigma_{ap}(T^*) \), see [9, Proposition 1.3.1]. Let \( p := p(T) \) denote the \textit{ascent} of an operator \( T \), i.e., the smallest non-negative integer \( p \) such that \( \ker T^p = \ker T^{p+1}(X) \). If such integer does not exist we put \( p(T) = \infty \). Analogously, let \( q := q(T) \) denote the \textit{descent} of an operator \( T \), i.e., the smallest non-negative integer \( q \) such that \( T(X) = T^{q+1}(X) \), if such integer does not exist we put \( q(T) = \infty \) exactly whenever \( \lambda_0 \) is a pole of the resolvent \( R(\lambda, T) := (\lambda I - T)^{-1} \) [7, Proposition 50.2].

The classes of operators defined above motivate the definition of several spectra. The \textit{upper semi-Fredholm spectrum} is defined by
\[
\sigma_{uf}(T) := \{ \lambda \in \mathbb{C}: \lambda I - T \notin \Phi_+(X) \},
\]
the \textit{lower semi-Fredholm spectrum} is defined by
\[
\sigma_{lf}(T) := \{ \lambda \in \mathbb{C}: \lambda I - T \notin \Phi_-(X) \},
\]
while the \textit{semi-Fredholm spectrum} and the \textit{Fredholm spectrum} are defined, respectively, by
\[
\sigma_{sf}(T) := \{ \lambda \in \mathbb{C}: \lambda I - T \notin \Phi_\pm(X) \}
\]
and
\[
\sigma_{f}(T) := \{ \lambda \in \mathbb{C}: \lambda I - T \notin \Phi(X) \}.
\]
Clearly,
\[
\sigma_{sf}(T) = \sigma_{sf}(T) \cap \sigma_{lf}(T), \quad \sigma_{f}(T) = \sigma_{sf}(T) \cup \sigma_{lf}(T).
\]

**Definition 1.1.** Let \( X \) be a complex Banach space and \( T \in L(X) \). The operator \( T \) is said to have the \textit{single-valued extension property} at \( \lambda_0 \in \mathbb{C} \) (SVEP at \( \lambda_0 \) for brevity), if for every open disc \( D_{\lambda_0} \) centered at \( \lambda_0 \) the only analytic function \( f : D_{\lambda_0} \to X \) which satisfies the equation
\[
(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in D_{\lambda_0}
\]
is the function \( f \equiv 0 \).

An operator \( T \in L(X) \) is said to have the SVEP if \( T \) has the SVEP at every point \( \lambda \in \mathbb{C} \).
Trivially, an operator $T \in L(X)$ has the SVEP at every point of the resolvent $\rho(T) := C \setminus \sigma(T)$. Moreover, from the identity theorem for analytic function it easily follows that $T \in L(X)$ has the SVEP at every point of the boundary $\partial \sigma(T)$ of the spectrum. In particular, $T$ has the SVEP at every isolated point of the spectrum. Hence, we have the implication

$$
\sigma(T) \text{ does not cluster at } \lambda_o \Rightarrow T \text{ has the SVEP at } \lambda_o.
$$

(2)

We have analogous statements if, instead of $\sigma(T)$, we consider some distinguished parts of the spectrum, for instance,

$$
\sigma_{ap}(T) \text{ does not cluster at } \lambda_o \Rightarrow T \text{ has the SVEP at } \lambda_o,
$$

(3)

and

$$
\sigma_{su}(T) \text{ does not cluster at } \lambda_o \Rightarrow T^* \text{ has the SVEP at } \lambda_o.
$$

(4)

Indeed, if $\sigma_{ap}(T)$ does not cluster at $\lambda_o$, then there is an open disc $U_{\lambda_o}$ centered at $\lambda_o$ such that $\lambda I - T$ is injective for every $\lambda \in U_{\lambda_o}$, $\lambda \neq \lambda_o$. Let $f : D_{\lambda_o} \rightarrow X$ be an analytic function defined on another open disc $D_{\lambda_o}$ centered at $\lambda_o$ for which the equation $(\lambda I - T) f(\lambda) = 0$ holds for every $\lambda \in D_{\lambda_o}$. Obviously, we may assume that $D_{\lambda_o} \subseteq U_{\lambda_o}$. Then $f(\lambda) \in \ker(\lambda I - T) = \{0\}$ for every $\lambda \in D_{\lambda_o}$, $\lambda \neq \lambda_o$, thus $f(\lambda) = 0$ for every $\lambda \in D_{\lambda_o}$, $\lambda \neq \lambda_o$, and from the continuity of $f$ at $\lambda_o$, we conclude that $f(\lambda_o) = 0$. Hence $f \equiv 0$ in $D_{\lambda_o}$ and therefore $T$ has the SVEP at $\lambda_o$. The second implication is an immediate consequence of the equality $\sigma_{su}(T) = \sigma_{ap}(T^*)$.

Note that none of the implications (2)–(4) may be reversed. Indeed, the boundary $\partial \sigma(T)$ of $\sigma(T)$ is contained in $\sigma_{ap}(T)$, as well as in $\sigma_{su}(T)$, see [9, Proposition 3.1.6]. Consequently, if $\lambda_o$ is a non-isolated boundary point of $\sigma(T)$ then $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ cluster at $\lambda_o$, but, as observed before, $T$ and $T^*$ have the SVEP at $\lambda_o$. An example of operator $T$ having the SVEP and such that every spectral point is limit of points of $\sigma_{ap}(T)$ may be found among the unilateral weighted right shift operators. Indeed, there exist unilateral weighted right shift operators $T$ on $\ell^p(\mathbb{N})$ for which $\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T)$ and $\sigma(T)$ is a closed ball centered at 0 with radius $r > 0$, see the remarks after Theorem 6.1 of [5].

Recall that $T \in L(X)$ is said to be decomposable if for every open cover $\{U_1, U_2\}$ of $C$ there exist $T$-invariant closed linear subspaces $X_1$ and $X_2$ of $X$ for which $X = X_1 + X_2$, $\sigma(T \mid X_1) \subseteq U_1$ and $\sigma(T \mid X_2) \subseteq U_2$.

The basic role of SVEP arises in local spectral theory, since a decomposable operators $T$ enjoy this property, as well as its adjoint $T^*$. Indeed, the decomposability of an operator may be viewed as the union of two properties, the so-called Bishop’s property ($\beta$), which implies the SVEP, and the property ($\delta$), which implies the SVEP for the adjoint $T^*$ of $T$, see Proposition 1.2.19, Theorems 2.5.18 and 2.5.19 of Laursen and Neumann [9]. The notion of the localized SVEP at a point dates back to Finch [6] and it has been pursued further in the most recent papers [1–3,5,10]. In particular, it has been shown that if $\lambda_o I - T$ is of Kato type then the SVEP at a point $\lambda_o \in C$ is equivalent to a variety of conditions that involve some kernel-type and range-type subspaces of $\lambda_o I - T$, as the hyperrange and the generalized kernel [2], as well as the analytical core and quasi-nilpotent part [3]. In this paper we shall give further characterizations of the SVEP at $\lambda_o$, always in the case that
\( \lambda_o I - T \) is of Kato type. Precisely, we shall see that if \( \lambda_o I - T \) is of Kato type then \( T \) has the SVEP at \( \lambda_o \) if and only if \( \sigma_{ap}(T) \) does not cluster at \( \lambda_o \). A dual result shows that, always if \( \lambda_o I - T \) is of Kato type, \( T^* \) has the SVEP at \( \lambda_o \) precisely when \( \sigma_{su}(T) \) does not cluster at \( \lambda_o \). In particular, these characterizations of the SVEP at \( \lambda_o \) hold when \( \lambda_o I - T \) is a semi-Fredholm operator. As consequence we shall deduce several results on cluster points of distinguished parts of the spectrum. These results are applied to some concrete cases, as isometries and Cesáro operators.

### 2. Results

We focus our attention to the implications (3) and (4) in the case that \( T \in L(X) \) admits an important decomposition property.

**Definition 2.1.** An operator \( T \in L(X) \), \( X \) a Banach space, is said to be semi-regular if \( T(X) \) is closed and \( \ker T \subseteq T^\infty(X) \). An operator \( T \in L(X) \) is said to admit a generalized Kato decomposition, abbreviated GKD, if there exists a pair of \( T \)-invariant closed subspaces \( (M,N) \) such that \( X = M \oplus N \), the restriction \( T \mid M \) is semi-regular and \( T \mid N \) is quasi-nilpotent.

A relevant case is obtained if we assume in the definition above that \( T \mid N \) is nilpotent. In this case \( T \) is said to be of Kato type, see [4]. A very important class of operators of Kato type is the class of semi-Fredholm operators, see [8, Theorem 4] or West [13, Proposition 2.5]. Obviously, every semi-regular operator has the GKD \( M = X \) and \( N = \{0\} \), thus is of Kato type. Note that a semi-Fredholm operator \( T \) is semi-regular if and only if its jump \( j(T) \) is zero, see [13, Proposition 2.2].

For every \( T \in L(X) \), let \( T^\infty(X) \) denote the hyperrange of \( T \) defined by

\[
T^\infty(X) = \bigcap_{n=1}^{\infty} T^n(X).
\]

Note that \( T^\infty(X) \) is \( T \)-invariant. Furthermore, if \( T \) is of Kato type then \( T^\infty(X) \) is closed and \( T(T^\infty(X)) = T^\infty(X) \), see Theorems 2.3 and 2.4 of [2].

The SVEP at a point \( \lambda_o \) has been characterized in several ways in the case that \( \lambda_o I - T \) is of Kato type, see [2,3,5]. The following result gives a further characterization.

**Theorem 2.2.** Suppose that \( \lambda_o I - T \), \( X \) a Banach space, is of Kato type. Then the following statements are equivalent:

(i) \( T \) has the SVEP at \( \lambda_o \).
(ii) \( p(\lambda_o I - T) < \infty \).
(iii) \( \sigma_{ap}(T) \) does not cluster at \( \lambda_o \).

**Proof.** The equivalence (i) \( \Leftrightarrow \) (ii) has been proved in [1, Theorem 2.6]. To prove the equivalence (i) \( \Leftrightarrow \) (iii) we need only to prove the implication (i) \( \Rightarrow \) (iii). We may suppose
that \( \lambda_0 = 0 \). Assume that \( T \) has the SVEP at 0. First we show that there exists \( \epsilon > 0 \) such that \( \lambda I - T \) has closed range for every \( 0 < |\lambda| < \epsilon \). Indeed, let \((M, N)\) be a GKD for \( T \) such that \( T \mid N \) is nilpotent. From the nilpotency of \( T \mid N \) we know that \( \lambda I - T \mid N \) is bijective for every \( \lambda \neq 0 \), thus \( N = (\lambda I - T)(N) \) for every \( \lambda \neq 0 \), and therefore

\[
(\lambda I - T)(X) = (\lambda I - T)(M) \oplus (\lambda I - T)(N)
\]

\[
= (\lambda I - T)(M) \oplus N \quad \text{for every } \lambda \neq 0.
\]

From assumption \( T \mid M \) is semi-regular, so there exists \( \epsilon > 0 \) for which \( (\lambda I - T) \mid M \) is semi-regular for every \( |\lambda| < \epsilon \), see [9, Proposition 3.1.9], and hence \( (\lambda I - T)(M) \) is closed for every \( |\lambda| < \epsilon \). The following standard argument shows that \( (\lambda I - T)(X) \) is closed for every \( 0 < |\lambda| < \epsilon \). Consider the Banach space \( M \times N \) provided with the canonical norm

\[
\| (x, y) \| := \| x \| + \| y \|, \quad x \in M, \ y \in N,
\]

and let

\[
\Psi : M \times N \to M \oplus N = X
\]

denote the topological isomorphism defined by \( \Psi(x, y) := x + y \) for every \( x \in M \) and \( y \in N \). Then, for every \( 0 < |\lambda| < \epsilon \), the set

\[
\Psi[(\lambda I - T)(M) \times N] = (\lambda I - T)(M) \oplus N = (\lambda I - T)(X)
\]

is closed, since the product \( (\lambda I - T)(M) \times N \) is closed in \( M \times N \). From this we conclude that for every \( 0 < |\lambda| < \epsilon \) then \( \lambda \in \sigma(T) \) if and only if \( \lambda \) is an eigenvalue of \( T \).

Now, it is easy to see that

\[
\ker(\lambda I - T) \subseteq T^\infty(X) \quad \text{for every } \lambda \neq 0,
\]

so every non-zero eigenvalue of \( T \) belongs to the spectrum of the restriction \( T \mid T^\infty(X) \). Finally, assume that 0 is a cluster point of \( \sigma(T) \). Let \( (\lambda_n)_{n \in \mathbb{N}} \) be a sequence of non-zero eigenvalues which converges to 0. Then \( \lambda_n \in \sigma(T) \mid T^\infty(X) \) for every \( n \in \mathbb{N} \) and hence \( 0 \in \sigma(T \mid T^\infty(X)) \), since the spectrum of an operator is closed. But \( T \) has the SVEP at 0, thus \( T \mid M \) is injective, see [2, Theorem 2.5]. By Theorems 2.3 and 2.4 of [2] we have

\[
\{0\} = \ker T \mid M = \ker T \cap T^\infty(X),
\]

so also the restriction \( T \mid T^\infty(X) \) is injective. On the other hand, from the equality \( T(T^\infty(X)) = T^\infty(X) \) we know that \( T \mid T^\infty(X) \) is surjective, so \( 0 \notin \sigma(T \mid T^\infty(X)) \); a contradiction. \( \Box \)

For a bounded operator \( T \in L(X) \) let us consider the following parts of the spectrum:

\[
\sigma_k(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular} \},
\]

\[
\sigma_k(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not of Kato type} \}.
\]

It is known that the two sets \( \sigma_k(T), \sigma_k(T) \) are closed, for the first set see [9, Proposition 3.1.9], for second set see [4, Corollary 1]. The set \( \sigma_k(T) \) is known in literature as the \textit{Kato spectrum}. Clearly,

\[
\sigma_k(T) \subseteq \sigma_k(T) \subseteq \sigma_k(T)
\]

(5)
and
\[ \sigma_{kt}(T) \subseteq \sigma_{sf}(T) \subseteq \sigma_{ap}(T). \] (6)

The result of Theorem 2.2 is quite useful for establishing the membership of cluster points of some distinguished parts of the spectrum to the spectrum \( \sigma_{kt}(T) \). A first application is given from the following result, which improves a classical Putnam theorem about the non-isolated boundary points of the spectrum as a subset of the Fredholm spectrum.

**Corollary 2.3.** For every operator \( T \in L(X) \) on a Banach space \( X \), every non-isolated boundary point of \( \sigma(T) \) belongs to \( \sigma_{kt}(T) \). In particular, every non-isolated boundary point of \( \sigma(T) \) belongs to the Fredholm spectrum \( \sigma_f(T) \).

**Proof.** If \( \lambda_o \in \partial \sigma(T) \) is non-isolated in \( \sigma(T) \) then \( \sigma_{ap}(T) \) clusters at \( \lambda_o \) since, as observed before, \( \partial \sigma(T) \subseteq \sigma_{ap}(T) \). Since \( T \) has the SVEP at every point of \( \partial \sigma(T) \) then, by Theorem 2.2, \( \lambda_oI - T \) is not of Kato type. The last assertion is obvious, since \( \sigma_{kt}(T) \subseteq \sigma_f(T) \).

**Corollary 2.4.** Suppose that \( T \in L(X) \), \( X \) a Banach space, has the SVEP. Then all cluster points of \( \sigma_{ap}(T) \) belong to \( \sigma_{kt}(T) \).

**Proof.** Suppose that \( \lambda_o \notin \sigma_{kt}(T) \). Since \( T \) has the SVEP, and in particular has the SVEP at \( \lambda_o \), then \( \sigma_{ap}(T) \) does not cluster at \( \lambda_o \), by Theorem 2.2.

Note that since \( \sigma_{kt}(T) \subseteq \sigma_{sf}(T) \subseteq \sigma_{f}(T) \), Corollaries 2.3 and 2.4 improve Proposition 3.7.8 and part (b) of Proposition 3.7.6 of [9], respectively.

The next result is dual, in a sense, to Theorem 2.2.

**Theorem 2.5.** Suppose that \( \lambda_oI - T, X \) a Banach space, is of Kato type. Then the following properties are equivalent:

(i) \( T^* \) has the SVEP at \( \lambda_o \).
(ii) \( q(\lambda_oI - T) < \infty \).
(iii) \( \sigma_{su}(T) \) does not cluster at \( \lambda_o \).

**Proof.** The equivalence (i) \( \Leftrightarrow \) (ii) has been proved in [1]. The equivalence (i) \( \Leftrightarrow \) (iii) immediately follows from Theorem 2.2, once observed that \( \sigma_{ap}(T^*) = \sigma_{sa}(T) \).

**Corollary 2.6.** Suppose that for \( T \in L(X) \), \( X \) a Banach space, \( T^* \) has the SVEP. Then all cluster points of \( \sigma_{sa}(T) \) belong to \( \sigma_{kt}(T) \).

**Proof.** Suppose that \( \lambda_o \notin \sigma_{kt}(T) \). Since \( T^* \) has the SVEP at \( \lambda_o \), then \( \sigma_{sa}(T) \) does not cluster at \( \lambda_o \), by Theorem 2.5.

Again, Corollary 2.4 improves part (c) of Proposition 3.7.6 of [9].
Corollary 2.7. Assume that both \( T \) and \( T^* \) have the SVEP. If \( \lambda_o \in C \) is a cluster point of \( \sigma(T) \) then \( \lambda \in \sigma_{kt}(T) \). In particular, this holds if \( T \) is decomposable.

Proof. This is obvious from Corollaries 2.4 and 2.6, since \( \sigma(T) = \sigma_{ap}(T) \cup \sigma_{su}(T) \). \( \square \)

It should be noted that if \( T \) and \( T^* \) have the SVEP then \( \sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T) \), see Proposition 1.3.2 of [9].

The two spectra \( \sigma_{kt}(T) \) and \( \sigma_{kt}(T) \) are subsets of \( \sigma_{ap}(T) \), thus one may ask if \( T \) has the SVEP at a point \( \lambda_o \) whenever one of these spectra does not cluster at \( \lambda_o \). Generally this is not true. To see this it suffices to consider the case that \( \sigma_{kt}(T) \) does not cluster at \( \lambda_o \). Let \( T \in L(X) \) be any non-injective semi-regular operator. Then 0 is a point of the Kato resolvent \( \rho_k(T) := C \setminus \sigma_k(T) \), which is an open set, see [9, Proposition 3.1.9], so that \( \sigma_k(T) \), and hence also \( \sigma_{kt}(T) \), does not cluster at 0. On the other hand, since \( T \) is not injective then \( T \) does not have the SVEP at 0, see [1, Theorem 2.11].

Theorem 2.8. Suppose that for \( T \in L(X) \), \( X \) a Banach space, the Kato spectrum \( \sigma_k(T) \) clusters at \( \lambda_o \). Then \( \lambda_o \in \sigma_{sf}(T) \).

Proof. Suppose that \( \lambda_oI - T \in \Phi_{\pm}(X) \) is not semi-regular. Then \( \lambda_oI - T \) has jump \( j(\lambda_oI - T) > 0 \). [13, Proposition 2.2]. It is well known that the semi-Fredholm region \( \rho_{sf}(T) := C \setminus \sigma_{sf}(T) \) is open, so it may be decomposed into open maximal connected components. Let \( \Omega \) be the component which contains \( \lambda_o \). The set \( \Gamma := \{ \lambda \in \Omega : j(\lambda I - T) > 0 \} \), is discrete [12]. Consequently, there is a punctured disc \( D(\lambda_o) \) centered at \( \lambda_o \) such that the jump \( j(\lambda I - T) \) is 0 for every \( \lambda \in D(\lambda_o) \). This is equivalent to saying that \( D(\lambda_o) \cap \sigma_k(T) = \emptyset \), again by [13, Proposition 2.2]. \( \square \)

From Theorem 2.8 we can obtain other examples of operators for which \( \sigma_k(T) \) does not cluster at \( \lambda_o \in \sigma_k(T) \) and \( T \) does not have the SVEP at \( \lambda_o \). Indeed, Theorem 2.8 shows that if \( \lambda_oI - T \) is semi-Fredholm then \( \sigma_k(T) \) does not cluster at \( \lambda_o \). If, additionally, \( \lambda_oI - T \) has infinite ascent then \( T \) fails to have the SVEP at \( \lambda_o \), by Theorem 2.2.

The next result gives a clear description of the points \( \lambda_o \notin \sigma_k(T) \) which belong to the boundary of \( \sigma(T) \). Recall first that if both ascent and descent of an operator are finite then they are equal [7, Proposition 38.3], and that \( \lambda_o \) is a pole of the resolvent precisely when \( 0 < p(\lambda_oI - T) = q(\lambda_oI - T) < \infty \) [7, Proposition 50.2].

Theorem 2.9. Let \( T \in L(X) \), \( X \) a Banach space, and suppose that \( \lambda_o \in \partial \sigma(T) \). Then

\[ \lambda_oI - T \text{ is of Kato type} \iff \lambda_o \text{ is a pole of the resolvent } R(\lambda, T). \]

Proof. If \( \lambda_oI - T \) is of Kato type then, by Corollary 2.3, \( \lambda_o \) is isolated in \( \sigma(T) \). Moreover, both \( T \) and \( T^* \) have the SVEP at \( \lambda_o \in \partial \sigma(T) = \partial \sigma(T^*) \), thus \( p(\lambda_oI - T) \) and \( q(\lambda_oI - T) \) are finite, by Theorems 2.2 and 2.5. Conversely, assume that \( \lambda_o \) is a pole \( R(\lambda, T) \) or, equivalently, that \( d := p(\lambda_oI - T) = q(\lambda_oI - T) < \infty \). Let \( M := (\lambda_oI - T)^d(X) \) and
N := \ker(\lambda_o I - T)^d. Then \lambda_o I - T \mid M is bijective, and hence semi-regular, by [7, Proposition 38.4], and obviously \lambda_o I - T \mid N is nilpotent. □

All results established above have a number of interesting applications. In the next theorem we consider a situation which occurs in some concrete cases.

**Theorem 2.10.** Let \( T \in L(X) \) be an operator for which \( \sigma_{ap}(T) = \partial \sigma(T) \) and every \( \lambda \in \partial \sigma(T) \) is not isolated in \( \sigma(T) \). Then

\[
\sigma_{sf}(T) = \sigma_{uf}(T) = \sigma_{ap}(T) = \sigma_{kt}(T).
\]  

(7)

**Proof.** \( T \) has the SVEP at every point of the boundary as well as at every point \( \lambda \) which belongs to the remaining part of the spectrum, since \( \sigma_{ap}(T) \) does not cluster at \( \lambda \). Hence \( T \) has the SVEP and hence if \( \lambda I - T \) is semi-Fredholm then \( \alpha(\lambda I - T) \leq \beta(\lambda I - T) \), see Corollary 2.7 of [2]. From this it easily follows that every semi-Fredholm is upper and hence \( \sigma_{sf}(T) = \sigma_{uf}(T) \). On the other hand, by Corollary 2.3 we have \( \sigma_{ap}(T) \subseteq \sigma_{kt}(T) \) so, from the inclusions (5) and (6), we obtain \( \sigma_{kt}(T) = \sigma_{k}(T) = \sigma_{ap}(T) \). □

**Corollary 2.11.** Suppose that \( \sigma_{su}(T) = \partial \sigma(T) \) and every \( \lambda \in \partial \sigma(T) \) is not isolated in \( \sigma(T) \). Then

\[
\sigma_{sf}(T) = \sigma_{lf}(T) = \sigma_{k}(T) = \sigma_{su}(T) = \sigma_{kt}(T).
\]  

(8)

**Proof.** From assumption we obtain \( \sigma_{su}(T) = \sigma_{ap}(T^*) = \partial \sigma(T) = \partial \sigma(T^*) \), so we can apply Theorem 2.10. The equalities (8) then follows once observed that \( \sigma_{lf}(T) = \sigma_{uf}(T^*) \) and \( \sigma_{kt}(T) = \sigma_{kt}(T^*) \). □

Theorem 2.10 applies to arbitrary non-invertible isometries. To see that, let \( r(T) \) denote the spectral radius of an arbitrary operator \( T \in L(X) \) and let \( k(T) \) denote the lower bound

\[
k(T) := \lim_{n \to \infty} k(T^n)^{1/n} = \sup_{n \in \mathbb{N}} k(T^n)^{1/n}.
\]

Generally, \( i(T) \leq r(T) \). From Proposition 1.6.2 of [9] the approximate point spectrum \( \sigma_{ap}(T) \) of \( T \) is contained in the possibly degenerated annulus \( \{ \lambda \in \mathbb{C} : i(T) \leq |\lambda| \leq r(T) \} \). From this it follows that the condition \( i(T) = r(T) \) entails that \( \sigma_{ap}(T) \subseteq \partial \sigma(T) \). If we suppose that \( T \) is non-invertible then

\[
\sigma(T) = \mathfrak{D}(0, r(T)) \quad \text{and} \quad \sigma_{ap}(T) = \partial \mathfrak{D}(0, r(T)),
\]

where \( \mathfrak{D}(0, r(T)) \) denotes the closed unit disc of \( \mathbb{C} \) centered at 0 with radius \( r(T) \), see Proposition 1.3.2 of [9]. If \( T \) is an isometry then \( i(T) = r(T) = 1 \), so that Theorem 2.10 applies to non-invertible isometries.

Theorem 2.10 also applies to the Cesàro operator \( C_p \) defined on the classical Hardy space \( H_p(\mathfrak{D}) \), \( \mathfrak{D} \) the open unit disc and \( 1 < p < \infty \), by

\[
(C_p f)(\lambda) := \frac{1}{\lambda} \int_0^\lambda \frac{f(\mu)}{1 - \mu} \, d\mu, \quad \text{for all } f \in H_p(\mathfrak{D}) \text{ and } \lambda \in \mathfrak{D}.
\]
As noted in [11], the spectrum of the operator $C_p$ is the closed disc $\Gamma_p$ centered at $p/2$ with radius $p/2$, and
\[ \sigma_f(C_p) \subseteq \sigma_{ap}(C_p) = \partial \Gamma_p. \]
In this case we can say more than the results of Theorem 2.10. Indeed, from Corollary 2.3 we have
\[ \sigma_{kt}(C_p) = \sigma_{ap}(C_p) = \partial \Gamma_p \]
and this implies that $\sigma_f(C_p) = \partial \Gamma_p$, since the inclusion $\sigma_{kt}(T) \subseteq \sigma_f(T)$ holds for every bounded operator $T$.

References