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Parabolic and quasiparabolic subgroups of free partially commutative groups ☆

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Abstract

Let Γ be a finite graph and G be the corresponding free partially commutative group. In this paper we study subgroups generated by vertices of the graph Γ, which we call canonical parabolic subgroups. A natural extension of the definition leads to canonical quasiparabolic subgroups. It is shown that the centralisers of subsets of G are the conjugates of canonical quasiparabolic centralisers satisfying certain graph theoretic conditions.

Keywords: Partially commutative groups; Right-angled Artin groups; Centraliser lattice

1. Preliminaries

Free partially commutative groups arise naturally in many branches of mathematics and computer science and consequently have many aliases: they are known as semifree groups [1,2], graph groups [14,22,24,26,31,33], right-angled Artin groups [4–6,8,10,23,30,35], trace groups

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[13,29], locally free groups [9,28,34] and of course (free) partially commutative groups [3,7,11, 12,15,17,18,21,25,27,32]. We refer the reader to [5,13,21,22] for further references, more comprehensive surveys, introductory material and discussion of the various manifestations of these groups.

In this section we give a brief overview of some definitions and results from [20,21]. We begin with the basic notions of the theory of free partially commutative groups: which, for the sake of brevity we refer to simply as partially commutative groups. Let Γ be a finite, undirected, simple graph. Let $X = V(\Gamma) = \{x_1, \dots, x_n\}$ be the set of vertices of Γ and let $F(X)$ be the free group on X. Let

$$R = \bigl\{[x_i, x_j] \in F(X) \bigm| x_i, x_j \in X \text{ and there is an edge of } \Gamma \text{ joining } x_i \text{ to } x_j\bigr\}.$$

We define the *partially commutative group* with (*commutation*) *graph* Γ to be the group $G(\Gamma)$ with presentation $\langle X \mid R\rangle$. When the underlying graph is clear from the context we write simply G.

Denote by $l(g)$ the minimum of the lengths words that represent the element g. If w is a word representing g and w has length $l(g)$ we call w a *minimal form* for g. When the meaning is clear we shall say that w is a minimal element of G when we mean that w is a minimal form of an element of G. We say that $w \in G$ is *cyclically minimal* if and only if

$$l\bigl(g^{-1}wg\bigr) \geqslant l(w)$$

for every $g \in G$. We write $u \circ w$ to express the fact that $l(uw) = l(u) + l(w)$, where $u, w \in G$. We will need the notions of a divisor and the greatest divisor of a word w with respect to a subset $Y \subseteq X$, defined in [21]. Let u and w be elements of G. We say that u is a *left* (*right*) *divisor* of w if there exists $v \in G$ such that $w = u \circ v$ ($w = v \circ u$). We order the set of all left (right) divisors of a word w as follows. We say that u_2 is greater than u_1 if and only if u_1 left (right) divides u_2. It is shown in [21] that, for any $w \in G$ and $Y \subseteq X$, there exists a unique maximal left divisor of w which belongs to the subgroup $\langle Y\rangle < G$ which is called the *greatest left divisor* $\mathrm{gd}^l_Y(w)$ of w in Y. The *greatest right divisor* of w in Y is defined analogously. We omit the indices when no ambiguity occurs.

The *non-commutation* graph of the partially commutative group $G(\Gamma)$ is the graph Δ, dual to Γ, with vertex set $V(\Delta) = X$ and an edge connecting x_i and x_j if and only if $[x_i, x_j] \neq 1$. The graph Δ is a union of its connected components $\Delta_1, \dots, \Delta_k$ and words that depend on letters from distinct components commute. For any graph Γ, if S is a subset of $V(\Gamma)$ we shall write $\Gamma(S)$ for the full subgraph of Γ with vertices S. Now, if the vertex set of Δ_k is I_k and $\Gamma_k = \Gamma(I_k)$ then $G = G(\Gamma_1) \times \cdots \times G(\Gamma_k)$. For $g \in G$ let $\alpha(g)$ be the set of elements x of X such that $x^{\pm 1}$ occurs in a minimal word w representing g. It is shown in [21] that $\alpha(g)$ is well defined. Now suppose that the full subgraph $\Delta(\alpha(w))$ of Δ with vertices $\alpha(w)$ has connected components $\Delta_1, \dots, \Delta_l$ and let the vertex set of Δ_j be I_j. Then, since $[I_j, I_k] = 1$, we can split w into the product of commuting words, $w = w_1 \circ \cdots \circ w_l$, where $w_j \in G(\Gamma(I_j))$, so $[w_j, w_k] = 1$ for all j, k. If w is cyclically minimal then we call this expression for w a *block-decomposition* of w and say w_j is a *block* of w, for $j = 1, \dots, l$. Thus w itself is a block if and only if $\Delta(\alpha(w))$ is connected. In general let v be an element of G which is not necessarily cyclically minimal. We may write $v = u^{-1} \circ w \circ u$, where w is cyclically minimal and then w has a block-decomposition $w = w_1 \cdots w_l$, say. Then $w_j^u = u^{-1} \circ w_j \circ u$ and we call the expression $v = w_1^u \cdots w_l^u$ the *block-*

decomposition of v and say that w_j^u is a *block* of v, for $j = 1, \ldots, l$. Note that this definition is slightly different from that given in [21].

Let Y and Z be subsets of X. As in [20] we define the *orthogonal complement* of Y *in* Z to be

$$\mathcal{O}^Z(Y) = \{u \in Z \mid d(u, y) \leqslant 1, \text{ for all } y \in Y\}.$$

By convention we set $\mathcal{O}^Z(\emptyset) = Z$. If $Z = X$ we call $\mathcal{O}^X(Y)$ the orthogonal complement of Y, and if no ambiguity arises then we write $Y^\perp$ instead of $\mathcal{O}^X(Y)$ and $x^\perp$ for $\{x\}^\perp$. Let $\mathrm{CS}(\Gamma)$ be the set of all subsets Z of X of the form $Y^\perp$ for some $Y \subseteq X$. The set $\mathrm{CS}(\Gamma)$ is shown in [20] to be a lattice, the *lattice of closed sets* of Γ.

The *centraliser* of a subset S of G is

$$C(S) = C_G(S) = \{g \in G\colon\ gs = sg, \text{ for all } s \in S\}.$$

The set $\mathfrak{C}(G)$ of centralisers of a group is a lattice. An element $g \in G$ is called a *root element* if g is not a proper power of any element of G. If $h = g^n$, where g is a root element and $n \geqslant 1$, then g is said to be a *root* of h. As shown in [16] every element of the partially commutative group G has a unique root, which we denote $r(g)$. If $w \in G$ define $A(w) = \langle Y\rangle = G(Y)$, where $Y = \alpha(w)^\perp \setminus \alpha(w)$. Let w be a cyclically minimal element of G with block-decomposition $w = w_1 \cdots w_k$ and let $v_i = r(w_i)$. Then, from [16, Theorem 3.10],

$$C(w) = \langle v_1\rangle \times \cdots \times \langle v_k\rangle \times A(w). \tag{1.1}$$

We shall use [19, Corollary 2.4] several times in what follows, so for ease of reference we state it here: first recalling the necessary notation. It follows from [19, Lemma 2.2] that if g is a cyclically minimal element of G and $g = u \circ v$ then vu is cyclically minimal. For a cyclically minimal element $g \in G$ we define $\tilde{g} = \{h \in G \mid h = vu, \text{ for some } u, v \text{ such that } g = u \circ v\}$. (We allow $u = 1$, $v = g$ so that $g \in \tilde{g}$.)

Lemma 1.1. *(See [19, Corollary 2.4].) Let w, g be (minimal forms of) elements of G and $w = u^{-1} \circ v \circ u$, where v is cyclically minimal. Then there exist minimal forms a, b, c, d_1, d_2 and e such that $g = a \circ b \circ c \circ d_2$, $u = d_1 \circ a^{-1}$, $d = d_1 \circ d_2$, $w^g = d^{-1} \circ e \circ d$, $\tilde{e} = \tilde{v}$, $e = v^b$, $\alpha(b) \subseteq \alpha(v)$ and $[\alpha(b \circ c), \alpha(d_1)] = [\alpha(c), \alpha(v)] = 1$.*

Figure 1 expresses the conclusion of Lemma 1.1 as a Van Kampen diagram. In this diagram we have assumed that $v = b \circ f$ and so $e = f \circ b$. The regions labelled B are tessellated using relators corresponding to the relation $[\alpha(b \circ c), \alpha(d_1)] = 1$ and the region labelled A with relators corresponding to $[\alpha(c), \alpha(v)] = 1$. Reading anticlockwise from the vertex labelled 0 the boundary label of the exterior region is $g^{-1}wg$ and the label of the interior region (not labelled A or B) is e^d.

2. Parabolic subgroups

2.1. Parabolic and block-homogeneous subgroups

As usual let Γ be a graph with vertices X and $G = G(\Gamma)$. If Y is a subset of X denote by $\Gamma(Y)$ the full subgraph of Γ with vertices Y. Then $G(\Gamma(Y))$ is the free partially commutative

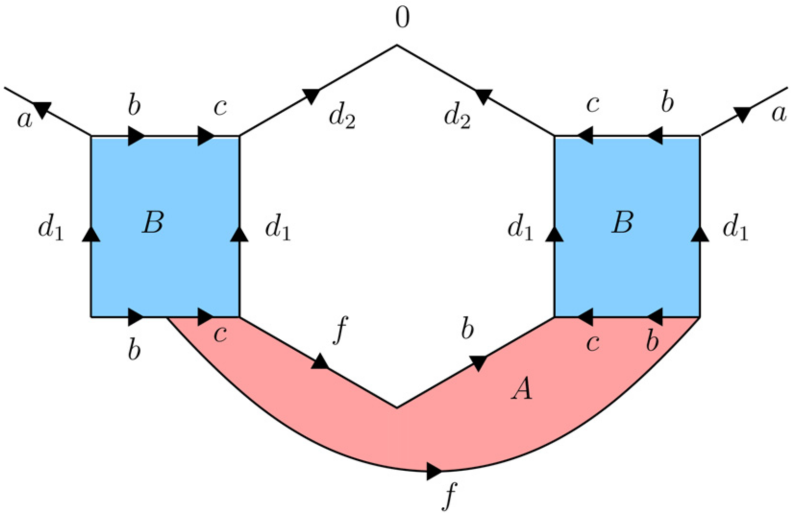

Fig. 1. A Van Kampen diagram for Lemma 1.1.

group with graph $\Gamma(Y)$. As Baudisch [1] observed $G(\Gamma(Y))$ is the subgroup $\langle Y\rangle$ of $G(\Gamma)$ generated by Y. We call $G(\Gamma(Y))$ a *canonical parabolic* subgroup of $G(\Gamma)$ (in keeping with the terminology for analogous subgroups of Coxeter groups) and, when no ambiguity arises, denote it $G(Y)$. Note that such subgroups are called *graphical* in [31], *full* in [23] and *special* in [5]. The elements of Y are termed the *canonical* generators of $G(Y)$.

Definition 2.1. A subgroup P of G is called *parabolic* if it is conjugate to a canonical parabolic subgroup $G(Y)$ for some $Y \subseteq X$. The *rank* of P is the cardinality $|Y|$ and Y is called a set of *canonical generators* for P.

To see that the definition of rank of a parabolic subgroup is well-defined note that if $Y, Z \subseteq X$ and $G(Y) = G(Z)^g$, for some $g \in G$, then we have $y = g^{-1}w_y g$, for some $w_y \in G(Z)$, for all $y \in Y$. It follows, from [21, Lemma 2.5], by counting the exponent sums of letters in a geodesic word representing $g^{-1}wg$, that $y \in \alpha(w_y)$, so $y \in Z$. Hence $Y \subseteq Z$ and similarly $Z \subseteq Y$ so $Y = Z$.

Definition 2.2. A subgroup H is called *block-homogeneous* if, for all $h \in H$, if h has block-decomposition $h = w_1 w_2 \cdots w_k$ then $w_i \in H$, for $i = 1, \dots, k$.

Lemma 2.3. *An intersection of block-homogeneous subgroups is again a block-homogeneous subgroup. If H is block-homogeneous and $g \in G$ then H^g is block-homogeneous. In particular parabolic subgroups are block-homogeneous.*

Proof. The first statement follows directly from the definition. Let H be block-homogeneous and $g \in G$ and let $w^g \in H^g$, where $w \in H$. Write $w = u^{-1} \circ v \circ u$, where v is cyclically reduced and has block-decomposition $v = v_1 \cdots v_k$. Then the blocks of w are v_j^u, so $v_j^u \in H$, for $j = 1, \dots, k$. From Lemma 1.1 there exist a, b, c, d_1, d_2, e such that $g = a \circ b \circ c \circ d_2$, $u = d_1 \circ a^{-1}$, $d = d_1 \circ d_2$, $w^g = d^{-1} \circ e \circ d$, $\tilde{e} = \tilde{v}$, $e = v^b$, $\alpha(b) \subseteq \alpha(v)$ and $[\alpha(b \circ c), \alpha(d_1)] = [\alpha(c), \alpha(v)] = 1$. As $\tilde{e} = \tilde{v}$

it follows that $\Delta(\alpha(e)) = \Delta(\alpha(v))$ so e has block-decomposition $e = e_1 \cdots e_k$, where $\tilde{e_j} = \tilde{v_j}$. Therefore w^g has block-decomposition $w^g = e^d = e_1^d \cdots e_k^d$. Moreover $e = v^b$ so $e_j = v_j^b$. Thus

$$e_j^d = e_j^{cd} = v_j^{bcd} = v_j^{d_1 bcd_2} = v_j^{ug} \in H^g,$$

which implies that H^g is block-homogeneous. It follows from [21, Lemma 2.5] that any canonical parabolic subgroup is block-homogeneous and this gives the final statement. $\square$

2.2. *Intersections of parabolic subgroups*

In this section we show that an intersection of parabolic subgroups is again a parabolic subgroup. To begin with we establish some preliminary results.

Lemma 2.4. *Let $Y, Z \subseteq X$, let $w \in G(Y)$ and let $g \in G(X)$ be such that*

$$\mathrm{gd}_Y^l(g) = \mathrm{gd}_Z^r(g) = 1.$$

1. *If $w^g \in G(Z)$ then $g \in A(w)$ and $w \in G(Y) \cap G(Z) = G(Y \cap Z)$.*
2. *If $Y = Z$ and $g \in C(w)$ then $g \in A(w)$.*

Proof. For 1, in the notation of Lemma 1.1 we have $w = u^{-1} \circ v \circ u$, $w^g = d_2^{-1} \circ d_1^{-1} \circ e \circ d_1 \circ d_2$ and $g = a \circ b \circ c \circ d_2$. Applying the conditions of this lemma we obtain $a = b = d_2 = 1$, $u = d_1$ and $e = v$ so $w^g = w$ and $g = c$. Moreover, from Lemma 1.1 again we obtain $[\alpha(g), \alpha(w)] = 1$. If $x \in \alpha(w) \cap \alpha(g)$ this means that $g = x \circ g'$, with $x \in Y$, contradicting the hypothesis on g. Hence $\alpha(w) \cap \alpha(g) = \emptyset$ and $g \in A(w)$. Statement 2 follows from 1. $\square$

Corollary 2.5. *Let $Y, Z \subseteq X$ and $g \in G$. If $G(Y)^g \subseteq G(Z)$ and $\mathrm{gd}_{Y^\perp}^l(g) = 1$ then $Y \subseteq Z$ and $\alpha(g) \subseteq Z$.*

Proof. Assume first that $\mathrm{gd}_Y^l(g) = \mathrm{gd}_Z^r(g) = 1$. Let $y \in Y$ and $w = y$ in Lemma 2.4; so $y^g \in G(Z)$ implies that $g \in A(y)$ and $y \in Z$. This holds for all $y \in Y$ so we have $Y \subseteq Z$ and $g \in A(Y)$. Hence, in this case, $g = 1$. Now suppose that $g = g_1 \circ d$, where $g_1 = \mathrm{gd}_Y^l(g)$. Then $\mathrm{gd}_{Y^\perp}^l(g) = 1$ implies that $\mathrm{gd}_{Y^\perp}^l(d) = 1$. Now write $d = e \circ g_2$, where $g_2 = \mathrm{gd}_Z^r(d)$. Then $G(Y)^g = G(Y)^d$ and $G(Y)^d \subseteq G(Z)$ implies $G(Y)^e \subseteq G(Z)$. As $\mathrm{gd}_{Y^\perp}^l(d) = 1$ the same is true of e and from the above we conclude that $e = 1$ and that $Y \subseteq Z$. Now $g = g_1 \circ g_2$, where $\alpha(g_1) \subseteq Y \subseteq Z$ and $\alpha(g_2) \subseteq Z$. Thus $\alpha(g) \subseteq Z$, as required. $\square$

Proposition 2.6. *Let P_1 and P_2 be parabolic subgroups. Then $P = P_1 \cap P_2$ is a parabolic subgroup. If $P_1 \nsubseteq P_2$ then the rank of P is strictly smaller than the rank of P_1.*

This lemma follows easily from the next more technical result.

Lemma 2.7. *Let $Y, Z \subset X$ and $g \in G$. Then*

$$G(Y) \cap G(Z)^g = G(Y \cap Z \cap T)^{g_2},$$

where $g = g_1 \circ d \circ g_2$, $\mathrm{gd}_Z^l(d) = \mathrm{gd}_Y^r(d) = 1$, $g_1 \in G(Z)$, $g_2 \in G(Y)$ and $T = \alpha(d)^\perp$.

Derivation of Proposition 2.6 from Lemma 2.7. Let $P_1 = G(Y)^a$ and $P_2 = G(Z)^b$, for some $a, b \in G$. Then $P = (G(Y) \cap G(Z)^{ba^{-1}})^a$, which is parabolic since Lemma 2.7 implies that $G(Y) \cap G(Z)^{ba^{-1}}$ is parabolic. Assume that the rank of P is greater than or equal to the rank of P_1. Let $g = ba^{-1}$. The rank of P is equal to the rank of $G(Y) \cap G(Z)^g$ and, in the notation of Lemma 2.7, $G(Y) \cap G(Z)^g = G(Y \cap Z \cap T)^{g_2}$, where $g = g_1 \circ d \circ g_2$, with $T = \alpha(d)^\perp$, $g_2 \in G(Y)$ and $g_1 \in G(Z)$. Therefore $Y \subseteq Y \cap Z \cap T$ which implies $Y \subseteq Z \cap T$. Thus $G(Y) \subseteq G(Z \cap T) = G(Z \cap T)^d$ so $G(Y) = G(Y)^{g_2} \subseteq G(Z \cap T)^{dg_2} \subseteq G(Z)^{dg_2} = G(Z)^g$. Hence $P_1 \subseteq P_2$. □

Proof of Lemma 2.7. Let $g_1 = \mathrm{gd}^l_Z(g)$ and write $g = g_1 \circ g'$. Let $g_2 = \mathrm{gd}^r_Y(g')$ and write $g' = d \circ g_2$. Then g_1, g_2 and d satisfy the conditions of the lemma. Set $T = \alpha(d)^\perp$. As $G(Y) \cap G(Z)^g = G(Y) \cap G(Z)^{dg_2} = G(Y)^{g_2} \cap G(Z)^{dg_2} = (G(Y) \cap G(Z)^d)^{g_2}$ it suffices to show that $G(Y) \cap G(Z)^d = G(Y \cap Z \cap T)$. If $d = 1$ then $T = X$ and $G(Y) \cap G(Z) = G(Y \cap Z)$, so the result holds. Assume then that $d \neq 1$. Let $p = w^d \in G(Y) \cap G(Z)^d$, with $w \in G(Z)$. Applying Lemma 2.4 to $w^d \in G(Y)$ we have $d \in A(w)$ and $w \in G(Z) \cap G(Y) = G(Y \cap Z)$. Thus $w \in \alpha(d)^\perp = T$ and so $w \in G(Y \cap Z \cap T)$. This shows that $G(Y) \cap G(Z)^d \subseteq G(Y \cap Z \cap T)$ and as the reverse inclusion follows easily the proof is complete. □

Proposition 2.8. *The intersection of parabolic subgroups is a parabolic subgroup and can be obtained as an intersection of a finite number of subgroups from the initial set.*

Proof. In the case of two parabolic subgroups the result follows from Proposition 2.6. Consequently, the statement also holds for a finite family of parabolic subgroups. For the general case we use Proposition 2.6 again, noting that a proper intersection of two parabolic subgroups is a parabolic subgroup of lower rank, and the result follows. □

As a consequence of this proposition we obtain: given two parabolic subgroups P and Q the intersection R of all parabolic subgroups containing P and Q is the unique minimal parabolic subgroup containing both P and Q. Define $P \vee Q = R$ and $P \wedge Q = P \cap Q$.

Corollary 2.9. *The parabolic subgroups of G with the operations $\vee$ and $\wedge$ above form a lattice.*

2.3. The lattice of parabolic centralisers

Let $Z \subseteq X$. Then the subgroup $C_G(Z)^g$ is called a parabolic centraliser. As shown in [20, Lemma 2.3] every parabolic centraliser is a parabolic subgroup: in fact $C_G(Z)^g = G(Z^\perp)^g$. The converse also holds as the following proposition shows.

Proposition 2.10. *A parabolic subgroup $G(Y)^g$, $Y \subseteq X$ is a centraliser if and only if there exists $Z \subseteq X$ so that $Z^\perp = Y$. In this case $G(Y)^g = C_G(Z^g)$.*

Proof. It suffices to prove the proposition for $g = 1$ only. Suppose that there exists such a Z. It is then clear that $G(Y) \subseteq C_G(Z)$. If $w \in G$, w is a reduced word and $\alpha(w) \nsubseteq Y$ then there exists $x \in \alpha(w)$ and $z \in Z$ so that $[x, z] \neq 1$ and consequently, by [21, Lemma 2.4], $[w, z] \neq 1$. Assume further that $G(Y)$ is a centraliser of a set of elements $w_1, \dots, w_k$ written in reduced form. Since $[y, w_i] = 1$, for any $y \in Y$, then by [21, Lemma 2.4] again, $[y, \alpha(w_i)] = 1$. Denote $Z = \bigcup_{i=1}^k \alpha(w_i)$. We have $[y, z] = 1$ for all $z \in Z$ and consequently $Y \subseteq Z^\perp$. Conversely if $x \in Z^\perp$ then $x \in C_G(w_1, \dots, w_k)$ so $x \in Y$. □

We now introduce the structure of a lattice on the set of all parabolic centralisers. As we have shown above the intersection of two parabolic subgroups is a parabolic subgroup. So, we set $P_1 \wedge P_2 = P_1 \cap P_2$. The most obvious way to define $P_1 \vee P_2$ would be as in Section 2.2 above. However, in this case $P_1 \vee P_2$ is not necessarily a centraliser, though it is a parabolic subgroup. For any $S \subseteq G$ we define $\overline{S} = \bigcap\{P\colon\ P$ is a parabolic centraliser and $S \subseteq P\}$. Then $\overline{S}$ is the minimal parabolic centraliser containing S; since intersections of centralisers are centralisers and intersections of parabolic subgroups are parabolic subgroups. We now define $P_1 \vee P_2 = \overline{P_1 \cup P_2}$.

3. Quasiparabolic subgroups

3.1. Preliminaries

As before let Γ be a finite graph with vertex set X and $G = G(\Gamma)$ be the corresponding partially commutative group.

Definition 3.1. Let w be a cyclically minimal root element of G with block-decomposition $w = w_1 \cdots w_k$ and let Z be a subset of X such that $Z \subseteq \alpha(w)^{\perp}$. Then the subgroup $Q = Q(w, Z) = \langle w_1\rangle \times \cdots \times \langle w_k\rangle \times G(Z)$ is called a *canonical quasiparabolic subgroup* of G.

Note that we may choose $w = 1$ so that canonical parabolic subgroups are canonical quasiparabolic subgroups. Given a canonical quasiparabolic subgroup $Q(w, Z)$, with w and Z as above, we may reorder the w_i so that $l(w_i) \geqslant 2$, for $i = 1, \ldots, s$ and $l(w_i) = 1$, for $i = s+1, \ldots, k$. Then setting $w' = w_1 \cdots w_s$ and $Z' = Z \cup \{w_{s+1}, \ldots, w_k\}$ we have $Z' \subseteq \alpha(w)^{\perp}$ and $Q(w, Z) = Q(w', Z')$. This prompts the following definition.

Definition 3.2. We say that a canonical quasiparabolic subgroup $Q = \langle w_1\rangle \times \cdots \times \langle w_k\rangle \times G(Z)$ is written in standard form if $|\alpha(w_i)| \geqslant 2$, $i = 1, \ldots, k$, or $w = 1$.

There are two obvious advantages to the standard form which we record in the following lemma.

Lemma 3.3. *The standard form of a canonical quasiparabolic subgroup Q is unique, up to reordering of blocks of w. If $Q(w, Z)$ is the standard form of Q then $Z \subseteq \alpha(w)^{\perp} \setminus \alpha(w)$.*

Proof. That the standard form is unique follows from uniqueness of roots of elements in partially commutative groups. The second statement follows directly from the definitions. □

Definition 3.4. A subgroup H of G is called quasiparabolic if it is conjugate to a canonical quasiparabolic subgroup.

Let $H = Q^g$ be a quasiparabolic subgroup of G, where Q is the canonical quasiparabolic subgroup of G in standard form

$$Q = \langle w_1\rangle \times \cdots \times \langle w_k\rangle \times G(Z).$$

We call $(|Z|, k)$ the *rank* of H. We use the left lexicographical order on ranks of quasiparabolic subgroups: if H and K are quasiparabolic subgroups of ranks $(|Z_H|, k_H)$ and $(|Z_K|, k_K)$, respectively, then $\operatorname{rank}(H) < \operatorname{rank}(K)$ if $(|Z_H|, k_H)$ precedes $(|Z_K|, k_K)$ in left lexicographical order.

The centraliser $C_G(g)$ of an element $g \in G$ is a typical example of a quasiparabolic subgroup [16]. We shall see below (Theorem 3.12) that the centraliser of any set of elements of the group G is a quasiparabolic subgroup.

Lemma 3.5. *A quasiparabolic subgroup is a block-homogeneous subgroup and consequently any intersection of quasiparabolic subgroups is again block-homogeneous.*

Proof. Let $Q(w, Z)$ be a canonical quasiparabolic subgroup. Since w is a cyclically minimal root element it follows that $Q(w, Z)$ is block-homogeneous. An application of Lemma 2.3 then implies $Q(w, Z)^g$ is also block-homogeneous. $\square$

We shall need the following lemma in Section 4.

Lemma 3.6. *Let $Q_1 = Q(u, Y)$ and $Q_2 = Q(v, Z)$ be canonical quasiparabolic subgroups in standard form and let $g \in G$. If $Q_2^g \subseteq Q_1$, $g \in G(Z^\perp)$ and $\operatorname{gd}_Y^r(g) = 1$ then Q_2^g is a canonical quasiparabolic subgroup.*

Proof. Let u and v have block-decompositions $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$, respectively. As $g \in G(Z^\perp)$ we have

$$Q_2^g = \langle v_1^g \rangle \times \cdots \times \langle v_l^g \rangle \times G(Z).$$

Therefore, for $j = 1, \dots, l$, either $v_j^g = u_i$ for some $i = 1, \dots, k$, or $v_j^g \in G(Y)$. If $v_j^g = u_i$ then v_j^g is a cyclically minimal root element. If, on the other hand, $v_j^g \in G(Y)$ then, from Lemma 1.1, there exist elements b, c, d and e such that $g = b \circ c \circ d$, $v_j^g = d^{-1} \circ e \circ d$ and $e = v_j^b$ is a cyclically minimal root element. As $v_j^g \in G(Y)$ and $\operatorname{gd}_Y^r(g) = 1$ we have $d = 1$ and so $v_j^g = e$ and is a cyclically minimal root element. Therefore Q_2^g is a canonical quasiparabolic subgroup. $\square$

3.2. Intersections of quasiparabolic subgroups

The main result of this section is the following

Theorem 3.7. *An intersection of quasiparabolic subgroups is a quasiparabolic subgroup.*

We shall make use of the following results.

Lemma 3.8. *Let $A = A_1 \times \cdots \times A_l$ and $B = B_1 \times \cdots \times B_k$, A_i, B_j, $i = 1, \dots, l$, $j = 1, \dots, k$ be block-homogeneous subgroups of G and $C = A \cap B$. Then*

$$C = \prod_{\substack{i=1,\dots,l;\\ j=1,\dots,k}} (A_i \cap B_j).$$

Proof. If $C = 1$ then the result is straightforward. Assume then that $C \neq 1$, $w \in C$ and $w \neq 1$ and let $w = w_1 \cdots w_t$ be the block-decomposition of w. Since C is a block-homogeneous subgroup, $w_i \in C$, $i = 1, \ldots, t$. As w_i is a block element we have $w_i \in A_r$ and $w_i \in B_s$ and consequently w_i lies in $\prod_{i,j}(A_i \cap B_j)$. As it is clear that $C \geqslant \prod_{i,j}(A_i \cap B_j)$ this proves the lemma. $\square$

Lemma 3.9. *Let $Z \subseteq X$, $w \in G(Z)$, $g \in G$. Suppose that $u = g^{-1}wg$ is cyclically minimal and $\mathrm{gd}^l_{\alpha(w)}(g) = 1$, then g and w commute.*

Proof. Let $g = d \circ g_1$, where $d = \mathrm{gd}^l_{\alpha(w)^\perp}(g)$. If $g_1 = 1$ then $g \in C(w)$. Suppose $g_1 \neq 1$. Then $\mathrm{gd}^l_{\alpha(w)}(g_1) = 1$ so we write $g_1 = x \circ g_2$, where $x \in (X \cup X^{-1}) \setminus (\alpha(w) \cup \alpha(w)^\perp)$ and thus $u = g_2^{-1}x^{-1}wxg_2$ is written in geodesic form. This is a contradiction for $l(w) < l(u)$. $\square$

Lemma 3.10. *Let*

$$Q_1 = \langle u_1\rangle \times \cdots \times \langle u_l\rangle \times G(Y) \quad \textit{and} \quad Q_2 = \langle v_1\rangle \times \cdots \times \langle v_k\rangle \times G(Z)$$

be canonical quasiparabolic subgroups in standard form and let $g \in G$ such that $\mathrm{gd}^l_Z(g) = 1$. Write $g = d \circ h$, where $h = \mathrm{gd}^r_Y(g)$ and set $T = \alpha(d)^\perp$. Then, after reordering the u_i and v_j if necessary, there exist m, s, t such that

$$Q_1 \cap Q_2^g = \left(\prod_{i=1}^{s}\langle v_i\rangle \times \prod_{i=s+1}^{t}\langle v_i\rangle \times \prod_{j=s+1}^{m}\langle u_i\rangle \times G(Y \cap Z \cap T)\right)^g \tag{3.1}$$

and

(i) $\langle u_i\rangle = \langle v_i\rangle^g$, *for* $i = 1, \ldots, s$;
(ii) $\langle v_i\rangle^g \subseteq G(Y)$, *for* $i = s+1, \ldots, t$; *and*
(iii) $\langle u_i\rangle \subseteq G(Z)$, *for* $i = s+1, \ldots, m$.

Proof. As $Q_1 \cap Q_2^g = (Q_1 \cap Q_2^d)^h$ we may assume that $h = 1$ and $d = g$, so $\mathrm{gd}^r_Y(g) = 1$. As Q_1 and Q_2^g are block-homogeneous we may apply Lemma 3.8 to compute their intersection. Therefore we consider the various possible intersections of factors of Q_1 and Q_2^g.

(i) If $\langle u_i\rangle \cap \langle v_j\rangle^g \neq 1$ then, as u_i and v_j are root elements, $\langle u_i\rangle = \langle v_j\rangle^g$. Suppose that this is the case for $u_1, \ldots, u_s$ and $v_1, \ldots, v_s$ and that $\langle u_i\rangle \cap \langle v_j\rangle^g = 1$, if $i > s$ or $j > s$.
(ii) If $\langle v_j\rangle^g \cap G(Y) \neq 1$ then, since v_j is cyclically minimal, $\langle v_j\rangle^g \subset G(Y)$. This cannot happen if $j \leqslant s$ so suppose it is the case for $v_{s+1}, \ldots, v_t$, and that $\langle v_j\rangle^g \cap G(Y) = 1$, for $j > t$.
(iii) If $\langle u_i\rangle \cap G(Z)^g \neq 1$ then $u_i = w^g$, $w \in G(Z)$ and by Lemma 3.9, w and g commute so does $u_i = w = u_i^g$. This cannot happen if $i \leqslant s$ so suppose that it is the case for $u_{s+1}, \ldots, u_m$, and not for $i > m$.
(iv) Finally, using Lemma 2.7 and the assumption that $\mathrm{gd}^r_Y(g) = \mathrm{gd}^l_Z(g) = 1$, we have $G(Y) \cap G(Z)^g = G(Y \cap Z \cap T) = G(Y \cap Z \cap T)^g$, where $T = \alpha(g)^\perp$.

Combining these intersections (3.1) follows from Lemma 3.8. $\square$

Corollary 3.11. *Let H_1 and H_2 be quasiparabolic subgroups of G. Then $H_1 \cap H_2$ is quasiparabolic. If $H_1 \nsubseteq H_2$ then* $\operatorname{rank}(H_1 \cap H_2) < \operatorname{rank}(H_1)$.

Proof. Let $H_1 = Q_1^f$ and $H_2 = Q_2^g$, where $Q_1 = Q(u, Y)$ and $Q_2 = Q(v, Z)$ are quasiparabolic subgroups in standard form, as in Lemma 3.10. As in the proof of Proposition 2.6 we may assume that $f = 1$ and $\mathrm{gd}_Z^l(g) = 1$ and so Lemma 3.10 implies $H_1 \cap H_2$ is quasiparabolic. If $\operatorname{rank}(H_1 \cap H_2) \geqslant \operatorname{rank}(H_1)$ then $|Y| \leqslant |Y \cap Z \cap T|$ so $Y \subseteq Z \cap T$. In this case (ii) of Lemma 3.10 cannot occur. Therefore, in the notation of Lemma 3.10, $\operatorname{rank}(H_1 \cap H_2) = s + m$. If $\operatorname{rank}(H_1 \cap H_2) \geqslant \operatorname{rank}(H_1)$ then $s + m \geqslant l$ which implies $m = l - s$ and so $u_i \in G(Z)^g$, for $i = s+1, \ldots, l$. As $u_i = v_i^g$, for $i = 1, \ldots, s$ it follows that $H_1 \subseteq H_2$. $\square$

Proof of Theorem 3.7. Given Corollary 3.11 the intersection of an infinite collection of quasiparabolic subgroups is equal to the intersection of a finite sub-collection. From Corollary 3.11 again such an intersection is quasiparabolic and the result follows. $\square$

3.3. A criterion for a subgroup to be a centraliser

Theorem 3.12. *A subgroup H of G is a centraliser if and only if the two following conditions hold.*

1. *H is conjugate to some canonical quasiparabolic subgroup Q.*
2. *If Q is written in standard form*

$$Q = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Y),$$

where $w = w_1 \cdots w_k$ is the block-decomposition of a cyclically minimal element w, w_i is a root element and $|\alpha(w_i)| \geqslant 2$, $i = 1, \ldots, k$, then

$$Y \in \mathrm{CS}(\Gamma) \quad \textit{and} \quad Y \in \mathrm{CS}(\Gamma_w) \quad \textit{where } \Gamma_w = \Gamma\big(\alpha(w)^\perp \setminus \alpha(w)\big).$$

Proof. Let $H = C(u_1, \ldots, u_l)$. Then $H = \bigcap_{i=1}^k C(u_i)$ and we may assume that each u_i is a block root element. Since $C(u_i)$ is a quasiparabolic subgroup, then by Theorem 3.7, H is also a quasiparabolic subgroup and is conjugate to a canonical quasiparabolic subgroup $Q = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Y)$ written in standard form. Thus condition 1 is satisfied.

Then $H = Q^g$ and, after conjugating all the u_i's by g^{-1} we have a centraliser $H^{g^{-1}} = Q$. Thus we may assume that $H = Q$. Let $w = w_1 \cdots w_k$, set $Z = \alpha(w)^\perp \setminus \alpha(w)$ and $T = \bigcup_{i=1}^l \alpha(u_i)$. As w has block-decomposition $w = w_1 \cdots w_k$ we have $C(w) = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Z)$. For all $y \in Y$ we have $y \in C(u_i)$ so $y \in C(\alpha(u_i))$ and thus $y \in C(\alpha(u_i))$ and $Y \subseteq T^\perp$. Conversely if $y \in T^\perp$ then $y \in C(u_i)$ so $y \in Q$ and, by definition of standard form, $y \in Y$. Therefore $Y = T^\perp$. It follows that $Y \in \mathrm{CS}(\Gamma)$ and since by Lemma 3.3 we have $Y \cap \alpha(w) = \emptyset$ we also have $Y \subseteq Z$.

It remains to prove that $Y \in \mathrm{CS}(\Gamma_w) = \mathrm{CS}(\Gamma(Z))$. Set $W = \alpha(w)$. We show that $T \cup Z \subseteq W \cup Z$. Take $t \in T = \bigcup_{i=1}^l \alpha(u_i)$, $t \notin W$, and suppose that $t \in \alpha(u_m)$. Since $w \in C(u_i)$, we have $u_m \in C(w) = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Z)$. Now u_m is a root block element and $C(w)$ is a block-homogeneous subgroup so if $u_m = w_j^{\pm 1}$ for some j then $t \in W = \alpha(w)$, contrary to the choice of t. Therefore $u_m \in G(Z)$, so $t \in Z$ and $T \cup Z \subseteq W \cup Z$, as claimed.

Assume now that $Y \notin \mathrm{CS}(\Gamma(Z))$. In this case there exists an element $z \in Z \setminus Y$ such that $z \in \mathrm{cl}_Z(Y)$. Since $z \notin Y = T^\perp$, there exists u_m such that $[u_m, z] \neq 1$ and so there exists $t \in \alpha(u_m)$

such that $[t, z] \neq 1$. As $[z, W] = 1$, we have $t \notin W$ and since $W \cup Z \supseteq T \cup Z$, we get $t \in Z$. This together with $t \in \alpha(u_m) \subseteq Y^\perp$ implies that $t \in \mathcal{O}^Z(Y)$. Since $[z, t] \neq 1$, we obtain $z \notin \mathrm{cl}_Z(Y)$, in contradiction to the choice of z. Hence $\mathrm{cl}_Z(Y) = Y$ and $Y \in \mathrm{CS}(\Gamma(Z))$.

Conversely, let $Q = \langle w_1\rangle \times \cdots \times \langle w_k\rangle \times G(Y)$ be a canonical quasiparabolic subgroup written in the standard form, $Y \in \mathrm{CS}(X)$ and $Y \in \mathrm{CS}(\Gamma(Z))$, where $Z = \alpha(w)^\perp \setminus \alpha(w)$. We shall prove that $Q = C(w, z_1, \ldots, z_l)$, where $z_1, \ldots, z_l$ are some elements of Z. If $Y = Z$ then $Q = C(w)$. If $Y \subsetneq Z$ then, since $Y = \mathrm{cl}_Z(Y)$, there exist $z_1, u \in Z$ so that $z_1 \in \mathcal{O}^Z(Y)$ and $[z_1, u] \neq 1$. In which case $C(w, z_1) = \langle w_1\rangle \times \cdots \times \langle w_k\rangle \times G(Y_1)$, $Y \subseteq Y_1 \subsetneq Z$ (the latter inclusion is strict for $u \notin Y_1$). If $Y_1 = Y$ then $Q = C(w, z_1)$, otherwise iterating the procedure above, the statement follows. □

A centraliser which is equal to a canonical quasiparabolic subgroup is called a *canonical quasiparabolic centraliser*.

4. Height of the centraliser lattice

In this section we will give a new shorter proof of the main theorem of [19].

Theorem 4.1. *Let $G = G(\Gamma)$ be a free partially commutative group, let $\mathfrak{C}(G)$ be its centraliser lattice and let $L = \mathrm{CS}(\Gamma)$ be the lattice of closed sets of Γ. Then the height $h(\mathfrak{C}(G)) = m$ equals the height $h(L)$ of the lattice of closed sets L.*

In order to prove this theorem we introduce some notation for the various parts of canonical quasiparabolic subgroups.

Definition 4.2. Let $Q = \langle w_1\rangle \times \cdots \times \langle w_k\rangle \times G(Z)$ be a quasiparabolic subgroup in standard form. Define the *block set* of Q to be $\mathcal{B}(Q) = \{w_1, \ldots, w_k\}$ and the *parabolic part* of Q to be $\mathcal{P}(Q) = G(Z)$. Let Q' be a quasiparabolic subgroup with block set $\{v_1, \ldots, v_l\}$ and parabolic part $G(Y)$. Define the *block difference* of Q and Q' to be $b(Q, Q') = |\mathcal{B}(Q) \setminus \mathcal{B}(Q')|$, that is the number of blocks occurring in the block set of Q but not Q'. Define the *parabolic difference* of Q and Q' to be $p(Q, Q') = |Z \setminus Y|$.

The following lemma is the key to the proof of the theorem above.

Lemma 4.3. *Let C and D be canonical quasiparabolic centralisers such that $C > D$ and $b = b(D, C) > 0$. Then $p(C, D) > 0$ and there exists a strictly descending chain of canonical parabolic centralisers*

$$\mathcal{P}(C) > C_b > \cdots > C_1 > \mathcal{P}(D) \tag{4.1}$$

of length $b + 1$.

Proof. Let C and D have parabolic parts $\mathcal{P}(C) = G(Y)$ and $\mathcal{P}(D) = G(Z)$, for closed subsets Y and Z in $\mathrm{CS}(\Gamma)$. Let the block sets of C and D be $\mathcal{B}(C) = \{u_1, \ldots, u_k\}$ and $\mathcal{B}(D) = \{v_1, \ldots, v_l\}$. Let $i \in 1 \leqslant i \leqslant l$. As $D < C$, either $\langle v_i\rangle = \langle u_j\rangle$, for some j, or $\langle v_i\rangle \subseteq G(Y)$. As $b(D, C) > 0$ there exists i such that $\langle v_i\rangle \subseteq G(Y)$. Moreover, for such i, we have $\alpha(v_i) \subseteq Y \setminus Z$, so $p(C, D) > 0$.

Assume that, after relabelling if necessary, $\langle v_i\rangle = \langle u_i\rangle$, for $i = 1,\dots,s$, and that $\langle v_{s+1}\rangle,\dots,\langle v_l\rangle \subseteq G(Y)$, so $b = l - s$. Choose $t_i \in \alpha(v_{s+i})$ and let $Y_i = \mathrm{cl}(Z \cup \{t_1,\dots,t_i\})$, for $i = 1,\dots,l-s = b$. Let $C_i = G(Y_i) = C_G(Y_i^{\perp})$, so C_i is a canonical parabolic centraliser. We claim that the chain (4.1) is strictly descending. To begin with, as $t_1 \in Y_1 \setminus Z$ we have $G(Z) < C_1$. Now fix i and n such that $1 \leqslant i < n \leqslant b$. If $a \in \alpha(v_{s+n})$ then $a \in Z^{\perp}$ and $a \in \alpha(v_{s+j})^{\perp} \subseteq t_j^{\perp}$, for $1 \leqslant j < n$, by definition of the standard form of quasiparabolic subgroups. Hence $a \in (Z \cup \{t_1,\dots,t_i\})^{\perp}$. Thus $[a,b] = 1$, for all $b \in Y_i$. This holds for all $a \in \alpha(v_{s+n})$ so $Y_i \subseteq \alpha(v_{s+n})^{\perp}$. As v_{s+n} is a block of length at least 2 we have $\alpha(v_{s+n}) \cap \alpha(v_{s+n})^{\perp} = \emptyset$, so $Y_i \cap \alpha(v_{s+n}) = \emptyset$. Hence $t_n \notin Y_i$ and it follows that $C_i < C_{i+1}$, $i = 1,\dots,b-1$. Now choose $c \in \alpha(v_{s+1})$ such that $[c,t_1] \neq 1$. Then $c \in Y$, as $\alpha(v_{s+1}) \subseteq Y$, however $c \notin Y_b$, since $t_1 \in Z^{\perp} \cap t_1^{\perp} \cap \cdots \cap t_b^{\perp} = Y_b^{\perp}$ and $Y_b = Y_b^{\perp\perp}$. As $D < C$ we have $Z \subseteq Y$ so $C_b = G(Y_b) < G(Y)$. $\square$

We can use this lemma to prove the following about chains of canonical quasiparabolic subgroups.

Lemma 4.4. *Let $C_0 > \cdots > C_d$ be a strictly descending chain of canonical quasiparabolic centralisers such that C_0 and C_d are canonical parabolic centralisers. Then there exists a strictly descending chain $C_0 > P_1 > \cdots > P_{d-1} > C_d$, of canonical parabolic centralisers.*

Proof. First we divide the given centraliser chain into types depending on block differences. Then we replace the chain with a chain of canonical parabolic centralisers, using Lemma 4.3. A simple counting argument shows that the new chain has length at least as great as the old one. In detail let $I = \{0,\dots,d-1\}$ and

$$\begin{aligned} I_+ &= \bigl\{i \in I\colon\ b(C_{i+1},C_i) > 0\bigr\},\\ I_0 &= \bigl\{i \in I\colon\ b(C_{i+1},C_i) = 0 \text{ and } p(C_i,C_{i+1}) > 0\bigr\}, \quad \text{and}\\ I_- &= \bigl\{i \in I\colon\ b(C_{i+1},C_i) = p(C_i,C_{i+1}) = 0\bigr\}. \end{aligned}$$

Then $I = I_+ \sqcup I_0 \sqcup I_-$. For $i \in I_+$ let Δ_i be the strictly descending chain of canonical parabolic centralisers of length $b(C_{i+1},C_i)+1$ from $\mathcal{P}(C_i)$ to $\mathcal{P}(C_{i+1})$, constructed in Lemma 4.3. For $i \in I_0$ let Δ_i be the length one chain $\mathcal{P}(C_i) > \mathcal{P}(C_{i+1})$ and for $i \in I_-$ let Δ_i be the length zero chain $\mathcal{P}(C_i) = \mathcal{P}(C_{i+1})$. This associates a chain Δ_i of canonical parabolic centralisers to each $i \in I$ and we write l_i for the length of Δ_i. If $\Delta_i = P_0 > \cdots > P_{l_i}$ and $\Delta_{i+1} = P'_0 > \cdots > P'_{l_{i+1}}$ then by definition $P_{l_i} = P'_0$, for $i = 1,\dots,d-1$. We may therefore concatenate Δ_i and Δ_{i+1} to give a chain of canonical parabolic centralisers

$$P_0 > \cdots > P_{l_i} = P'_0 > \cdots > P'_{l_{i+1}}$$

of length $l_i + l_{i+1}$. Concatenating $\Delta_1,\dots,\Delta_{d-1}$ in this way we obtain a strictly descending chain of canonical parabolic centralisers of length $l = \sum_{i=0}^{d-1} l_i$. Moreover

$$l = \sum_{i\in I_+} b(C_{i+1},C_i) + |I_+| + |I_0|,$$

since $l_i = b(C_{i+1}, C_i) + 1$, for all $i \in I_+$, $l_i = 1$, for all $i \in I_0$ and $l_i = 0$, for all $l_i \in I_-$. As $|I| = d$ we have now

$$l - d = \sum_{i \in I_+} b(C_{i+1}, C_i) - |I_-|.$$

To complete the argument we shall show that

$$\sum_{i \in I_+} b(C_{i+1}, C_i) = \left| \bigcup_{i=0}^{d} \mathcal{B}(C_i) \right| \geqslant |I_-|.$$

As $\mathcal{B}(C_0) = \emptyset$ we have $b(C_1, C_0) = |\mathcal{B}(C_0) \cup \mathcal{B}(C_1)|$. Assume inductively that

$$\sum_{i=0}^{k} b(C_{i+1}, C_i) = \left| \bigcup_{i=0}^{k+1} \mathcal{B}(C_i) \right|,$$

for some $k \geqslant 0$. Then

$$\sum_{i=0}^{k+1} b(C_{i+1}, C_i) = \left| \bigcup_{i=0}^{k+1} \mathcal{B}(C_i) \right| + \left| \mathcal{B}(C_{k+2}) \setminus \mathcal{B}(C_{k+1}) \right|.$$

Moreover, if $w \in \mathcal{B}(C_{k+2}) \setminus \mathcal{B}(C_{k+1})$ then $w \in \mathcal{P}(C_j)$, for all $j \leqslant k+1$, so $w \notin \mathcal{B}(C_j)$, for $j = 0, \dots, k+1$. Hence

$$\mathcal{B}(C_{k+2}) \setminus \mathcal{B}(C_{k+1}) = \mathcal{B}(C_{k+2}) \Big\backslash \bigcup_{i=0}^{k+1} \mathcal{B}(C_i)$$

and it follows that

$$\sum_{i=0}^{k+1} b(C_{i+1}, C_i) = \left| \bigcup_{i=0}^{k+2} \mathcal{B}(C_i) \right|.$$

As $b(C_{i+1}, C_i) = 0$ if $i \notin I_+$ it follows that

$$\sum_{i \in I_+} b(C_{i+1}, C_i) = \left| \bigcup_{i=0}^{d} \mathcal{B}(C_i) \right|,$$

as required. If $i \in I_-$ then $b(C_{i+1}, C_i) = p(C_i, C_{i+1}) = 0$, so $b(C_i, C_{i+1}) > 0$. Therefore there is at least one element $w \in \mathcal{B}(C_i) \setminus \mathcal{B}(C_{i+1})$. It follows that $w \notin \mathcal{B}(C_j)$, for all $j \geqslant i+1$ and so $I_- \leqslant |\bigcup_{i=0}^{d} \mathcal{B}(C_i)|$. Therefore $l - d \geqslant 0$ and the proof is complete. □

Proof of Theorem 4.1. Let

$$G = C_0 > \cdots > C_d = Z(G)$$

be a maximal descending chain of centralisers of G. By Theorem 3.12, each of the C_i's is a quasiparabolic subgroup. If each C_i is canonical then, since G and $Z(G)$ are both canonical parabolic centralisers the result follows from Lemma 4.4.

Suppose that now $C_1, \dots, C_s$ are canonical quasiparabolic and C_{s+1} is not: say $C_{s+1} = Q^g$, where Q is a canonical quasiparabolic subgroup. Let $C_s = Q(u, Y)$ and $Q = Q(v, Z)$ both in standard form. Write $g = f \circ h$, where $f = \mathrm{gd}^l_{Z^\perp}(g)$ and let $f = e \circ d$, where $d = \mathrm{gd}^r_Y(f)$, so $d \in G(Y \cap Z^\perp)$. Then $G(Z)^h = G(Z)^{dh} = G(Z)^g \subseteq G(Y)$ and $\alpha(h) \subseteq Y$, from Corollary 2.5. Hence $\alpha(d \circ h) \subseteq Y$ which implies that $\alpha(d \circ h) \subseteq \mathcal{P}(C_s) \subseteq \cdots \subseteq \mathcal{P}(C_0)$. It follows that $C_r^{dh} = C_r$, for $r = 0, \dots, s$. Therefore conjugating $C_0 > C_1 > \cdots > C_d$ by $(dh)^{-1}$ we obtain a chain in which $C_0, \dots, C_s$ are unchanged and $C_{s+1} = Q^e = \langle v_1 \rangle^e \times \langle v_l \rangle^e \times G(Z)$, with $\mathrm{gd}^r_Y(e) = 1$, $e \in G(Z^\perp)$. As Lemma 3.6 implies that Q^e is a canonical quasiparabolic subgroup we now have a chain in which $C_0, \dots, C_{s+1}$ are canonical quasiparabolic. Continuing this way we eventually obtain a chain, of length d, of canonical quasiparabolic centralisers to which the first part of the proof may be applied. $\square$

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