Abstract

This paper presents a set of hypotheses and a mathematical formulation in order to model the equilibrium between supply and demand on an urban housing market. Housing supply is assumed to be price-elastic, in a way that can vary for each location and each dwelling type. Household housing demand is disaggregated on the basis of employment area, category of activity, income and household size. The market equilibrium is characterized by a set of primal-dual conditions, then by a variational inequality whose continuity guarantees the existence of an equilibrium. In the monocentric case, the primal-dual conditions are the saddle point conditions of a constrained maximization program the concavity of which implies the uniqueness of the equilibrium. We propose a resolution algorithm that is derived from traffic assignment on a transportation network. Last, we explore the outcomes of the model in a demonstration application and analyze the sensitivity of the equilibrium to our supply and demand hypotheses.

Keywords: Housing market; Disaggregate demand; Supply and demand equilibrium; Income distribution; Variational inequality

1. Introduction

1.1. Context

Two socio-economic systems are of particular interest for urban planning: the land use system (which comprehends the real estate system) and the transportation system. These are strongly intertwined, as established by a now considerably large body of literature (e.g. [12] and [5] for recent contributions, [11] for an ancient one). The relationship between the two is not symmetrical however, with urban change processes working on different time scales ([14]). Land Use – Transport Interaction (LUTI) models, which very aim is to model these urban
change processes linking those two systems, are therefore especially helpful for the design and ex-ante evaluation of urban development scenarios.

The typology proposed by David Simmonds Consultancy et al [14] distinguishes two main families of LUTI models: static models such as Relu-Tran ([11]) or Pirandello ([3]), and which focus on the long term supply – demand equilibrium: versus quasi – dynamic models such as UrbanSim ([15]), or Ilute ([13]), and which analyze the dynamics of the two systems. The former are the simpler and are therefore the natural choice in order to simulate an area and assist the planning process – before using a more detailed and more complex dynamic model if this proves necessary.

It is above all vital for the model’s hypotheses to be realistic and to be verifiable by an analyst. Most static models assume that the total floor area of the dwellings in each zone in the area can be influenced by the demands of households and their accumulation, and also that each demander is immutable (1). The dynamic framework removes these limitations, but at the cost of an increase in the number of hypotheses whose interactions are not only more difficult to check but even to identify qualitatively.

1.2. Objective

This paper presents a static model for the equilibrium between supply and demand on an urban housing market. This model disaggregates demand according to place of work, category of activity and household size and income. It is designed for the private rented property market, so social housing and home purchase are not dealt with at this stage but may be covered by subsequent extensions. The size of each dwelling is fixed, and the number of dwellings, broken down on a qualitative basis (which includes size) and with respect to location, is elastic to price according to a given specification.

We shall provide an efficient mathematical formulation for this model: starting from primal-dual equilibrium conditions, a variational inequality problem will be formulated; a concave maximization program is available also for a monocentric instance. This has enabled us to demonstrate the existence and uniqueness of the equilibrium and propose a robust computation method.

The model is intended for simple applications, providing an approximate result in the context of an operational study or for teaching purposes.

1.3. The approach

Overall, an economic model is specified for which a mathematical and algorithmic treatment is provided: the paper is situated at the crossroads between urban economics and operations research. The mathematical treatment is derived from traffic assignment on a transport network, in particular, dual criteria assignment with a cost-time trade-off. Demand will be disaggregated discretely according to the employment area and household size, and continuously according to income. For each household, we will model the discrete choice between the housing options, according to zone and quality (which includes size), assuming that the household is sensitive to the price of housing and the cost of transport between home and work. Transport between home and work has a monetary cost and a temporal cost which determine budgetary accessibility and affect utility in relation to income. Thus, for each employment area and discrete type of household, the choice of residential zone and quality of dwelling is analogous to the choice of a route on an origin-destination link on the basis of cost and time, for a population of demanders who are differentiated on the basis of the cost-time trade-off. We will adapt our previous treatment of the dual criteria assignment of traffic onto a network ([6]) to the microeconomic choice of housing: this adaptation is straightforward for a simple utility function e.g. Cobb-Douglas or similar.

1 i.e. which does not change over time, without the concept of life-cycle changes.
1.4. The structure of the paper

The main body of the paper is in four parts followed by a conclusion. We shall begin by stating the hypotheses of the model (Section 2). From these we shall derive a “grouped” demand function by aggregating microeconomic agents without restricting their individual behaviour (Section 3). Next, we will formulate the supply-demand equilibrium, give mathematical formulations, establish the properties of existence and uniqueness, and provide a resolution algorithm (Section 4). Last, we shall provide a toy instance with a single employment area, two housing zones, two sizes of dwellings and two sizes of households for which we will determine a reference equilibrium and analyze the sensitivity to a number of supply and demand hypotheses (Section 5). The conclusion will set out some possibilities for extension (Section 6).

2. Modelling hypotheses

The model’s hypotheses relate to job supply (§ 2.1), housing supply (§ 2.2), transport supply (§ 2.3) and demand for housing and transport (§ 2.4).

2.1. Job supply

Let us assume that the number of jobs per zone for each category of activity (which can include the nature of the activity and the job category) and for each level of salary is exogenous in the medium term. Let \( W = J \times K \) be the set of pairs \((j,k)\) for an employment area \( j \in J \) and a category of activity \( k \in K \). The number of jobs in the category is denoted by \( Q_w \), and the salaries are distributed according to a distribution function \( H_w \). Thus, economic activity in the area does not depend on the economic conditions that apply to the housing and transport markets.

2.2. Housing supply

Let us assume that housing is distributed according to quality and locational zone. Quality includes, in particular, the floor area of the dwelling, the quality of construction, the quality of the neighbourhood and the available local amenities. \( R \) denotes all the types of property, \( s_r \) denotes the floor area of the type \( r \) dwellings, \( q_r \) the residual attributes of quality, \( S_r \) the quantity supplied and \( p_r \) the price per unit of floor area. Assume also that in the medium term supply is elastic to the market price according to the function:

\[
S_r = O_r(p_r)
\]  
(2.1)

By assumption, \( O_r \) is an increasing function. \( O_r^{-1} \) is the reciprocal function, which is also increasing.

In principle, these hypotheses are valid for the private rented property sector. The social rented property sector operates differently, with a supply function that depends both on the production cost and private sector prices, while demand depends both on the private sector price and the waiting delay. The owner-occupier sector bears a closer resemblance to the private rented sector than the social sector, but the transaction costs depend on the length of occupation and the price for the demander depends on the cost of credit and the initial deposit (therefore capital as well as income).

2.3. Transport supply

It is assumed that transport services are available between the different types of residential areas (by zone and housing mode) \( i \in I \) and the different types of job \( w \in W \) (making a distinction according to the zone and, if necessary, the category of activity which can generate specific modal requirements such as car access). Servicing the pair \((i,w)\) involves a travel time \( t_{iw} \) and a monetary cost \( t_{iw} \). These terms are taken as exogenous, i.e.
independent of network flows and the income derived from the job held.

2.4. Demand for housing and transport

Household demand is the keystone of the system as it links jobs, housing and transport. We have assumed that demand consists of a set of households, each of which has one job and a size \( v_m \) that is expressed as the number of individuals or consumption units. The household’s job is specified in terms of location, category of activity \( k_m \) and salary \( \rho_m \).

For the household, a dwelling of type \( r \) is a residential option with a floor area \( s_r \) and a quality \( q_r \), and carries with it housing expenditure of \( p_r \), and travel expenditure (\( t_{rfj} \) in money and \( t_{tj} \) in time). These expenditures reduce the amount of the household’s income that is available for all types of consumption apart from housing and transport. These types of consumption other than housing and transport are known as the “numéraire good” which is monetized and denoted by \( z \).

We have made the assumption that the household desires space in its dwelling, quality (apart from the floor area) denoted by \( q \), the numéraire good \( z \) and free time \( \tau \), and that its preferences for a “bundle” \( (s,q,z,\tau) \) are represented by a utility function \( U_m(s,q,z,\tau) \). This function models the household’s interest in the bundle. Each housing option constitutes a specific bundle. The household is assumed to be micro-economically rational, meaning that he selects the option with the maximum utility from those on the market which are accessible to him.

The household evaluates a residential option in relation to two budgetary constraints, one of which is monetary and the other temporal. Denote by \( c_r(q_r,s_r,v_m) \) the running cost of the dwelling in each period with respect to household size, including possibly a reserve price from the supplier. The monetary budgetary constraint is for the household’s expenditure on housing, transport and the numéraire good to be compatible with his income:

\[
p_r s_r + c_r(q_r,s_r,v_m) + t_{rfj} + z \leq \rho_m. \tag{2.2}
\]

The temporal budgetary constraint compares the time at work \( \overline{\rho}_m \), travel time \( \overline{t}_{tj} \) and the residual time \( \tau \) to the available time \( \theta_m \):

\[
\overline{\rho}_m + \overline{t}_{tj} + \tau \leq \theta_m. \tag{2.3}
\]

Let \( R_m \) denote all the housing options which are within the household’s budgetary reach, i.e. which satisfy (2.2) and (2.3). An option of this type leaves the household with an amount of the numéraire good and free time that are formulated as follows:

\[
\bar{z}_{mr} = \rho_m - p_r s_r - c_r(q_r,s_r,v_m) - t_{rfj}, \tag{2.4}
\]

\[
\bar{\tau}_{mr} = \theta_m - \overline{\rho}_m - \overline{t}_{tj}. \tag{2.5}
\]

Its utility for the household is therefore

\[
\bar{U}_m(r) = U_m(s_r,q_r,\bar{z}_{mr},\bar{\tau}_{mr}). \tag{2.6}
\]

The demander’s microeconomic behaviour consists of selecting the best available option, i.e.

\[
\text{Find } r^* \in R_m, \forall r \in R_m, \quad \bar{U}_m(r^*) \geq \bar{U}_m(r). \tag{2.7}
\]

3. The demand function for each class (segment)

We have specified the microeconomic model for an individual demander of housing and transport. This model is disaggregated on the basis of job location \( j \), category of activity \( k \), income level \( \rho \) and household size \( v \). Disaggregation is performed discretely for \( j,k,v \) and continuously for \( \rho \). A demand class is defined by the
triple $\sigma = (j,k,v)$ and a distribution function $H_\sigma$ for income. Income is thus the central parameter of our analysis. By considering its distribution we treat a class collectively in order to formulate its overall demand for a given set of supplied options.

### 3.1. Residual consumption and residential utility

The relationship between the residual amount of the numéraire good and income is affine: denoting $P_{\sigma r} = p_r s_r + c_r (q_r, s_r) + t_r j$, equation (2.4) yields that

$$\bar{z}_{\sigma r} = \rho - P_{\sigma r},$$

which is an increasing function of income.

The amount of free time $\bar{z}_{mr}$ does not depend directly on income and can be denoted by $\bar{z}_{\sigma r}$.

The residential utility $\bar{U}_m(r)$ of a residential option $r$ for a demander $m = (\sigma, \rho)$ is the utility function $U_m(s_r, q_r, \bar{z}_{mr}, \bar{z}_{mr})$, on condition the option is compatible with the temporal and monetary budgetary constraints. This compatibility depends simultaneously on $r$, $\sigma$ and $\rho$. When this compatibility exists (each constraint must be examined in relation to $\rho$), under given $r$ and $\sigma$, the residential utility is an increasing function of the residual consumption $\bar{z}_{\sigma r}$, and therefore increases with income as $\bar{z}_{\sigma r}$ increases with $\rho$.

Let $\hat{U}_{\sigma r}(\rho)$ denote the utility of the option $r$ in relation to income $\rho$, within the segment $\sigma$. This function is increasing and continuous if the initial function $m U$ is continuous.

Let us also define the maximum residential utility function out of all the budgetarily feasible options,

$$U_{\rho\sigma}^*(\rho) = \max \{ \hat{U}_{\sigma r}(\rho) : r \in R_{\sigma\rho} \}.$$  (3.2)

As an increase in income makes it easier to satisfy a budgetary constraint, a higher income makes it possible to access more residential options. A higher income therefore provides higher maximum residential utility for two reasons: first because the utility of each accessible option is higher and second because the choice set is larger.

### 3.2. Efficiency domain of a housing option

For a given option $r$, we shall define the feasibility domain $F_{\sigma r} = \{ \rho : r \in R_{\sigma\rho} \}$ and the efficiency domain, which is the set of incomes for which the option provides the maximum utility:

$$E_{\sigma r} = \{ \rho : r \in R_{\sigma\rho} \text{ and } \hat{U}_{\sigma r}(\rho) = U_{\rho\sigma}^*(\rho) \}.$$  (3.3)

The assignment of each demander to their preferred option amounts to specify the efficiency domains of the various options. If the utility functions have a simple form the efficiency domains do too, as they are intervals that are bounded by special values, the break-off points. Let us make the specific hypothesis that there is an increasing function $\varphi_\sigma$ such that, for every option, the compound function $u_{\sigma r} = \varphi_\sigma \circ \bar{U}_{\sigma r}$ is affine in relation to income. As $\varphi_\sigma$ is an increasing function, so is $\rho \mapsto u_{\sigma r}(\rho)$, which is a utility function that depends ultimately on $s_r$, $q_r$, $\bar{z}_{\sigma r}$ and $\bar{z}_{\sigma r}$. Let us define the affinity parameters $a$ and $b$ according to income such that

$$u_{\sigma r}(\rho) = a_{\sigma r} + b_{\sigma r} \rho = b_{\sigma r} (\rho - \xi_{\sigma r}).$$  (3.4)

The efficiency domain of an option is the intersection between its relative efficiency domains compared to any other option: $E_{\sigma r} = \bigcap_{\rho \in R_{\rho}} E_{\sigma r/s}$, for the relative domain

$$E_{\sigma r/s} = \{ \rho \in F_{\sigma r} : u_{\sigma r}(\rho) \geq u_{\sigma s}(\rho) \text{ if } \rho \in F_{\sigma s} \}.$$  (3.5)

Under the hypothesis of affinity, we can re-express the condition of superiority thus:
If \( b_{\sigma r} \neq b_{\sigma s} \), both sides of the previous line are equal at \( \rho_{rs}^* = \left( b_{\sigma r} \xi_{\sigma r} - b_{\sigma s} \xi_{\sigma s} \right) / \left( b_{\sigma r} - b_{\sigma s} \right) \), and
\[
u_{\sigma r}(\rho) = \frac{b_{\sigma r}(\rho - \xi_{\sigma r}) - b_{\sigma s}(\rho - \xi_{\sigma s})}{(b_{\sigma r} - b_{\sigma s})}.
\]

So, if \( b_{\sigma r} > b_{\sigma s} \), \( r \) is preferred when \( \rho \geq \rho_{rs}^* \) and \( s \) when \( \rho \leq \rho_{rs}^* \). If \( b_{\sigma r} < b_{\sigma s} \), \( s \) is better above \( \rho_{rs}^* \) and \( r \) is better below. In addition, it is necessary to include the admissibility conditions for each option: for \( r \) the financial constraint is \( \xi_{\sigma r} \geq 0 \) so \( \rho \geq \rho_{ar} \) and the temporal constraint is \( \tau_{mr} \geq 0 \) so \( \theta_m - \bar{\rho}_m \geq \tilde{t}_{ij} \).

Finally, denoting by \( D_{\sigma,r/s} \) the efficiency interval in the absence of admissibility constraints, it holds that
\[
E_{\sigma,r/s} = F_{\sigma r} \setminus (D_{\sigma,r/s} \cap F_{\sigma s}) = F_{\sigma r} \cap F_{\sigma s} \setminus (F_{\sigma r} \setminus F_{\sigma s}).
\] (3.6)

The efficiency domains \( E_{\sigma} \) of the options can be determined in a comprehensive manner, integrating the options in the order of increasing values of \( \varphi_{\sigma} \). Including a new option conserves or reduces the domain of each of the previous options, but it remains an interval.

Ultimately, \( E_{\sigma} = ]P_{\sigma}, \bar{P}_{\sigma}[ \), perhaps with closed brackets and/or \( \bar{P}_{\sigma} = +\infty \). \( E_{\sigma} \) may be empty, which we consider to be equivalent to \( P_{\sigma} = \bar{P}_{\sigma} \) if the distribution \( H_{\sigma} \) is continuous. The intervals are arranged in increasing order of \( b_{\sigma r} \), so between two consecutive non-punctual intervals \( s < r \), \( \bar{P}_{\sigma} = \bar{P}_{\sigma} = \rho_{sr}^* \).

### 3.3. The demand function for each class

Under these conditions, the option \( r \) is optimum for \( E_{\sigma r} \) and selected for every value of \( \rho \in E_{\sigma r} \) (save possibly for the break-off points). Therefore it attracts customers from segment \( \sigma \) in number of
\[
n_{\sigma r} = Q_{\sigma} \Pr(\rho \in E_{\sigma r}) = Q_{\sigma} [H_{\sigma}(\bar{P}_{\sigma}) - H_{\sigma}(\rho_{ar})].
\] (3.7)

The \( [n_{\sigma r} : r \in R_{\sigma}] \) make up the demand function of segment \( \sigma \) for the residential options. They depend on the conditions of supply.

### 3.4. Quasi Cobb-Douglas utility function

The framework of hypotheses is particularly appropriate for the following utility function:
\[
U_{\sigma}(s, q, z, \tau) = K_{\sigma} ((s - v_{\sigma, s0})^+]^\alpha q^\beta z^\gamma \tau^\delta,
\] (3.8)
which is of the Cobb-Douglas type with the parameters \( (\alpha, \beta, \gamma, \delta) \) which may themselves depend on \( \sigma \), and that is modified by requiring a minimum floor area, denoted by \( s_0 \), for each household member. The utility of an option whose floor area is too small is reduced to zero.

The function \( \varphi_{\sigma} : x \mapsto x^{1/\gamma} \), applied to \( U_{\sigma} \), gives the following affine function:
\[
u_{\sigma r}(\rho) = [K_{\sigma} ((s - v_{\sigma, s0})^+]^\alpha q^\beta z^\gamma]^{1/\gamma} \xi_{\sigma r}(\rho) = b_{\sigma r}(\rho - P_{\sigma r}).
\] (3.9)

Therefore \( b_{\sigma r} = K_{\sigma} [((s - v_{\sigma, s0})^+]^\alpha q^\beta z^\gamma]^{1/\gamma} \) and \( \xi_{\sigma} = P_{\sigma} \) and \( a_{\sigma} = -b_{\sigma r} \xi_{\sigma r} = -b_{\sigma r} P_{\sigma r} \).

The order of the \( b_{\sigma r} \) values depends on the characteristics \( q_r \) and \( s_r \) as well as the travel time \( \tilde{t}_{ij} \) between the home \( r \) and zone of work \( j \) for segment \( \sigma \), via \( \tau_{\sigma r} = \theta_m - \bar{\rho}_m - \tilde{t}_{ij} \). A longer travel time reduces the \( b_{\sigma r} \) and therefore acts in the opposite direction to a higher floor area \( s_r \). All other things being equal, options which are identical in all ways except for travel time are arranged in order of decreasing travel time so, a priori, the efficiency domain of high incomes favours short home-to-work distances.
4. Mathematical analysis

4.1. Primal-dual equilibrium conditions

To option \( r \) and segment \( \sigma \) let us associate the sum \( N_{\sigma r} = \sum_{\ell=1}^{r} n_{\sigma \ell} \) and the attractiveness function

\[
I_{\sigma r} = a_{\sigma r} + \sum_{\ell=1}^{r} (b_{\sigma \ell} - b_{\sigma , \ell+1}) H_\sigma^{-1}(N_{\sigma \ell} / Q_\sigma).
\] (4.1)

Let us assume that each demander is assigned to their optimum option. Then, between two options \( s \) and \( r \) that are used consecutively,

\[
I_{\sigma r} - I_{\sigma s} = a_{\sigma r} - a_{\sigma s} - \sum_{\ell=1}^{r-1} (b_{\sigma \ell} - b_{\sigma , \ell+1}) H_\sigma^{-1}(N_{\sigma \ell} / Q_\sigma) = a_{\sigma r} - a_{\sigma s} + (b_{\sigma r} - b_{\sigma s}) \rho_{sr}^* = 0 \text{ from the definition of } \rho_{sr}^*.
\]

For an unused option \( t \), the values \( \rho_{\sigma t} \) and \( \overline{\rho}_{\sigma t} \) are fixed at the break-off value between the two used options \( s \) and \( r \) which are respectively lower and higher in the order of increasing values of \( b_{\sigma r} \), so

\[
I_{\sigma r} - I_{\sigma t} = a_{\sigma r} - a_{\sigma t} + (b_{\sigma r} - b_{\sigma t}) \rho_{sr}^* \geq 0 \text{ since } u_{\sigma r}(\rho_{sr}^*) \geq u_{\sigma t}(\rho_{sr}^*).
\]

Thus, the attractiveness function is at a maximum for any option that is used, and an option with a value that is strictly lower than the maximum value cannot be efficient (i.e. have a non-trivial efficiency domain). We can therefore define the equilibrium conditions with a dual variable \( \mu_\sigma \) as follows:

\[
n_{\sigma r} \geq 0 \text{ and } \sum_{\ell=1}^{r} n_{\sigma \ell} = Q_\sigma \\
I_{\sigma r} - \mu_\sigma \leq 0 \text{ and } n_{\sigma r} (I_{\sigma r} - \mu_\sigma) = 0.
\] (4.2a,b,c,d)

It can easily be shown that these conditions are sufficient to characterize a local equilibrium for the segment \( \sigma \). Coordination between segments is achieved by adjusting the price and quantity of the supply: \( a_{\sigma r} \) depends on the price \( p_r \) which in turn depends on \( S_r = \sum_{\sigma} n_{\sigma r} \). This provides the basis for the general characterization that follows.

4.2. Variational inequality and property of existence

In vector terms, let us denote \( n_\sigma = [n_{\sigma r} : r \in R_\sigma] \) and \( I_\sigma = [I_{\sigma r} : r \in R_\sigma] \) with components that are arranged in order of increasing \( b_{\sigma r} \) values.

**Theorem.** The residential equilibrium resolves the following variational inequality problem in \( n_S = [n_\sigma : \sigma \in S] \), for \( I_S = [I_\sigma : \sigma \in S] \):

\[
\text{Find } n_\sigma^* \text{ such that, } \forall n_S \text{ that is admissible, } I_S(n_\sigma^*) . (n_S - n_\sigma^*) \geq 0.
\] (4.3)

If the functions \( O_r \) and \( H_\sigma \) are continuous, \( I_S \) is too, which guarantees the existence of a solution if the problem is feasible (in particular, total supply must not be lower than total demand). This demonstrates the existence of a residential equilibrium.

4.3. Extremal formulation and property of uniqueness

The above treatment has been adapted from the model for traffic assignment on a transport network developed
by Leurent ([6], [9]). In the case where there is only one demand segment $\sigma$, which makes it possible to represent a monocentric model with a distribution of incomes but not household sizes, the function $I_S$ is derived from a potential function, namely:

$$J_\sigma = \sum_{r=1}^{R_\sigma} a_{\sigma r}(\theta) d\theta + Q_\sigma \sum_{r=1}^{R_\sigma} b_{\sigma r}\left[\eta(x)\left(\frac{N_{\sigma r}}{Q_\sigma}\right) - \eta(x)\left(\frac{N_{\sigma r-1}}{Q_\sigma}\right)\right],$$

wherein $\eta(x) = \int H_\sigma^1(\theta) d\theta$.

(4.4)

In fact, the function $J_\sigma$ has as the following partial derivative:

$$\frac{\partial J_\sigma}{\partial q_{\sigma r}} = a_{\sigma r}(S_r) + \sum_{r=1}^{R_\sigma} b_{\sigma r}\left[H_\sigma^1\left(\frac{N_{\sigma r}}{Q_\sigma}\right)1_{(\sigma r<\sigma \geq r)} - H_\sigma^1\left(\frac{N_{\sigma r-1}}{Q_\sigma}\right)1_{(\sigma r>\sigma \geq r)}\right]$$

$$= a_{\sigma r} + \sum_{r=1}^{R_\sigma} b_{\sigma r} H_\sigma^1\left(\frac{N_{\sigma r}}{Q_\sigma}\right) - \sum_{r=1}^{R_\sigma} b_{\sigma r} H_\sigma^1\left(\frac{N_{\sigma r-1}}{Q_\sigma}\right)$$

$$= a_{\sigma r} + \sum_{r=1}^{R_\sigma} b_{\sigma r} H_\sigma^1\left(\frac{N_{\sigma r}}{Q_\sigma}\right)$$

Furthermore

$$\frac{\partial^2 J_\sigma}{\partial q_{\sigma r} \partial q_{\sigma s}} = \sum_{r,s} \frac{\partial a_{\sigma r}}{\partial q_{\sigma s}} x_s x_r = \sum_{r,s} \frac{\partial a_{\sigma r}}{\partial q_{\sigma s}} x_s^2 + Q_\sigma \sum_{r=1}^{R_\sigma} b_{\sigma r} H_\sigma^1\left(\frac{N_{\sigma r}}{Q_\sigma}\right) X_r^2 \text{ for } X_r = \sum_{r=1}^{R_\sigma} x_r.$$

All the terms in the sum are below zero because $\Delta b \leq 0$, $H_\sigma$ is an increasing function, so its reciprocal is increasing too and it has a non-negative derivative $H_\sigma^1$, which is non positive since the inverse supply function is increasing hence its derivative is non negative. $J_\sigma$ is therefore concave. This entails the uniqueness of the residential equilibrium (with regard to the components whose diagonal coefficient in the Hessian matrix $[\partial^2 J_\sigma / \partial q_{\sigma r} \partial q_{\sigma s} : r, s \in R_\sigma]$ is strictly negative).

In the case where there are several demand segments, the coefficients $[b_{\sigma r} : \sigma \in S]$ for a given option $r$ are heterogeneous, so the term $a_{\sigma r}$ in the attractiveness function $I_{\sigma r}$ can no longer be included in an objective function. The reason for this is that $\partial a_{\sigma r} / \partial q_{\sigma r} = -s_r \hat{O}^{-1}$ which is non positive since the inverse supply function is increasing hence its derivative is non negative. $J_\sigma$ is therefore concave. This entails the uniqueness of the residential equilibrium (with regard to the components whose diagonal coefficient in the Hessian matrix $[\partial^2 J_\sigma / \partial q_{\sigma r} \partial q_{\sigma s} : r, s \in R_\sigma]$ is strictly negative).

4.4. Resolution algorithm

Residential equilibrium can be found by an algorithm that is taken from the assignment of traffic on a network. A natural first approach is to apply the “historical” algorithm developed by Beckmann et al [2], i.e. the Method of Successive Averages (MSA) as it has been adapted for the dual criteria assignment model [7].

The MSA algorithm performs successive iterations, which are given the index $k$. The price of the options is updated during each iteration on the basis of the clienteles $[S_r^{(k)} : r \in R]$. Each segment $\sigma \in S$ is then processed as follows:
• determination of the coefficients \( b_{\sigma r} \) and \( a_{\sigma r} \) \( \forall r \in R_{\sigma} \).

• determination of the efficiency intervals \([\bar{p}_{\sigma r}^{(k)}, \underline{p}_{\sigma r}^{(k)}]\).

• from which an auxiliary state for the clientele is derived: \( \bar{n}_{\sigma r}^{(k)} = Q_{\sigma}(\bar{p}_{\sigma r}^{(k)}) - H_{\sigma}(\underline{p}_{\sigma r}^{(k)}) \).

Once all the segments have been processed, the next state is determined by a convex combination of the present state of order \( k \) and the auxiliary state, whose combination coefficient \( \zeta_k \in ]0,1[ \) is predetermined:

\[
\forall \sigma \in S, \forall r \in R_{\sigma}, \quad \bar{n}_{\sigma r}^{(k+1)} = (1- \zeta_k)\bar{n}_{\sigma r}^{(k)} + \zeta_k\bar{n}_{\sigma r}^{(k)\prime},
\]

\[
\forall r \in R, \quad S_r^{(k+1)} = \sum_{\sigma: R_{\sigma} \ni r} \bar{n}_{\sigma r}^{(k+1)} = (1- \zeta_k)S_r^{(k)} + \zeta_kS_r^{(k)\prime}.
\]

Convergence towards equilibrium is assessed on the basis of the duality gap of the variational inequality, \( DG^{(k)} = \sum_{\sigma \in S} \sum_{r \in R_{\sigma}} (\bar{n}_{\sigma r}^{(k)} - \underline{n}_{\sigma r}^{(k)}) \). This criterion is non-negative by construction, is only cancelled at one equilibrium, and by continuity a very small value indicates that an equilibrium state has almost been reached. If the criterion is sufficiently small, the algorithm is stopped, otherwise another iteration is performed.

For the initial state at \( k = 0 \), to generate values of \( S_r^{(0)} \) therefore the prices \( p_r^{(0)} \) and attractiveness values \( I_{\sigma r}^{(1)} \), it is possible to fix the values of \( \bar{n}_{\sigma r}^{(0)} \) for instance as a uniform distribution, \( \bar{n}_{\sigma r}^{(0)} = Q_{\sigma}/\text{Card}(R_{\sigma}) \).

The sequence \( \{\zeta_k\} \) must decrease towards zero, but not too rapidly in order to redistribute demanders efficiently between the options at each iteration.

5. Demonstration application

We have designed a toy instance with a single employment area (cf the monocentric model), two residential zones, two sizes of dwelling and two segments of demand according to whether the household is small or large (§ 5.1). This case allows us to show the application of the model in a reference situation (§ 5.2), and explore sensitivities to variations in supply (§ 5.3) and demand (§ 5.4).

5.1. Case design

Let us consider an urban area in which all the jobs are concentrated in a single employment area, as in the monocentric model of urban economics. Two residential zones are specified which differ with regard to travel cost and time to the employment area: one zone which is close, which is called the Centre, and one “distant zone” called the Suburbs. Each zone contains two types of dwelling, small (40 m²) and large (100 m²). For each zone and type of dwelling, supply depends on price with a constant elasticity of 0.05 in the Centre and 0.3 in the Suburbs. The quality of all the dwellings is homogeneous.

There are two classes of demand: small households (\( v = 2 \)) and large ones (\( v = 4 \)). The minimum floor area per person is \( s_0 = 15 \) m², which means that large households cannot live in the small dwellings. Each household has one working member, and the incomes obey a shifted log-normal distribution whose parameters are \( \rho_{\min} \) for offset, \( m_{\sigma} \) and \( s_{\sigma} \) for the mean and standard deviation of the logs of the shifted value (which are distributed Gaussian). The amount of time usable each day is fixed at \( \theta_m = 15 \) h and the amount of time at work is fixed at \( \tilde{\rho}_m = 9 \) h. A quasi Cobb-Douglas utility function is used: \( U \propto (s - v s_0)^{\alpha} \tilde{z}^\gamma \tau^\delta \), where \( \alpha = .3, \gamma = .5 \) and \( \delta = .2 \).
Table 2. Parameters of the numerical application

<table>
<thead>
<tr>
<th>Topic</th>
<th>Parameter</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Job supply</td>
<td>( Q = N_{\text{small}} + N_{\text{large}} )</td>
<td>5,000 = 2,000 + 3,000</td>
</tr>
<tr>
<td>Housing supply</td>
<td>Category, according to location and size</td>
<td>C-sm / C-lg / B-sm / B-lg</td>
</tr>
<tr>
<td></td>
<td>( S_{\text{ref}} ) reference number</td>
<td>1000 / 1,500 / 1,500 / 1,500</td>
</tr>
<tr>
<td></td>
<td>( p_{\text{ref}} ) reference price (€/m²/day)</td>
<td>0.5 / 0.5 / 0.5 / 0.5</td>
</tr>
<tr>
<td></td>
<td>( \varepsilon ) price elasticity of volume</td>
<td>0.05 / 0.05 / 0.3 / 0.3</td>
</tr>
<tr>
<td></td>
<td>( c_{\text{r}} ) cost of use (€/day)</td>
<td>4 / 9 / 3 / 8</td>
</tr>
<tr>
<td>Transport supply</td>
<td>( I_{\text{rj}} ) price of return journey (€/day)</td>
<td>Centre 1.2 / Suburbs 5.0</td>
</tr>
<tr>
<td></td>
<td>( I_{\text{rj}} ) duration of it (hours per day)</td>
<td>Centre 0.4 / Suburbs 1.2</td>
</tr>
<tr>
<td>Demand</td>
<td>( N_{\text{seg}} ) number of households per segment</td>
<td>Small 2,000 / Large 3,000</td>
</tr>
<tr>
<td></td>
<td>( p_{\text{min}} ) minimum income (€/day)</td>
<td>Small 60 / Large 60</td>
</tr>
<tr>
<td></td>
<td>( m_{\text{seg}} ) average of the log of incomes</td>
<td>Small 3 / Large 4</td>
</tr>
<tr>
<td></td>
<td>( s_{\text{seg}} ) standard deviation of the log of incomes</td>
<td>Small 1 / Large 1</td>
</tr>
</tbody>
</table>

C Centre, B Suburbs, sm Small dwelling, lg Large dwelling

5.2. Residential equilibrium: reference situation

Small households can access all the residential options. These are classified in the following order of increasing slope \( b_{\text{rg}} \), which is given in brackets:

Small Suburbs (7.46) / Small Centre (7.93) / Large Suburbs (23.9) / Large Centre (25.5).

Only the large dwellings are suitable for the large households: the order is: Suburbs (17.1) then Centre (18.2).

Figure 1 depicts the quantities and prices of the residential options when the housing market is in equilibrium. The small households choose a small or large dwelling in the suburbs or a large dwelling in the centre. The cut-off incomes between the options are respectively 75.6 and 125.2 Euros/day. The large households are distributed between large dwellings in the two zones, with a cut-off income of 124.7 €/day.

![Fig 1. The conditions of exchange on the housing market](image-url)
to house the large households at prices affordable to all of them, and those with the lowest incomes divert to the suburbs. The price is fixed by the number of such households and their solvency. Small households with high incomes afford themselves with large dwellings in the centre; those with lower incomes live in the small dwellings in the centre (medium-low incomes) or in the suburbs (low incomes) with much lower unitary rents but lower utility too.

![Graph](image)

Fig. 2. Utility by residential option according to income (in €/day), Small household segment

The aggregation of microeconomic demands thus produces spatialized macroscopic effects that are linked to the economic behaviour and macroscopic structure of supply for each type of dwelling. The deterministic nature of the utility functions makes it possible to differentiate clearly between the options (2) and focus on their principal characteristics, namely floor area and position in relation to the employment area. In this case, the difference in the floor areas of dwellings has a greater influence than individual travel conditions (3); consequently, the overall effect of demand that is heterogeneous with regard to income and household size leads to a degree of heterogeneity in unitary rents according to category of floor area, so it is this characteristic that has the greatest influence. Travel conditions exert a second order influence within each floor area category.

5.3. Sensitivity analysis of supply

The model enables us to specify housing supply for each type and zone and impose the resulting supply conditions on demand. It is thus possible to specify a particular medium- or long-term change in supply in order to link a short- or medium-term demand reaction to it (which occurs on a shorter time scale).

Let us consider sensitivity to an incremental increase in the quantity of supply, starting at the reference state, by changing the number of large dwellings in the centre from 1536 to 1636, i.e. 6%. All prices fall, leading to a proportional relative reduction of 6%. The reductions for large dwellings are therefore greater in absolute terms.

A more dynamic urbanization policy would make supply more elastic to demand: in the case of dwellings in the centre, let us increase the elasticity of production to the same level as in the suburbs, i.e. 0.3. Starting in the reference situation, which is already in equilibrium, more dynamic supply does not result in a marked change in

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2 While a random utility model with a distributed income coefficient confuses the signal somewhat, making it difficult to determine the conditions of equilibrium precisely.

3 Cf. on Figure 2 the proximity between the utility functions for the options with the same floor area but which are located in different zones.
the equilibrium. When the starting situation is out of equilibrium, for supply functions that have been recalibrated from an alternative reference point taken at 90% of the volume produced in the reference situation, the effect is radical: the number of large dwellings increases by 40% in the centre and the price drops by 40%; these large dwellings attract 61% of the large households and 24% of the small ones. Consequently, there is a general reduction in prices; small households also choose large dwellings in the suburbs and small ones in the centre, and the latter in larger numbers too. The small dwellings in the suburbs are abandoned and their price is cancelled.

Last, supply in each zone could also be adjusted by restructuring the dwellings – merging small dwellings together to form a large one or dividing up a large dwelling to form several small ones. In the reference situation, the heterogeneity of rents per unit floor area even within a given zone could result in changes, let us say from 500 small dwellings (40 m²) to 200 large dwellings (100 m²) in the centre. The effect on the equilibrium is to lower the price of large dwellings in the centre, and, indirectly, in the suburbs, but to increase the price of small dwellings. Therefore, large households as well as small households with high incomes win, but the small households with lower income lose.

5.4. Sensitivity analysis of demand

On the demand side, we can vary demographic factors, the number and size of the households, or socio-economic parameters in terms of the level and distribution of incomes.

A homothetic increase of 5% in the number of households generates general price tension, with disparities between different properties that remain at the same level within the same floor area category but a greater gap between large dwellings and small ones. Large households which are subject to space requirements pay higher rents and small households with moderately high incomes leave their large dwellings in the centre and move into the suburbs.

Let us now simulate a structural transformation in households by varying their composition. If 400 small households are replaced by 200 large ones, the number of persons remains the same but there are 200 less workers (because the number of workers is limited to one per household). In the equilibrium state the price tension is therefore greater for large dwellings than small ones. The large households are the losers, due to excessive competition for large dwellings. The proportion of small households that give up their large dwellings in the centre exceeds the proportion by which their number has been reduced, as a result of price changes.

With respect to incomes, we have simulated two variations which at first sight might look equivalent because they have the same impact on mean income. One of these maintains the same level of relative dispersion (i.e. the ratio between the standard deviation and the mean value, which is determined by the \( s \) parameter of a log-normal distribution) and increases the median income (determined by the \( m \) parameter); the other maintains the median but increases the relative dispersion. In the first case, a 10% increase in mean income is obtained by increasing \( m_{\text{small}} \) from 3 to 3.248 and \( m_{\text{large}} \) from 4 to 4.154, keeping the \( s \) parameters at the same level. The effect is to increase the price of large dwellings as there are more wealthy bidders, while at the same time slightly reducing the price of small dwellings, demand for which has fallen as there are more large dwellings on the market in response to the higher prices.

In the second case, the 10% increase in mean incomes is determined by an increase in \( s_{\text{small}} \) from 1 to 1.122 and for \( s_{\text{large}} \) from 1 to 1.144. The Gini coefficients are therefore modified. The effect is almost identical to above, but there are some minor differences: the price of large dwellings increases very slightly less in the centre and very slightly more in the suburbs.
Fig. 3. Price according to type of housing and scenario

Fig. 4. The quantity of different types of housing according to the scenario

Fig. 5. The quantity of housing type by household segment and scenario
6. Conclusion

We have modelled the structure of housing supply according to type and zone, and the structure of demand according to household size, job location and income. We have modelled the microeconomic behaviours of demanders and macroeconomic behaviours on the supply side. The structural features that are represented are fundamental for the medium-term operation of an urban housing market: they provide a minimum core which must feature in any operational (or pre-decisional) application. Our model deals with the disaggregation of demand. The inherent complexity has been reduced mathematically and algorithmically owing to a specific aggregated treatment.

The model could be extended in many ways — economically, mathematically or algorithmically. We shall simply mention, from the economic standpoint, the representation of households with no working member or several workers, the inclusion of social housing, the distinction between the rented sector and the “owner-occupier” sector in the case of the private sector property market, and the inclusion of complex supply behaviours, etc.

As it stands, the model provides a good compromise between explanatory power and processing simplicity. It can be used to study the housing market in a conurbation and the effects of a variety of housing policies, as well as to simulate planning schemes and transport plans. The evaluation of the costs and benefits for the different categories of actors is covered in a companion paper.

References