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Improving bounds for nonmaximal eigenvalues of positive matrices ☆

Stephen G. Walker *

Institute of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, CT2 7NZ, UK

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Abstract

We provide a method for improving bounds for nonmaximal eigenvalues of positive matrices. A numerical example indicates the improvements can be substantial.

Keywords: Nonmaximal eigenvalues; Positive matrix

1. Introduction and preliminary results

Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be a positive matrix, that is $a_{i,j} > 0$ for all i, j, with positive right and left eigenvectors u and v, with $v'u = 1$. Let $\rho(A)$ denote the spectral radius of A and denote the eigenvalues of A by $\lambda_i(A)$ with

$$\rho(A) = \lambda_n(A) > \mathrm{Re}(\lambda_{n-1}(A)) > \cdots > \mathrm{Re}(\lambda_1(A)).$$

This paper is concerned with bounds for

$$\tau(A) = \mathrm{Re}(\lambda_{n-1}(A)) < \rho(A)$$

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* Tel.: +44 1227 823800; fax: +44 1227 827932.
E-mail address: s.g.walker@kent.ac.uk

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and, in particular, provides a simple approach to improving current bounds for $\tau(A)$. Such bounds are important for determining the convergence of powers of the matrix; see, for example, [4].

The idea is to consider the positive matrix

$$A_c = A - \rho(A)uv'/(1+c)$$

for any $1 + c > c^*(A) = \rho(A)\max_{i,j} u_i v_j/a_{ij}$. It is easy to show that

$$\rho(A)\max_{i,j} u_i v_j/a_{ij} \geqslant 1.$$

Assume the contrary, so that $a_{ij} > \rho(A)u_i v_j$ for all i, j. Then

$$\rho(A)u_i = \sum_{j=1}^{n} a_{ij}u_j > \rho(A)\sum_{j=1}^{n} u_i v_j u_j = \rho(A)u_i$$

which is a clear contradiction.

The eigenvalues of A_c are given by

$$\frac{c\rho(A)}{1+c}, \lambda_{n-1}(A), \ldots, \lambda_1(A).$$

So, if $c\rho(A)/(1+c) > \tau(A)$ and $1 + c > c^*(A)$, then we have

$$\tau(A_c) = \tau(A)$$

which forms the basis of the paper. To ensure the former constraint we can take $c\rho(A)/(1+c) > \xi(A) \geqslant \tau(A)$, where $\xi(A)$ is an upper bound for $\tau(A)$. Therefore, we require $c > c_*(A)$ where

$$c_*(A) = \max\left\{c^*(A) - 1, \xi(A)/(\rho(A) - \xi(A))\right\}.$$

So, we have denoted an upper bound for $\tau(A)$ as $\xi(A)$, assumed to be applicable when A is a positive matrix. For example, [1] has

$$\xi(A) = \sqrt{\rho^2(A) - h^2(A)/\delta^2},$$

where $\delta = \max_i u_i v_i$,

$$h(A) = \min_{U\in S} \frac{\sum_{i\in U, j\in U'} a_{ij}v_i u_j + a_{ji}v_j u_i}{2|U|},$$

$$S = \{U : \emptyset \neq U,\ |U| \leqslant \lfloor n/2 \rfloor\},$$

$U' = \langle n \rangle - U$ and $\langle n \rangle = \{1 \ldots n\}$.

Our intention is to apply this bound ξ, and others, to the matrix A_c. The main result is as follows.

Lemma 1. *Let $\xi(A)$ be an upper bound for $\tau(A)$. If $c > c_*(A)$ satisfies $\xi(A_c) < \xi(A)$ then $\xi(A_c)$ is an improved bound for $\tau(A)$, in the sense that*

$$\tau(A) \leqslant \xi(A_c) < \xi(A).$$

Proof. By virtue of the fact that $\tau(A_c) = \tau(A)$, we know that $\tau(A) = \tau(A_c) \leqslant \xi(A_c) < \xi(A)$. Hence, we obtain an improved bound for $\tau(A)$.

Clearly, this method will only be applicable if A_c is a positive matrix and this can only be the case if A is itself a positive matrix. If $a_{i,j} = 0$ for some i, j then there is clearly no finite c for which $A_c > 0$.

Applying the bound ξ to A_c, we obtain

$$\xi(A_c) = \xi\left(A - \rho(A)uv'/(1+c)\right).$$

Consequently, we are then interested in the existence of a $c \in (c_*(A), \infty)$ for which

$$\xi\left(A - \rho(A)uv'/(1+c)\right) < \xi(A).$$

The improved bound for $\tau(A)$ will then be

$$\xi(A - \rho(A)uv'/(1+c)).$$

In [2] a similar approach was described. Essentially [2] took the bound

$$\tau(A) \leqslant \rho(A_c).$$

Clearly, for non-trivial ξ, it will be that $\xi(A_c) < \rho(A_c)$.

We must work on specific bounds and in the next section we consider bounds recently obtained by [1] and also by [3] and show that we can obtain strict improvements. That is, we can find a $c_*(A) < c < \infty$ such that $\xi\left(A - \rho(A)uv'/(1+c)\right) < \xi(A)$. Note, however, that the bounds of [1] and [3] apply to non-negative matrices whereas the improvements are only available for positive matrices. In Section 3 a numerical example is presented which demonstrates significant improvements over a bound obtained by [3].

2. Illustrations

We present two examples of bounds ξ and show that using A_c it is possible to find strict improvements when $A > 0$.

2.1. Berman/Zhang bound

We first work on the [1] bound for $\tau(A)$ which was described in Section 1. Let us define

$$c_0(A) = \max_{U \in S} \left\{ \frac{2\gamma(U) - \delta}{2(\delta - \gamma(U))} \right\}$$

and

$$\gamma(U) = \frac{\sum_{i \in U} u_i v_i \left(1 - \sum_{i \in U} u_i v_i\right)}{|U|}.$$

Theorem 1. *If*

$$\xi(A) = \sqrt{\rho^2(A) - h^2(A)/\delta^2}$$

is an upper bound for $\tau(A)$ *and* $A > 0$ *then an improved upper bound for* $\tau(A)$ *is given by* $\xi(A_c)$ *for any*

$$c > \max\{c_*(A), c_0(A)\}.$$

Proof. It is convenient to also define

$$h(A, U) = \frac{\sum_{i \in U, j \in U'} a_{ij} v_i u_j + a_{ji} v_j u_i}{2|U|}$$

and

$$h(A_c, U) = \frac{\sum_{i \in U, j \in U'} a_{ij} v_i u_j + a_{ji} v_j u_i - 2\rho(A)(1+c)^{-1} u_i v_i u_j v_j}{2|U|}$$
$$= h(A, U) - \rho(A)\gamma(U)/(1+c).$$

For reasons explained in Section 1, we are looking for a finite c for which $\xi(A_c) < \xi(A)$; that is, for which

$$\frac{c^2 \rho^2(A)}{(1+c)^2} - \frac{\{h(A, U) - \rho(A)\gamma(U)/(1+c)\}^2}{\delta^2} < \rho^2(A) - \frac{h^2(A)}{\delta^2}$$

for all $U \in S$. This is equivalent to showing there is a finite c for which

$$2\rho(A)\gamma(U)h(A, U)(1+c) - \gamma^2(U) < (1+2c)\rho^2(A)\delta^2$$

for all $U \in S$.

Now

$$\sum_{j \in U'} a_{ij} u_j \leqslant \rho(A) u_i$$

and

$$\sum_{j \in U'} a_{ji} v_j \leqslant \rho(A) v_i$$

so

$$h(A, U) \leqslant \frac{2\rho(A) \sum_{i \in U} u_i v_i}{2|U|} = \rho(A) \frac{\sum_{i \in U} u_i v_i}{|U|} \leqslant \rho(A)\delta$$

and hence $h(A, U) \leqslant \rho(A)\delta$ for all $U \in S$.

Hence, removing the $\gamma^2(U)$ term, we wish to show that there exists a finite c for which

$$2\gamma(U)\delta(1+c) < (1+c)\delta^2 + c\delta^2$$

for all $U \in S$. This follows since $\gamma(U) < \delta$ and so we can find a finite c for which

$$2\gamma(U) < \delta + c\delta/(1+c)$$

for all $U \in S$. We would take such a c from the set

$$c \in \left(\max_{U\in S}\left\{\frac{2\gamma(U)-\delta}{2(\delta-\gamma(U))}\right\}, \infty\right),$$

completing the proof, since we also need $c > c_*(A)$. □

2.2. Nabben bound

Next we work on one of the bounds provided by [3]. Let us first define

$$l(A) = \min_{U\in S}\frac{\sum_{i\in U, j\in U'} a_{ij}v_i u_j + a_{ji}v_j u_i}{2\rho(A)\sum_{i\in U} u_i v_i},$$

where

$$S = \left\{U : \emptyset \neq U \neq \langle n\rangle, \sum_{i\in U} u_i v_i \leqslant \frac{1}{2}\right\}$$

and also define

$$c_1(A) = \max_{U\in S}\left\{\frac{2l(A,U)\gamma(U)-1}{2(1-l(A,U)\gamma(U))}\right\},$$

$$\gamma(U) = \frac{\sum_{i\in U, j\in U'} u_i v_i u_j v_j}{\sum_{i\in U} u_i v_i} = 1 - \sum_{i\in U} u_i v_i$$

and

$$l(A,U) = \frac{\sum_{i\in U, j\in U'} a_{ij}v_i u_j + a_{ji}v_j u_i}{2\rho(A)\sum_{i\in U} u_i v_i}.$$

Theorem 2. *If*

$$\xi(A) = \rho(A)\sqrt{1-l^2(A)}$$

is an upper bound for $\tau(A)$, then an improved upper bound for $\tau(A)$ is given by $\xi(A_c)$ for any

$$c > \max\{c_*(A), c_1(A)\}.$$

Proof. Following reasons outlined in Section 1, we are interested to show that there exists a finite c for which $\xi(A_c) < \xi(A)$, that is for which

$$\frac{c^2}{(1+c)^2}\rho^2(A)\left[1 - \frac{(1+c)^2}{c^2}\left\{l^2(A,U) - \frac{2\gamma(U)l(A,U)}{1+c} + \frac{\gamma^2(U)}{(1+c)^2}\right\}\right]$$
$$< \rho^2(A)\{1 - l^2(A,U)\}$$

for all $U \in S$. This reduces to finding a finite c for which

$$2l(A, U)\gamma(U) \leqslant 1 + c/(1 + c)$$

for all $U \in S$. Now

$$\gamma(U) = 1 - \sum_{i \in U} u_i v_i$$

which is strictly less than 1 for all $U \in S$ and $l(A, U) \leqslant 1$ for all $U \in S$ and hence such a c can be found. In fact, we can take

$$c \in \left(\max_{U \in S} \left\{ \frac{2l(A, U)\gamma(U) - 1}{2(1 - l(A, U)\gamma(U))} \right\}, \infty \right),$$

completing the proof. □

3. Numerical example

We consider the improvement over the Nabben bound with

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}.$$

Then $\rho(A) = 4$ and $\tau(A) = 1$. We take

$$u = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The $U \in S$ minimising $l(A, U)$ is $U = \{2\}$ and $l(A) = \frac{1}{2}$ giving $\xi(A) = 3.46$.

Now $c_*(A) = 1/3$ and $c_1(A) < c_*(A)$ and for illustrative purposes we take $c = 1$. Then it is easy to show that

$$A_1 = \frac{1}{3}\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$$

which gives (as we know) $\rho(A_1) = 2$ and $\tau(A_1) = 1$. In this case we obtain $l(A_1) = 1/3$ and hence $\xi(A_1) = 1.89$, which is a substantial improvement over 3.46.

In fact it is clear that as $c \downarrow 1/3$ we have $\rho(A_c) \downarrow 1$, $l(A_c) \downarrow 0$ and hence $\xi(A_c) \downarrow 1$.

4. Discussion

Applying bounds ξ to A_c has shown to lead to improvements in bounds for the real part of nonmaximal eigenvalues of positive matrices. If $c > c_*(A)$ and $\xi(A_c) < \xi(A)$ then $\xi(A_c)$ is an improved bound for $\tau(A)$. Applying ξ to A_c should be no more difficult than applying it to A. The additional piece of information is $c_*(A)$ which can be computed using the same pieces of knowledge required to compute ξ, namely $\rho(A)$, u and v.

Walker [5] used a similar technique when A is a positive stochastic matrix to provide improved bounds. In this case A_c needs to be a stochastic matrix and so

$$A_c = \frac{(1+c)A - uv'}{c}$$

was selected for large enough c to ensure A_c is nonnegative. Here u is a column vector of 1s and v is the invariant probability vector associated with the stochastic matrix A.

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